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SOME NEW OSCILLATION CRITERIA FOR A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Several oscillation criteria are established for nonlinear fractional differential equations of the form

$$\left\{a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}\right\}' - F\left(t, \int_t^{\infty}(v-t)^{-\alpha}x(v)dv\right) = 0,$$

where $D_{-}^{\alpha}x(t)$ is the Liouville right-side fractional derivative of order $\alpha \in (0,1)$ of $x(t), \eta = \frac{2n+1}{2m+1}$, and $n,m \in N$. $F(t,G) \in C([t_0,\infty)\times R; R)$, and there exists function $q(t) \in C^1([t_0,\infty); R_+)$ such that $\frac{F(t,G)}{G^{\eta}} \geqslant q(t)$ for $G \neq 0$ and $x \neq 0$, $t \geqslant t_0$. We also give some examples to illustrate the main results.

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order and can be found their wide applications in many fields of science and engineering, such as viscoelasticity, mechanics, dynamical processes in self-similar and porous structures, etc., see [1, 8, 11]. In the last few decades, research on various aspects of fractional differential equations, for example, the existence, uniqueness and stability of solutions of fractional differential equations, the numerical methods for fractional differential equations, and so on, has been paid much attention by scholars, refer [3, 4, 7, 9, 14]. Recently, many articles have investigated oscillation of various fractional differential equations, see [2, 5, 10, 12, 13].

In [5], Han et al. have established some oscillation criteria for the equations

$$\left[r(t)g\left(D_{-}^{\alpha}y(t)\right)\right]' - p(t)f\left(\int_t^{\infty}(s-t)^{-\alpha}y(s)ds\right) = 0,\ t>0,\ 0<\alpha<1.$$

In [10], by means of a generalized Ricaati transformation and inequality technique, Qi studied some new interval oscillation criteria for the following equations

$$\left(a(t)\left[r(t)D_{-}^{\alpha}x(t)\right]'\right)' + p(t)\left[r(t)D_{-}^{\alpha}x(t)\right]' - q(t)\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi = 0$$

for $t \in [t_0,\infty)$, $\alpha \in (0,1)$.

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In [2], Chen considered the oscillation of the fractional differential equations

$$\left[r(t)\left(D_-^\alpha x(t)\right)^\eta\right]' - q(t)f\left(\int_t^\infty (v-t)^{-\alpha}x(v)dv\right) = 0,\ t>0,\ 0<\alpha<1.$$

In [13], Zheng researched oscillation of the following nonlinear fractional differential equations with damping term

$$\left[a(t)\left(D_-^\alpha x(t)\right)^\gamma\right]' + p(t)\left(D_-^\alpha x(t)\right)^\gamma - q(t)f\left(\int_t^\infty (\xi-t)^{-\alpha}x(\xi)d\xi\right) = 0$$

for $t \in [t_0,\infty),\ 0<\alpha<1$.

In [12], Xu discussed the oscillation of the following equations

$$\left\{a(t)\left[(r(t)D_-^\alpha x(t))'\right]^\eta\right\}' - F\left(t, \int_t^\infty (v-t)^{-\alpha}x(v)dv\right) = 0,\ \ t \geqslant t_0 > 0,\ \alpha \in (0,1).$$

To the best of our knowledge, nothing is known regarding the oscillatory behaviour for the following fractional differential equations

$$\left\{a(t)\left[\left(r(t)g\left(D_-^\alpha x(t)\right)\right)'\right]^\eta\right\}' - F\left(t, \int_t^\infty (v-t)^{-\alpha}x(v)dv\right) = 0,\ \ t \geqslant t_0 > 0, \tag{1.1}$$

where $\eta = \frac{2n+1}{2m+1}$, and $n,m \in N$. In [2, 5, 10, 12, 13] and Eq. (1.1), $D_-^\alpha x(t)$ is the Liouville right-side fractional derivative of order $\alpha \in (0,1)$ of $x(t)$ defined by

$$D_-^\alpha x(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty (v-t)^{-\alpha}x(v)dv,\ \ t \in R_+ = (0,\infty),$$

here $\Gamma(\cdot)$ is the gamma function defined by $\Gamma(t) = \int_0^\infty e^{-s}s^{t-1}ds$, $t \in R_+$.

Throughout this paper, the following conditions are assumed to hold:

(H_1) $a(t) \in C^1([t_0,\infty);R_+)$, $r(t) \in C^2([t_0,\infty);R_+)$, $\int_{t_0}^\infty \frac{1}{a^{\frac{1}{\eta}}(s)}ds = \infty$.

(H_2) $g \in C^2(R;R)$, g is an increasing function and there exists positive k such that $\frac{x}{g(x)} \geqslant k > 0$ for $xg(x) \neq 0$. $g^{-1} \in C(R;R)$ are continuous functions with $ug^{-1}(u) > 0$ for $u \neq 0$, and there exits some positive constant γ_1 such that $g^{-1}(uv) \geqslant \gamma_1 g^{-1}(u)g^{-1}(v)$ for $uv \neq 0$.

(H_3) $F(t,G) \in C([t_0,\infty)\times R;R)$, there exists function $q(t) \in C^1([t_0,\infty);R_+)$ such that $\frac{F(t,G)}{G^\eta} \geqslant q(t)$ for $G \neq 0$ and $x \neq 0$, $t \geqslant t_0$.

