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GLOBAL EXISTENCE AND UNIQUENESS FOR IMPLICIT DIFFERENTIAL EQUATION OF ARBITRARY ORDER

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Abstract. The aim of this paper is to establish the existence result for implicit differential equation of fractional (arbitrary) order via topological transversality theorem known as Leray-Schauder alternative. Further we prove the uniqueness results. The Grownwall's lemma for singular kernels play an important role to prove our results. We verify our results by providing an example.

1. Introduction

It is well known that behavior of many physical systems can be properly described by using the various forms of fractional differential and hence the theory of fractional differential has a great importance. Detail basic theory of fractional calculus, fractional differential equations and its applications, can found in the monographs [5, 9, 12, 15]. We mention few papers [1, 2, 6, 10, 11] in which the interesting theory results of fractional differential equations has been investigated.

Recently, in papers [4, 13] the existence, uniqueness and other properties of solutions for implicit fractional differential equations with initial conditions have been established by using the technique of fixed point theorems and approximated methods.

Inspired by the works of [4, 13] here we study the existence and uniqueness of solution of initial value problem for implicit differential equation of fractional order with Caputo fractional derivative given in the form:

$$ {}^cD^{\alpha}x(t) = f(t,x(t),{}^cD^{\alpha}x(t)), \quad t\in[0,b],\ b>0,\ n-1<\alpha\leqslant n, \tag{1.1}$$

$$ x^k(0) = x_k \in \mathbb{R}^n, \quad k=0,1,\cdots,n-1, \tag{1.2}$$

where $f:[0,b]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a nonlinear continuous function, $x:[0,b]\to\mathbb{R}^n$ and ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α.

The initial value problem (1.1)–(1.2) and their special forms have been studied [3, 4, 13] by using different approaches.

The method which we have used here gives existence as well as the maximal interval of existence. Further, the uniqueness result is obtained with the same assumption of existence result.

We organize this paper as follows: Preliminaries, the basic definitions of fractional calculus, and the theorems which are required in this paper are given in Section 2. In Section 3, we establish existence and uniqueness theorems for problem (1.1)–(1.2). We present an example to illustrate the theorem in Section 3.

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2. Preliminaries

Let $\mathbb{R}^n$ is an Euclidean n-space with the norm $\|\cdot\|$ and denote by $B = C([0,b],\mathbb{R}^n)$ – the Banach space of all continuous functions from $[0,b]$ into R^n with the suprimum norm $\|x\|_B = \sup\{\|x(t)\| : t \in [0,b]\}$.

Let $C[a,b]$ denotes the space of all continuous functions defined over [a,b] and $C^n[a,b]$ be the set of all real valued functions on [a,b] having n^{th} order continuous derivatives.

Here we give some basic definitions and the results [5, 9, 12, 15] which are required throughout this paper.

DEFINITION 2.1. Let $f \in C[0,b]$ and $\alpha > 0$ then Riemann-Liouville fractional integral of order α of a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided the integral exists. Note that, $I^0 f(t) = f(t)$.

DEFINITION 2.2. For a function $f \in C^n[0,b]$, the Caputo derivative of order α is defined as

$$^cD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n-1 < \alpha \leqslant n.$$

LEMMA 2.3. *Let $f \in C^n[0,b]$, and $\alpha > 0$, then*

$$I^{\alpha}[^cD^{\alpha} f(t)] = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{\Gamma(k+1)} t^k, \quad n-1 < \alpha \leqslant n.$$

LEMMA 2.4. *Let $f(t) = t^{\beta}$, where $\beta \geqslant 0$ and let $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$, then*

$$^cD^{\alpha} f(t) = \begin{cases} 0 & \text{if } \beta \in \{0,1,\cdots,n-1\} \\ \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} & \text{if } \beta \in \mathbb{N},\ \beta \geqslant n \text{ or } \beta \notin \mathbb{N},\ \beta > n-1. \end{cases}$$