By a solution of Eq. (1.1), we mean a function $x(t) \in C(R_+;R)$ such that $\int_t^\infty (v-t)^{-\alpha}x(v)dv \in C^1(R_+;R)$, $r(t)g(D_-^\alpha x(t)) \in C^2(R_+;R)$, and satisfies Eq. (1.1) on $[t_0,\infty)$. A nontrivial solution of Eq. (1.1) is called oscillatory if it has arbitrary large zero. Otherwise, it is called nonoscillatory. Eq. (1.1) is called oscillatory if all of its solutions are oscillatory.

The organization of the rest of this paper is as follows: in Section 2, we establish some new oscillation criteria for Eq. (1.1) by a generalized Riccati transformation and inequality technique, and we present some applications for our results in Section 3.

2. Main results

First, we set

$$G(t)=\int_t^{\infty}(v-t)^{-\alpha}x(v)dv, \tag{2.1}$$

then it follows that

$$G'(t)=-\Gamma(1-\alpha)D_{-}^{\alpha}x(t). \tag{2.2}$$

The following lemmas are fundamental in the proofs of our main results.

LEMMA 1. (see [6]) *If X and Y are nonnegative, then*

$$mXY^{m-1}-X^m\leqslant(m-1)Y^m,\quad m>1. \tag{2.3}$$

LEMMA 2. *Assume that $x(t)$ is an eventually positive solution of Eq. (1.1). If*

$$\int_{t_0}^{\infty}g^{-1}\left(\frac{1}{r(s)}\right)ds=\infty, \tag{2.4}$$

$$\int_{t_0}^{\infty}g^{-1}\left(\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty}q(s)ds\right]^{\frac{1}{\eta}}d\tau\right)d\xi=\infty, \tag{2.5}$$

then there exists a sufficiently large T such that $\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'<0$ on $[T,\infty)$, and one of the following two conditions hold:

(i) $D_{-}^{\alpha}x(t)<0$ on $[T,\infty)$, (ii) $D_{-}^{\alpha}x(t)>0$ on $[T,\infty)$, and $\lim_{t\to\infty}G(t)=0$.

Proof. From the hypothesis, there exists $t_1\geqslant t_0$ such that $x(t)>0$ on $[t_1,\infty)$, so that $G(t)>0$ on $[t_1,\infty)$, thus we have

$$\left\{a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}\right\}'=F\left(t,\int_t^{\infty}(v-t)^{-\alpha}x(v)dv\right)\geqslant q(t)G^{\eta}(t)>0,\ t\in[t_1,\infty). \tag{2.6}$$

Then $a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}$ is strictly increasing on $[t_1,\infty)$, consequently we can conclude that $\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'$ is eventually of one sign. We claim that $\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'$ <0 on $[t_2,\infty)$, where $t_2>t_1$ is sufficiently large. Otherwise, there exists $t_3>t_2$ such that $\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'|_{t=t_3}>0$, then we can derive

$$a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}\geqslant a(t_3)\left[\left(r(t_3)g\left(D_{-}^{\alpha}x(t_3)\right)\right)'\right]^{\eta}=\delta>0,\quad t\in[t_3,\infty). \tag{2.7}$$

Hence, we have

$$\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\geqslant\frac{\delta^{\frac{1}{\eta}}}{a^{\frac{1}{\eta}}(t)}>0,\ t\in[t_3,\infty), \tag{2.8}$$

integrating two sides of above inequality from t_3 to t leads to

$$r(t)g\left(D_-^{\alpha}x(t)\right)-r(t_3)g\left(D_-^{\alpha}x(t_3)\right)\geqslant \delta^{\frac{1}{\eta}}\int_{t_3}^{t}\frac{1}{a^{\frac{1}{\eta}}(s)}ds. \tag{2.9}$$

From (H_1), we have $\lim\limits_{t\to\infty} r(t)g\left(D_-^{\alpha}x(t)\right)=\infty$, which implies that for certain constant $t_4\geqslant t_3$, $r(t)g\left(D_-^{\alpha}x(t)\right)>0$, $t\in[t_4,\infty)$, it is clear that

$$r(t)g\left(D_-^{\alpha}x(t)\right)\geqslant\left[r(t)g\left(D_-^{\alpha}x(t)\right)\right]|_{t=t_4}=c>0,\ \ t\in[t_4,\infty). \tag{2.10}$$

Thus, from (H_2), we have

$$-\frac{G'(t)}{\Gamma(1-\alpha)}=D_-^{\alpha}x(t)\geqslant g^{-1}\left(\frac{c}{r(t)}\right)\geqslant\gamma_1 g^{-1}(c)g^{-1}\left(\frac{1}{r(t)}\right),\ \ t\in[t_4,\infty), \tag{2.11}$$

then, we get

$$g^{-1}\left(\frac{1}{r(t)}\right)\leqslant-\frac{G'(t)}{\gamma_1\Gamma(1-\alpha)g^{-1}(c)},\quad t\in[t_4,\infty). \tag{2.12}$$

Integrating both sides of above inequality from t_4 to t, we have

$$\int_{t_4}^{t} g^{-1}\left(\frac{1}{r(s)}\right)ds\leqslant-\frac{G(t)-G(t_4)}{\gamma_1\Gamma(1-\alpha)g^{-1}(c)}\leqslant\frac{G(t_4)}{\gamma_1\Gamma(1-\alpha)g^{-1}(c)},\quad t\in[t_4,\infty). \tag{2.13}$$

Letting $t\to\infty$, we obtain

$$\int_{t_4}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right)ds\leqslant\frac{G(t_4)}{\gamma_1\Gamma(1-\alpha)g^{-1}(c)}<\infty,\quad t\in[t_4,\infty), \tag{2.14}$$

this contradicts (2.4), accordingly $\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'<0$ on $[t_2,\infty)$. Then from (H_2), we can get $D_-^{\alpha}x(t)$ is eventually of one sign, there are two possibilities: (i) $D_-^{\alpha}x(t)<0$ on $[T,\infty)$, (ii) $D_-^{\alpha}x(t)>0$ on $[T,\infty)$, where T is sufficiently large.

Suppose that $D_-^{\alpha}x(t)>0$, $t\in[T,\infty)$ for certain sufficiently large constant $T>t_2$, so $G'(t)<0$, $t\in[T,\infty)$, and we have $\lim\limits_{t\to\infty}G(t)=\beta\geqslant 0$. We claim that $\beta=0$, otherwise, let $\beta>0$, then $G(t)\geqslant\beta$ on $[T,\infty)$, by (2.6), we have

$$\left\{a(t)\left[\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'\right]^{\eta}\right\}'\geqslant q(t)G^{\eta}(t)\geqslant q(t)\beta^{\eta},\quad t\in[T,\infty). \tag{2.15}$$

Integrating two sides of above inequality from t to ∞ leads to

$$\begin{aligned}-a(t)\left[\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'\right]^{\eta}&\geqslant\beta^{\eta}\int_t^{\infty}q(s)ds-\lim_{t\to\infty}a(t)\left[\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'\right]^{\eta}\\&\geqslant\beta^{\eta}\int_t^{\infty}q(s)ds,\end{aligned} \tag{2.16}$$

which means

$$\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'\leqslant-\beta\left[\frac{1}{a(t)}\int_t^{\infty}q(s)ds\right]^{\frac{1}{\eta}},\quad t\in[T,\infty). \tag{2.17}$$

Integrating two sides of (2.17) from t to ∞ yields

$$\begin{aligned}-r(t)g\left(D_{-}^{\alpha}x(t)\right) &\leqslant -\lim_{t\to\infty} r(t)g\left(D_{-}^{\alpha}x(t)\right) - \beta\int_{t}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{1}{\eta}} d\tau \\ &\leqslant -\beta\int_{t}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{1}{\eta}} d\tau, \quad t\in[T,\infty), \qquad (2.18)\end{aligned}$$

that is

$$\begin{aligned}-\frac{G'(t)}{\Gamma(1-\alpha)} = D_{-}^{\alpha}x(t) &\geqslant g^{-1}\left\{\frac{\beta}{r(t)}\int_{t}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{1}{\eta}} d\tau\right\} \\ &\geqslant \gamma_1 g^{-1}(\beta)g^{-1}\left\{\frac{1}{r(t)}\int_{t}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{1}{\eta}} d\tau\right\}. \quad (2.19)\end{aligned}$$

Integrating both sides of (2.19) from T to t, we obtain

$$G(t) \leqslant G(T) - \gamma_1\Gamma(1-\alpha)g^{-1}(\beta)\int_{T}^{t} g^{-1}\left\{\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{1}{\eta}} d\tau\right\}d\xi, \qquad (2.20)$$

letting $t\to\infty$, from (2.5), we get $\lim_{t\to\infty} G(t) = -\infty$, which contradicts the fact that $G(t) > 0$. Therefore, we get that $\beta = 0$, that is $\lim_{t\to\infty} G(t) = 0$. The proof is complete. $\square$

LEMMA 3. *Assume that $x(t)$ is an eventually positive solution of Eq. (1.1) such that $\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)' < 0$, $D_{-}^{\alpha}x(t) < 0$ on $[t_1,\infty)$, where t_1 is sufficiently large and $t_1 > t_0$, then we have*

$$G'(t) \geqslant \frac{-k\Gamma(1-\alpha)A_1(t_1,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'}{r(t)}, \qquad (2.21)$$

$$G(t) \geqslant -k\Gamma(1-\alpha)A_2(t_1,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)', \qquad (2.22)$$

where $A_1(t_1,t) = \int_{t_1}^{t}\frac{1}{a^{\frac{1}{\eta}}(s)}ds$, $A_2(t_1,t) = \int_{t_1}^{t}\frac{A_1(t_1,s)}{r(s)}ds$.