LEMMA 2.5. [8] *Suppose $\delta \geqslant 0$, $\alpha > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leqslant t < T$ (some $T \leqslant \infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leqslant t < T$ with*

$$u(t) \leqslant a(t) + \delta \int_0^t (t-s)^{\alpha-1} u(s)ds$$

on this interval; then

$$u(t) \leqslant a(t) + \theta \int_0^t E'_{\alpha}(\theta(t-s)) a(s)ds, \ \ 0 \leqslant t < T,$$

where $\theta = (\delta\Gamma(\alpha))^{\frac{1}{\alpha}}$, $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^{n\alpha}}{\Gamma(n\alpha+1)}$, $E'_\alpha(z) = \dfrac{d}{dz} E_\alpha(z)$, $E'_\alpha(z) \simeq \frac{z^{\alpha-1}}{\Gamma(\alpha)}$ *as* $z \to 0^+$, $E'_\alpha(z) \simeq \frac{1}{\alpha} e^z$ *as* $z \to +\infty$ *(and* $E_\alpha(z) \simeq \frac{1}{\alpha} e^z$ *as* $z \to \infty^+$ *). If* $a(t) \equiv a$*, constant, then* $u(t) \leqslant aE_\alpha(\theta t)$.

DEFINITION 2.6. If $f \in C([0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, then $x \in B$ given by

$$x(t) = \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^c D^\alpha x(s)) ds, \ \ t \in [0,b],$$

is the solution of implicit fractional differential equation (1.1)–(1.2).

To prove the existence result, we use the topological transversality theorem known as Leray-Schauder alternative given below.

LEMMA 2.7. ([7]) *Let* S *be a convex subset of a normed linear space* E *and assume that* $0 \in S$. *Let* $F : S \to S$ *be a completely continuous operator, and let*

$$\varepsilon(F) = \{x \in S : x = \lambda F x, \ \ 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ *is unbounded or* F *has a fixed point in* S.

Study of integer order differential equations via topological transversality can be found in [14] and the references therein.

3. Existence and uniqueness of solution

THEOREM 3.1. (Existence) *Let* $f : [0,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous function that satisfies the conditions:*

(H1) $\|f(t,x,y)\| \leqslant p(t)\|x\| + L\|y\|$, $t \in [0,b]$ *and* $x, y \in \mathbb{R}^n$, *where* $p(t) : [0,b] \to (0,\infty)$ *is continuous function and* $0 < L < 1$.

(H2) There exist constants $M > 0$, $0 < N < 1$ *such that*

$$\|f(t,x,y) - f(t,\bar{x},\bar{y})\| \leqslant M\|x - \bar{x}\| + N\|y - \bar{y}\|, \ \ t \in [0,b], \ \ x, \bar{x}, y, \bar{y} \in \mathbb{R}^n.$$

Then the initial-value problem for implicit fractional differential equation (1.1)–(1.2) *has at least one solution on* $[0,b]$.

Proof. Firstly we establish the priori bounds on the solutions of implicit fractional differential equation of the form:

$$ {}^c D^\alpha x(t) = \lambda f(t, x(t), {}^c D^\alpha x(t)), \quad t \in [0,b], \ b > 0, \ n-1 < \alpha \leqslant n, \tag{3.1}$$

$$ x^k(0) = x_k \in \mathbb{R}^n, \quad k = 0, 1, \cdots, n-1, \tag{3.2}$$

for $\lambda \in (0,1)$.

Let $x(t)$ be the solution of (3.1)–(3.2) then it is equivalent to integral equation

$$x(t)=\sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}t^k+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s),{}^cD^{\alpha}x(s))ds.$$

Then for any $t\in[0,b]$ we have

$$\begin{aligned}\|x(t)\| &\leqslant \sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,x(s),{}^cD^{\alpha}x(s))\|ds\\ &\leqslant \sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(p(s)\|x(s)\|+L\|{}^cD^{\alpha}x(s)\|\right)ds \qquad (3.3)\end{aligned}$$

From (3.1) and the hypothesis $(H1)$ for any $t\in[0,b]$ we have,

$$\|{}^cD^{\alpha}x(t)\|\leqslant p(t)\|x(t)\|+L\|{}^cD^{\alpha}x(t)\|,$$

and hence $\|{}^cD^{\alpha}x(t)\|\leqslant\frac{p(t)}{1-L}\|x(t)\|$. Thus for any $t\in[0,b]$, (3.3) gives

$$\begin{aligned}\|x(t)\| &\leqslant \sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(p(s)\|x(s)\|+\frac{Lp(s)}{1-L}\|x(s)\|\right)ds\\ &\leqslant \sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}b^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\frac{p(s)}{1-L}\|x(s)\|ds\end{aligned}$$

As $p(t):[0,b]\to(0,\infty)$ is continuous on compact set $[0,b]$ there exist constant P such that $p(t)\leqslant P<\infty$, $t\in[0,b]$. Therefore,

$$\|x(t)\|\leqslant\sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}b^k+\frac{P}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x(s)\|ds.\quad t\in[0,b].$$

By an application of Lemma 2.5, with $a(t)=\sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}b^k$, a constant, and $\delta=\frac{P}{(1-L)\Gamma(\alpha)}$, we obtain

$$\begin{aligned}\|x(t)\| &\leqslant \sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}b^kE_{\alpha}\left(\left[\frac{P}{(1-L)\Gamma(\alpha)}\Gamma(\alpha)\right]^{\frac{1}{\alpha}}t\right)\\ &=\sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}b^kE_{\alpha}\left(\left[\frac{P}{1-L}\right]^{\frac{1}{\alpha}}t\right)\\ &\leqslant\xi,\ t\in[0,b],\end{aligned}$$

where ξ is some nonnegative constant. Therefore $\|x\|_B\leqslant\xi$. This proves the solution of (3.1)–(3.2) is bounded.

Now let $y\in B$ and define

$$x(t)=\sum_{k=0}^{k=n-1}\frac{x_k}{\Gamma(k+1)}t^k+y(t),\ t\in[0,b].$$

Then $y(t)$ satisfies

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}s^k + y(s), {}^cD^\alpha\left(\sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}s^k + y(s)\right)\right)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}s^k + y(s), {}^cD^\alpha y(s)\right)ds \end{aligned}$$

if and only if $x(t)$ satisfies

$$x(t) = \sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}t^k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^cD^\alpha x(s))ds, \quad t \in [0,b]$$

Denote $B_0 = \{y \in B : y^{(k)}(0) = 0,\ k = 0,1,\dots,n-1\}$ then clearly B_0 is a convex subset of the space B and $0 \in B_0$. Define an operator $F : B_0 \to B_0$ by:

$$(Fy)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}s^k + y(s), {}^cD^\alpha y(s)\right)ds.$$

Let $y_n \to y$ in B_0. Then using (H2) for any $t \in [0,b]$ we have

$$\begin{aligned} \|(Fy_n)(t) - (Fy)(t)\| \leqslant{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \left\| f\left(s, \sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}s^k + y_n(s), {}^cD^\alpha y_n(s)\right)\right. \\ &\left. - f\left(s, \sum_{k=0}^{n-1}\frac{x_k}{\Gamma(k+1)}s^k + y(s), {}^cD^\alpha y(s)\right)\right\| ds \\ \leqslant{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(M\|(y_n-y)(s)\| + N\|{}^cD^\alpha(y_n-y)(s)\|\right)ds. \end{aligned}$$

But (H1) gives

$$\|{}^cD^\alpha(y_n-y)(t)\| \leqslant \frac{p(t)}{1-L}\|(y_n-y)(t)\|, \quad t \in [0,b].$$

Therefore

$$\begin{aligned} \|(Fy_n)(t) - (Fy)(t)\| \leqslant{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(M\|(y_n-y)(s)\| + \frac{Np(s)}{1-L}\|(y_n-y)(s)\|\right)ds \\ \leqslant{}& \frac{M(1-L)+NP}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|(y_n-y)(s)\|ds \\ \leqslant{}& \frac{(M(1-L)+NP)b^\alpha}{(1-L)\Gamma(\alpha+1)}\|y_n-y\|_B, \quad t \in [0,b]. \end{aligned}$$

It follows that

$$\|Fy_n - Fy\|_B \leqslant \frac{(M(1-L)+NP)b^\alpha}{(1-L)\Gamma(\alpha+1)}\|y_n-y\|_B,$$

Thus $Fy_n \to Fy$ in B_0 whenever $y_n \to y$ in B_0. This proves F is continuous on B_0.