Proof. From (2.6), we deduce that $a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}$ is strictly increasing on $[t_1,\infty)$, so we get

$$\begin{aligned}r(t)g\left(D_{-}^{\alpha}x(t)\right) &\leqslant r(t)g\left(D_{-}^{\alpha}x(t)\right) - r(t_1)g\left(D_{-}^{\alpha}x(t_1)\right) = \int_{t_1}^{t}\frac{a^{\frac{1}{\eta}}(s)\left(r(s)g\left(D_{-}^{\alpha}x(s)\right)\right)'}{a^{\frac{1}{\eta}}(s)}ds \\ &\leqslant a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\int_{t_1}^{t}\frac{1}{a^{\frac{1}{\eta}}(s)}ds \\ &= A_1(t_1,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'.\end{aligned}$$

From (H_2), we obtain

$$\frac{1}{k}r(t)D_{-}^{\alpha}x(t) \leqslant r(t)g\left(D_{-}^{\alpha}x(t)\right) \leqslant A_1(t_1,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)',$$

that is

$$G'(t) \geqslant \frac{-k\Gamma(1-\alpha)A_1(t_1,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'}{r(t)}.$$

Then

$$\begin{aligned} G(t)-G(t_1) &\geqslant -\int_{t_1}^{t}\frac{k\Gamma(1-\alpha)A_1(t_1,s)a^{\frac{1}{\eta}}(s)\left(r(s)g\left(D_{-}^{\alpha}x(s)\right)\right)'}{r(s)}ds \\ &\geqslant -k\Gamma(1-\alpha)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\int_{t_1}^{t}\frac{A_1(t_1,s)}{r(s)}ds, \end{aligned}$$

consequently

$$G(t) \geqslant -k\Gamma(1-\alpha)A_2(t_1,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'. \quad \square$$

THEOREM 4. *Assume that* (2.4), (2.5) *hold. If there exist two functions* $\phi(t) \in C^1([t_0,\ \infty);R_+)$, $\varphi(t) \in C^1([t_0,\ \infty);[0,\infty))$ *such that*

$$\begin{aligned} \int_{T}^{\infty}\Bigg\{ &\phi(s)q(s)-\phi(s)\varphi'(s)+\frac{k\Gamma(1-\alpha)\phi(s)A_1(T,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)} \\ &-\frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(T,s)\varphi^{\frac{1}{\eta}}(s)+r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(T,s)\right]^{\eta}r(s)}\Bigg\}ds=\infty \end{aligned} \tag{2.23}$$

for all sufficiently large T, *where* $A_1(T,s)$ *is defined in Lemma 3, then every solution of Eq. (1.1) is oscillatory or satisfies* $\lim\limits_{t\to\infty}G(t)=0$.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$, without loss of generality, we assume that $x(t)>0$ on $[t_1,\infty)$, where $t_1>t_0$. By Lemma 2, we have $\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'<0$, $t\in[t_2,\infty)$, where $t_2>t_1$ is sufficiently large, and either $D_{-}^{\alpha}x(t)<0$ on $[t_2,\infty)$ or $\lim\limits_{t\to\infty}G(t)=0$.

If $D_{-}^{\alpha}x(t)<0$ on $[t_2,\infty)$, we define the generalized Riccati function

$$\omega(t)=\phi(t)\left\{-\frac{a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}}{G^{\eta}(t)}+\varphi(t)\right\}, \tag{2.24}$$

then $\omega(t) > 0$ on $[t_2,\infty)$, so

$$\begin{aligned}
\omega'(t) &= \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\left\{-\frac{a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}}{G^{\eta}(t)}\right\}' + \phi(t)\varphi'(t) \\
&= -\phi(t)\frac{\left\{a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}\right\}' G^{\eta}(t) - a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}\eta G^{\eta-1}(t)G'(t)}{G^{2\eta}(t)} \\
&\quad + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t) \\
&= -\phi(t)\frac{F(t,G)}{G^{\eta}(t)} + \frac{\eta G'(t)\phi(t)a(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta}}{G^{\eta+1}(t)} + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t).
\end{aligned} \tag{2.25}$$

By Lemma 3 and (H_3), we get that

$$\begin{aligned}
\omega'(t) &\leqslant -\phi(t)q(t) - \frac{\eta k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)a^{1+\frac{1}{\eta}}(t)\left[\left(r(t)g\left(D_{-}^{\alpha}x(t)\right)\right)'\right]^{\eta+1}}{r(t)G^{\eta+1}(t)} \\
&\quad + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t) \\
&= -\phi(t)q(t) - \frac{\eta k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)}{r(t)} \cdot \left[-\left(\frac{\omega(t)}{\phi(t)} - \varphi(t)\right)\right]^{1+\frac{1}{\eta}} \\
&\quad + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t) \\
&= -\phi(t)q(t) - \frac{\eta k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)}{r(t)} \cdot \left[\frac{\omega(t)}{\phi(t)} - \varphi(t)\right]^{1+\frac{1}{\eta}} \\
&\quad + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t).
\end{aligned} \tag{2.26}$$

Applying the following inequality

$$(u-v)^{1+\frac{1}{\eta}} \geqslant u^{1+\frac{1}{\eta}} + \frac{1}{\eta}v^{1+\frac{1}{\eta}} - \left(1+\frac{1}{\eta}\right)v^{\frac{1}{\eta}}u, \tag{2.27}$$

we obtain

$$\left[\frac{\omega(t)}{\phi(t)} - \varphi(t)\right]^{1+\frac{1}{\eta}} \geqslant \frac{\omega^{1+\frac{1}{\eta}}(t)}{\phi^{1+\frac{1}{\eta}}(t)} + \frac{1}{\eta}\varphi^{1+\frac{1}{\eta}}(t) - \left(1+\frac{1}{\eta}\right)\varphi^{\frac{1}{\eta}}(t)\frac{\omega(t)}{\phi(t)}. \tag{2.28}$$

A combination of (2.28) and (2.26) yields the following

$$
\begin{aligned}
\omega'(t) \leqslant & -\phi(t)q(t) + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t) - \frac{\eta k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)}{r(t)} \\
& \times \left[\frac{\omega^{1+\frac{1}{\eta}}(t)}{\phi^{1+\frac{1}{\eta}}(t)} + \frac{1}{\eta}\varphi^{1+\frac{1}{\eta}}(t) - \left(1+\frac{1}{\eta}\right)\varphi^{\frac{1}{\eta}}(t)\frac{\omega(t)}{\phi(t)} \right] \\
= & -\phi(t)q(t) + \phi(t)\varphi'(t) - \frac{k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\varphi^{1+\frac{1}{\eta}}(t)}{r(t)} \\
& - \frac{\eta k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)}{r(t)} \cdot \frac{\omega^{1+\frac{1}{\eta}}(t)}{\phi^{1+\frac{1}{\eta}}(t)} \\
& + \frac{(\eta+1)k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\varphi^{\frac{1}{\eta}}(t) + r(t)\phi'(t)}{r(t)\phi(t)}\omega(t),
\end{aligned}
\tag{2.29}
$$

letting

$$
m = 1 + \frac{1}{\eta}, \quad X^m = \frac{\eta k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)}{r(t)} \frac{\omega^{1+\frac{1}{\eta}}(t)}{\phi^{1+\frac{1}{\eta}}(t)},
$$

and

$$
Y^{m-1} = \eta^{\frac{1}{\eta+1}} \cdot \frac{(\eta+1)k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\varphi^{\frac{1}{\eta}}(t) + r(t)\phi'(t)}{(\eta+1)\left[k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\right]^{\frac{\eta}{\eta+1}} r^{\frac{1}{\eta+1}}(t)}.
$$

Applying Lemma 1 in (2.29), we get

$$
\begin{aligned}
\omega'(t) \leqslant & -\phi(t)q(t) + \phi(t)\varphi'(t) - \frac{k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\varphi^{1+\frac{1}{\eta}}(t)}{r(t)} \\
& + \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\varphi^{\frac{1}{\eta}}(t) + r(t)\phi'(t)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\right]^{\eta} r(t)}.
\end{aligned}
\tag{2.30}
$$

Substituting t with s in (2.30), then integrating it with respect to s from t_2 to t, we obtain

$$
\begin{aligned}
& \int_{t_2}^{t} \left\{ \phi(s)q(s) - \phi(s)\varphi'(s) + \frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)} \right. \\
& \left. - \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s) + r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta} r(s)} \right\} ds \\
& \leqslant \omega(t_2) - \omega(t) \leqslant \omega(t_2),
\end{aligned}
$$

letting $t \to \infty$, we get a contradiction to (2.23).

If $D_{-}^{\alpha}x(t) > 0$ on $[t_2,\infty)$, then from Lemma 2, we get that $\lim_{t\to\infty} G(t) = 0$. This completes the proof. $\square$

THEOREM 5. *Assume that* (2.4), (2.5) *hold. If there exist two functions* $\phi(t) \in C^1([t_0, \infty);R_+)$, $\varphi(t) \in C^1([t_0, \infty);[0,\infty))$ *such that*

$$\int_T^{\infty} \left\{ \phi(s)q(s) - \phi(s)\varphi'(s) + \frac{\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(s)A_1(T,s)A_2^{\eta-1}(T,s)}{r(s)}\varphi^2(s) \right.$$
$$\left. - \frac{\left\{2\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(s)\varphi(s)A_1(T,s)A_2^{\eta-1}(T,s) + r(s)\phi'(s)\right\}^2}{4\eta\,[k\Gamma(1-\alpha)]^{\eta}\,A_1(T,s)A_2^{\eta-1}(T,s)r(s)\phi(s)} \right\} ds = \infty \qquad (2.31)$$

for all sufficiently large T*, where* $A_1(T,s)$*,* $A_2(T,s)$ *are defined in Lemma 3, then every solution of Eq. (1.1) is oscillatory or satisfies* $\lim_{t\to\infty} G(t) = 0$.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$, without loss of generality, we assume that $x(t) > 0$ on $[t_1,\infty)$, where $t_1 > t_0$. By Lemma 2, we have $\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)' < 0$, $t \in [t_2,\infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} G(t) = 0$.