Next, we show that F is completely continuous. Let $\{u_n\}$ be a bounded sequence in B_0.Then there exists a constant $K > 0$ such that $\|u_n\| \leqslant K \ \forall n$.

Using the definition of F and the hypothesis $(H1)$, for any $t \in [0,b]$, we obtain

$$\begin{aligned}
\|(Fu_n)(t)\| &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) \right\| ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ p(s) \left(\sum_{k=0}^{n-1} \frac{\|x_k\|}{\Gamma(k+1)} b^k + \|u_n(s)\| \right) + L\|{}^cD^\alpha u_n(s)\|) \right\} ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ p(s) \left(\sum_{k=0}^{n-1} \frac{\|x_k\|}{\Gamma(k+1)} b^k + \|u_n(s)\| \right) + \frac{Lp(s)}{1-L} \|u_n(s)\| \right\} ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ P \sum_{k=0}^{n-1} \frac{\|x_k\|}{\Gamma(k+1)} b^k + \frac{PK}{1-L} \right\} ds \\
&\leqslant \frac{b^\alpha}{\Gamma(\alpha+1)} \left\{ P \sum_{k=0}^{n-1} \frac{\|x_k\|}{\Gamma(k+1)} b^k + \frac{PK}{1-L} \right\}.
\end{aligned}$$

This proves the set $\{(Fu_n)(t) : \|u_n\|_B \leqslant K, 0 \leqslant t \leqslant b\}$ is uniformly bounded in $\mathbb{R}^n$.

Next aim is to show that the sequence $\{Fu_n\}$ is equicontinuous. Let any $t_1, t_2 \in [0,b]$. By definition of F, assumption $(H1)$ and letting $0 \leqslant t_1 \leqslant t_2 \leqslant b$, we have

$$\begin{aligned}
&\|(Fu_n)(t_1) - (Fu_n)(t_2)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} (t_1-s)^{\alpha-1} f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) ds \right. \\
&\qquad \left. - \int_0^{t_2} (t_2-s)^{\alpha-1} f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) ds \right\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) ds \right. \\
&\qquad \left. - \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) ds \right\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] \left\| f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) \right\| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left\| f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) \right\| ds. \qquad (3.4)
\end{aligned}$$

Noting that,

$$\left\| f\left(s, \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} s^k + u_n(s), {}^cD^\alpha u_n(s)\right) \right\| \leqslant \left\{ P \sum_{k=0}^{n-1} \frac{\|x_k\|}{\Gamma(k+1)} b^k + \frac{PM}{1-L} \right\},$$

from the inequality (3.4) we obtain

$$\|(Fu_n)(t_1) - (Fu_n)(t_2)\| \leqslant \frac{1}{\Gamma(\alpha+1)} \left\{ P \sum_{k=0}^{n-1} \frac{\|x_k\|}{\Gamma(k+1)} b^k + \frac{PM}{1-L} \right\} [t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha]$$

$$\leqslant \frac{2}{\Gamma(\alpha+1)}\left\{P\sum_{k=0}^{n-1}\frac{\|x_k\|}{\Gamma(k+1)}b^k+\frac{PM}{1-L}\right\}(t_2-t_1)^{\alpha}. \tag{3.5}$$

From the inequality 3.5 we conclude that $\{Fu_n\}$ is equicontinuous family and hence by Arzela-Ascoli argument the operator F is completely continuous.

Finally we see that the set $\varepsilon(F)=\{y\in B_0 : y=\lambda Fy, 0<\lambda<1\}$ is bounded in B, since for any $y\in\varepsilon(F)$, the function

$$x(t)=\sum_{k=0}^{n-1}\frac{t^k}{\Gamma(k+1)}x_k+y(t),\ \ t\in[0,b],$$

is a solution of (3.1)–(3.2) for which we have shown that $\|x\|_B\leqslant\xi$ and hence $\|y\|_B\leqslant \xi+\sum_{k=0}^{n-1}\frac{b^k}{\Gamma(k+1)}\|x_k\|$.

Therefore by Lemma 2.7, the operator F has a fixed point $\widetilde{y}\in B_0$. Then

$$\widetilde{x}(t)=\sum_{k=0}^{n-1}\frac{t^k}{\Gamma(k+1)}x_k+\widetilde{y}(t),\ \ t\in[0,b],$$

is the solution of (1.1)–(1.2). This completes the proof. □

REMARK 3.1. Note that an application of the topological transversality theorem (Leray-Schauder alternative) does not guarantee an uniqueness of the solution. With the same assumption of Theorem 3.1 we have following uniqueness theorem.

THEOREM 3.2. (Uniqueness) *Let $f:[0,b]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the assumptions (H1) and (H2). Then the initial-value problem for implicit fractional differential equation* (1.1)–(1.2) *has a unique solution.*

Proof. Let $x_1(t)$ and $x_2(t)$ be any two solution of (1.1)–(1.2) then by using $(H2)$ for any $t\in[0,b]$ we have

$$\begin{aligned}\|x_1(t)-x_2(t)\| &\leqslant \int_0^t (t-s)^{\alpha-1}\left(\|f(s,x_1(s),{}^cD^{\alpha}x_1(s))\|-\|f(s,x_2(s),{}^cD^{\alpha}x_2(s))\|\right)ds\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(M\|(x_1-x_2)(s)\|+N\|{}^cD^{\alpha}(x_1-x_2)(s)\|\right)ds.\end{aligned}$$

By $(H1)$ for any $t\in[0,b]$,

$$\|{}^cD^{\alpha}(x_1-x_2)(t)\|\leqslant\frac{p(t)}{1-L}\|(x_1-x_2)(t)\|.$$

Hence for any $t\in[0,b]$ we have

$$\begin{aligned}\|(x_1-x_2)(t)\| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(M\|(x_1-x_2)(s)\|+\frac{Np(s)}{1-L}\|(x_1-x_2)(s)\|\right)ds\\ &\leqslant \frac{M(1-L)+NP}{\Gamma(\alpha)(1-L)}\int_0^t (t-s)^{\alpha-1}\|(x_1-x_2)(s)\|ds.\end{aligned}$$

Using Lemma 2.5 and noting that $a(t)=0$ for above inequality , we obtain

$$\|(x_1-x_2)(t)\|=0,\ \forall t\in[0,b].$$

This proves uniqueness of solutions of (1.1)–(1.2). □

4. Example

In this section, we give an example to illustrate the results.

EXAMPLE. Consider the implicit fractional differential equation

$$^cD^{\frac{5}{2}}x(t)=\frac{e^t}{e^t+5}x(t)+\frac{1}{2}\sin(\|^cD^{\frac{5}{2}}x(t)\|),\ \ t\in[0,b],\ \ 0<b<\infty. \tag{4.1}$$

Define $f:[0,b]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is given by

$$f\left(t,x(t),^cD^{\frac{5}{2}}x(t)\right)=\frac{e^t}{e^t+5}x(t)+\frac{1}{2}\sin(\|^cD^{\frac{5}{2}}x(t)\|),\ \ t\in[0,b]$$

Then

$$\begin{aligned}\left\|f\left(t,x(t),^cD^{\frac{5}{2}}x(t)\right)\right\| &\leqslant \frac{e^t}{e^t+5}\|x(t)\|+\frac{1}{2}\left|\sin(\|^cD^{\frac{5}{2}}x(t)\|)\right|\\ &\leqslant \frac{e^t}{e^t+5}\|x(t)\|+\frac{1}{2}\|^cD^{\frac{5}{2}}x(t)\|.\end{aligned}$$

We see that $p(t)=\frac{e^t}{e^t+5}:[0,b]\to(0,\infty)$ is a continuous function and $L=\frac{1}{2}$.