If $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$, let $\omega(t)$ be defined as in Theorem 4, proceeding as in Theorem 4, we obtain (2.25), by Lemma 3, we have the following observation

$$\begin{aligned}
\frac{G'(t)}{G(t)} &\geqslant \frac{-k\Gamma(1-\alpha)A_1(t_2,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'}{r(t)G(t)} \\
&\geqslant \frac{-k\Gamma(1-\alpha)A_1(t_2,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'}{r(t)G^{\eta}(t)} \cdot G^{\eta-1}(t) \\
&\geqslant \frac{-k\Gamma(1-\alpha)A_1(t_2,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'}{r(t)G^{\eta}(t)} \\
&\quad \times \left\{-k\Gamma(1-\alpha)A_2(t_2,t)a^{\frac{1}{\eta}}(t)\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'\right\}^{\eta-1} \\
&= -\frac{[k\Gamma(1-\alpha)]^{\eta}\,A_1(t_2,t)A_2^{\eta-1}(t_2,t)}{r(t)} \cdot \left\{ \frac{a(t)\left[\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'\right]^{\eta}}{G^{\eta}(t)} \right\}. \qquad (2.32)
\end{aligned}$$

Applying (2.32) in (2.25), we get that

$$\begin{aligned}
\omega'(t) &\leqslant -\phi(t)q(t) - \frac{\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)}{r(t)} \\
&\quad \times \left\{ \frac{a(t)\left[\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)'\right]^{\eta}}{G^{\eta}(t)} \right\}^2 + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t) \\
&= -\phi(t)q(t) - \frac{\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)}{r(t)} \cdot \left[\frac{\omega(t)}{\phi(t)} - \varphi(t)\right]^2 \\
&\quad + \frac{\phi'(t)}{\phi(t)}\omega(t) + \phi(t)\varphi'(t)
\end{aligned}$$

$$
\begin{aligned}
&= -\phi(t)q(t) - \frac{\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)}{r(t)}\varphi^2(t)\\
&\quad - \frac{\eta\,[k\Gamma(1-\alpha)]^{\eta}\,A_1(t_2,t)A_2^{\eta-1}(t_2,t)}{r(t)\phi(t)}\omega^2(t)\\
&\quad + \frac{2\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)\varphi(t)+r(t)\phi'(t)}{r(t)\phi(t)}\omega(t)+\phi(t)\varphi'(t)\\
&\leqslant -\phi(t)q(t)+\phi(t)\varphi'(t) - \frac{\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)}{r(t)}\varphi^2(t)\\
&\quad + \frac{\left\{2\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)\varphi(t)+r(t)\phi'(t)\right\}^2}{4\eta\,[k\Gamma(1-\alpha)]^{\eta}\,A_1(t_2,t)A_2^{\eta-1}(t_2,t)r(t)\phi(t)}. \qquad (2.33)
\end{aligned}
$$

Substituting t with s in (2.33), then integrating it with respect to s from t_2 to t, we obtain

$$
\begin{aligned}
&\int_{t_2}^{t}\Bigg\{\phi(s)q(s)-\phi(s)\varphi'(s)+\frac{\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(s)A_1(t_2,s)A_2^{\eta-1}(t_2,s)}{r(s)}\varphi^2(s)\\
&\quad -\frac{\left\{2\eta\,[k\Gamma(1-\alpha)]^{\eta}\,\phi(s)A_1(t_2,s)A_2^{\eta-1}(t_2,s)+r(s)\phi'(s)\right\}^2}{4\eta\,[k\Gamma(1-\alpha)]^{\eta}\,A_1(t_2,s)A_2^{\eta-1}(t_2,s)r(s)\phi(s)}\Bigg\}ds\\
&\leqslant \omega(t_2)-\omega(t)\leqslant \omega(t_2), \qquad (2.34)
\end{aligned}
$$

letting $t\to\infty$, we get a contradiction to (2.31).

If $D_-^{\alpha}x(t)>0$ on $[t_2,\infty)$, then from Lemma 2, we get that $\lim_{t\to\infty} G(t)=0$. This completes the proof. □

THEOREM 6. *Assume that* (2.4), (2.5) *hold, and there exist two functions* $\phi(t)\in C^1([t_0,\infty);R_+)$, $\varphi(t)\in C^1([t_0,\infty);[0,\infty))$. *Furthermore, define* $D=\{(t,s)\mid t\geqslant s\geqslant t_0\}$, *if there exists function* $H\in C^1(D;R)$ *such that* $H(t,t)=0$, $H(t,s)>0$ *for* $t>s\geqslant t_0$, H *has a nonpositive continuous partial derivative* $H_s'(t,s)$ *and*

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\cdot\Bigg\{\phi(s)q(s)-\phi(s)\varphi'(s)+\frac{k\Gamma(1-\alpha)\phi(s)A_1(T,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}\\
&\quad -\frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(T,s)\varphi^{\frac{1}{\eta}}(s)+r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\,[k\Gamma(1-\alpha)\phi(s)A_1(T,s)]^{\eta}\,r(s)}\Bigg\}ds=\infty \qquad (2.35)
\end{aligned}
$$

for all sufficiently large T, *where* $A_1(T,s)$ *is defined in Lemma 3, then every solution of Eq.* (1.1) *is oscillatory or satisfies* $\lim_{t\to\infty}G(t)=0$.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$, without loss of generality, we assume that $x(t)>0$ on $[t_1,\infty)$, where $t_1>t_0$. By Lemma 2, we have $\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)' < 0$, $t \in [t_2,\infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} G(t) = 0$.