Further, noting that $|\sin u-\sin v|=\left|2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)\right|\leqslant|u-v|$, for any $u,v\in\mathbb{R}$, we have

$$\begin{aligned}&\left\|f\left(t,x(t),^cD^{\frac{5}{2}}x(t)\right)-f\left(t,\bar{x}(t),^cD^{\frac{5}{2}}\bar{x}(t)\right)\right\|\\ &\leqslant \frac{e^t}{e^t+5}\|x(t)-\bar{x}(t)\|+\frac{1}{2}\left|\sin(\|^cD^{\frac{5}{2}}x(t)\|-\sin(\|^cD^{\frac{5}{2}}\bar{x}(t)\|)\right|\\ &\leqslant \frac{e^t}{e^t+5}\|x(t)-\bar{x}(t)\|+\frac{1}{2}\left|\|^cD^{\frac{5}{2}}x(t)\|-\|^cD^{\frac{5}{2}}\bar{x}(t)\|\right|\\ &\leqslant M\|x(t)-\bar{x}(t)\|+\frac{1}{2}\left\|^cD^{\frac{5}{2}}x(t)-{}^cD^{\frac{5}{2}}\bar{x}(t)\right\|,\end{aligned}$$

where $M>0$ is a constant such that $\left|\frac{e^t}{e^t+5}\right|\leqslant M$, $t\in[0,b]$.

Since all the assumptions of Theorems 3.1 and 3.2 are satisfied the initial value problem (4.1) has unique solution on $[0,b]$ for any $0<b<\infty$.

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REFERENCES

[1] R. P. AGARWAL, M. BELMEKKI, M. BENCHOHRA, *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Diff. Equ. **47** (2009), Article ID 981728.

[2] R. P. AGARWAL, YONG ZHOU, YUNYUN HE, *Existence of fractional neutral functional differential equations*, Computers and Mathematics with Applications, **59** (2010), 1095–1100.

[3] S. BHALEKAR, V. DAFTARDAR-GEJJI, *A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order*, Journal of Fractional Calculus and Applications, **1** (5) (2011), 1–9.

[4] M. BENCHOHRA, M. SAID SOUID, *Integrable solutions for implicit fractional order differential equations*, TJMM, **6** (2) (2014), 101–107.

[5] K. DIETHELM, *The analysis of fractional differential equations, Lecture Notes in Mathematics*, Springer-Verlag Berlin Heidelberg, 2010.

[6] K. DIETHELM, N. J. FORD, *Analysis of fractional differential equations*, J. Math. Anal. Appl., **265** (2002), 229–248.

[7] J. DUGUNDJI, A. GRANAS, *Fixed Point Theory*, Vol-I, Monographic Matematycane, PNW Warsa, 1982.

[8] D. HENRY, *Geometric theory of semi linear parabolic equations*, Springer-Verlag; Berlin, Heidelberge, New York, 1981.

[9] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[10] V. LAKSHIMIKANTHAM, S. LEELA, *A Krasnoselskii-Krein-type uniqueness result for fractional differential equations*, Nonlinear Analysis, **71** (2009), 3421–3424.

[11] V. LAKSHMIKANTHAM, A. S. VATSALA, *Basic theory of fractional differential equations*, Nonlinear Analysis, **69** (2008), 2677–2682.

[12] K. S. MILLER, B. ROSS, *An introduction to the fractional calculus and differential equations*, John Wiley, New York, 1993.

[13] JUAN J. NIETO, ABELGHANI OUAHAB, VENKTESH VENKTESH, *Implicit fractional differential equations via Liouville-Caputo derivative*, Mathematics, **3** (2015), 398–411.

[14] S. K. NTOUYAS, *Initial and boundary value problems for functional differential equations via the topological transversality method: A survey*, Bull. Greek Math. Soc. **40** (1998), 3–41.

[15] I. PODLUBNY, *Fractional differential equations*, Academic Press, San Diego, 1999.

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