If $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$, let $\omega(t)$ be defined as in Theorem 4, proceeding as in Theorem 4, by (2.30), we have

$$\begin{aligned} &\phi(t)q(t) - \phi(t)\varphi'(t) + \frac{k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\varphi^{1+\frac{1}{\eta}}(t)}{r(t)} \\ &\qquad - \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\varphi^{\frac{1}{\eta}}(t) + r(t)\phi'(t)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(t)A_1(t_2,t)\right]^{\eta} r(t)} \\ &\leqslant -\omega'(t). \end{aligned} \tag{2.36}$$

Substituting t with s in (2.36), multiplying both sides of (2.36) by $H(t,s)$, then integrating it with respect to s from t_2 to t, we derive

$$\begin{aligned} &\int_{t_2}^{t} H(t,s) \cdot \left\{\phi(s)q(s) - \phi(s)\varphi'(s) + \frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}\right. \\ &\qquad \left. - \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s) + r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta} r(s)}\right\} ds \\ &\leqslant -\int_{t_2}^{t} H(t,s)\omega'(s)ds = H(t,t_2)\omega(t_2) + \int_{t_2}^{t} \omega(s)H_s'(t,s)ds \\ &\leqslant H(t,t_2)\omega(t_2) \leqslant H(t,t_0)\omega(t_2). \end{aligned} \tag{2.37}$$

Then

$$\begin{aligned} &\int_{t_0}^{t} H(t,s) \cdot \left\{\phi(s)q(s) - \phi(s)\varphi'(s) + \frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}\right. \\ &\left. - \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s) + r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta} r(s)}\right\} ds \\ &= \int_{t_0}^{t_2} H(t,s) \cdot \left\{\phi(s)q(s) - \phi(s)\varphi'(s) + \frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}\right. \\ &\left. - \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s) + r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta} r(s)}\right\} ds \\ &+ \int_{t_2}^{t} H(t,s) \cdot \left\{\phi(s)q(s) - \phi(s)\varphi'(s) + \frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}\right. \\ &\left. - \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s) + r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta} r(s)}\right\} ds \end{aligned}$$

$$\leqslant H(t,t_0)\omega(t_2)+H(t,t_0)\cdot\int_{t_0}^{t_2}\Bigg|\phi(s)q(s)-\phi(s)\varphi'(s)+\frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}$$
$$-\frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s)+r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta}r(s)}\Bigg|ds. \tag{2.38}$$

Therefore

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\cdot\Bigg\{\phi(s)q(s)-\phi(s)\varphi'(s)+\frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}$$
$$-\frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s)+r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta}r(s)}\Bigg\}ds$$
$$\leqslant \omega(t_2)+\int_{t_0}^{t_2}\Bigg|\phi(s)q(s)-\phi(s)\varphi'(s)+\frac{k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)}$$
$$-\frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\varphi^{\frac{1}{\eta}}(s)+r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(t_2,s)\right]^{\eta}r(s)}\Bigg|ds<\infty, \tag{2.39}$$

which contradicts (2.35).

If $D_-^{\alpha}x(t)>0$ on $[t_2,\infty)$, then from Lemma 2, we get that $\lim_{t\to\infty}G(t)=0$. This completes the proof. □

THEOREM 7. *Assume that* (2.4), (2.5) *hold, and there exist two functions* $\phi(t)\in C^1([t_0,\infty);R_+)$, $\varphi(t)\in C^1([t_0,\infty);[0,\infty))$. *Furthermore, define* $D=\{(t,s)\mid t\geqslant s\geqslant t_0\}$, *if there exists function* $H\in C^1(D;R)$ *such that* $H(t,t)=0$, $H(t,s)>0$ *for* $t>s\geqslant t_0$, H *has a nonpositive continuous partial derivative* $H_s'(t,s)$ *and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\cdot\Bigg\{\phi(s)q(s)-\phi(s)\varphi'(s)$$
$$+\frac{\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(s)A_1(T,s)A_2^{\eta-1}(T,s)}{r(s)}\varphi^2(s)$$
$$-\frac{\left\{2\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(s)A_1(T,s)A_2^{\eta-1}(T,s)\varphi(s)+r(s)\phi'(s)\right\}^2}{4\eta\left[k\Gamma(1-\alpha)\right]^{\eta}A_1(T,s)A_2^{\eta-1}(T,s)r(s)\phi(s)}\Bigg\}ds=\infty \tag{2.40}$$

for all sufficiently large T, *where* $A_1(T,s)$, $A_2(T,s)$ *are defined in Lemma 3, then every solution of Eq. (1.1) is oscillatory or satisfies* $\lim_{t\to\infty}G(t)=0$.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$, without loss of generality, we assume that $x(t)>0$ on $[t_1,\infty)$, where $t_1>t_0$. By Lemma 2, we have $\left(r(t)g\left(D_-^{\alpha}x(t)\right)\right)' < 0$, $t \in [t_2,\infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} G(t) = 0$.

If $D_-^{\alpha}x(t) < 0$ on $[t_2,\infty)$, let $\omega(t)$ be defined as in Theorem 4, proceeding as in Theorem 5, by (2.33), we have

$$\begin{aligned}
&\phi(t)q(t) - \phi(t)\varphi'(t) + \frac{\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)}{r(t)}\varphi^2(t) \\
&-\frac{\left\{2\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(t)A_1(t_2,t)A_2^{\eta-1}(t_2,t)\varphi(t) + r(t)\phi'(t)\right\}^2}{4\eta\left[k\Gamma(1-\alpha)\right]^{\eta}A_1(t_2,t)A_2^{\eta-1}(t_2,t)r(t)\phi(t)} \leqslant -\omega'(t).
\end{aligned} \tag{2.41}$$

Substituting t with s in (2.41), multiplying both sides of (2.41) by $H(t,s)$, then integrating it with respect to s from t_2 to t yield

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)\cdot\left\{\phi(s)q(s) - \phi(s)\varphi'(s) + \frac{\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(s)A_1(t_2,s)A_2^{\eta-1}(t_2,s)}{r(s)}\varphi^2(s)\right. \\
&\qquad\left.-\frac{\left\{2\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(s)A_1(t_2,s)A_2^{\eta-1}(t_2,s)\varphi(s) + r(s)\phi'(s)\right\}^2}{4\eta\left[k\Gamma(1-\alpha)\right]^{\eta}A_1(t_2,s)A_2^{\eta-1}(t_2,s)r(s)\phi(s)}\right\}ds \\
&\quad\leqslant -\int_{t_2}^{t} H(t,s)\omega'(s)ds = H(t,t_2)\omega(t_2) + \int_{t_2}^{t}\omega(s)H_s'(t,s)ds \\
&\quad\leqslant H(t,t_0)\omega(t_2).
\end{aligned} \tag{2.42}$$

Then, similar to the process of Theorem 6, we get that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\cdot\left\{\phi(s)q(s) - \phi(s)\varphi'(s)\right. \\
&\quad+\frac{\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(s)A_1(t_2,s)A_2^{\eta-1}(t_2,s)}{r(s)}\varphi^2(s) \\
&\quad\left.-\frac{\left\{2\eta\left[k\Gamma(1-\alpha)\right]^{\eta}\phi(s)A_1(t_2,s)A_2^{\eta-1}(t_2,s)\varphi(s) + r(s)\phi'(s)\right\}^2}{4\eta\left[k\Gamma(1-\alpha)\right]^{\eta}A_1(t_2,s)A_2^{\eta-1}(t_2,s)r(s)\phi(s)}\right\}ds < \infty,
\end{aligned} \tag{2.43}$$

which contradicts (2.40).

If $D_-^{\alpha}x(t) > 0$ on $[t_2,\infty)$, then from Lemma 2, we get that $\lim_{t\to\infty} G(t) = 0$. This completes the proof. □

In Theorem 6 and Theorem 7, if we take $H(t,s)$ for a special function such as $H(t,s) = \ln\frac{t}{s}$, $t > s \geqslant t_0$, then we can obtain the following two corollaries.

COROLLARY 1. *Assume that* (2.4), (2.5) *hold, and there exist two functions*

$\phi(t) \in C^1([t_0, \infty); R_+)$, $\varphi(t) \in C^1([t_0, \infty); [0, \infty))$. *If*

$$\limsup_{t\to\infty} \frac{1}{\ln t - \ln t_0} \int_{t_0}^{t} \ln\frac{t}{s} \cdot \left\{ \phi(s)q(s) - \phi(s)\varphi'(s) + \frac{k\Gamma(1-\alpha)\phi(s)A_1(T,s)\varphi^{1+\frac{1}{\eta}}(s)}{r(s)} \right.$$
$$\left. - \frac{\left[(\eta+1)k\Gamma(1-\alpha)\phi(s)A_1(T,s)\varphi^{\frac{1}{\eta}}(s) + r(s)\phi'(s)\right]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(T,s)\right]^{\eta} r(s)} \right\} ds = \infty \tag{2.44}$$

for all sufficiently large T, where $A_1(T,s)$ is defined in Lemma 3, then every solution of Eq. (1.1) is oscillatory or satisfies $\lim_{t\to\infty} G(t) = 0$.

COROLLARY 2. *Assume that (2.4), (2.5) hold, and there exist two functions $\phi(t) \in C^1([t_0, \infty); R_+)$, $\varphi(t) \in C^1([t_0, \infty); [0, \infty))$. If*

$$\limsup_{t\to\infty} \frac{1}{\ln t - \ln t_0} \int_{t_0}^{t} \ln\frac{t}{s} \cdot \left\{ \phi(s)q(s) - \phi(s)\varphi'(s) \right.$$
$$+ \frac{\eta\left[k\Gamma(1-\alpha)\right]^{\eta} \phi(s)A_1(T,s)A_2^{\eta-1}(T,s)}{r(s)}\varphi^2(s)$$
$$\left. - \frac{\left\{2\eta\left[k\Gamma(1-\alpha)\right]^{\eta} \phi(s)A_1(T,s)A_2^{\eta-1}(T,s)\varphi(s) + r(s)\phi'(s)\right\}^2}{4\eta\left[k\Gamma(1-\alpha)\right]^{\eta} A_1(T,s)A_2^{\eta-1}(T,s)r(s)\phi(s)} \right\} ds = \infty \tag{2.45}$$

for all sufficiently large T, where $A_1(T,s)$, $A_2(T,s)$ are defined in Lemma 3, then every solution of Eq. (1.1) is oscillatory or satisfies $\lim_{t\to\infty} G(t) = 0$.

3. Applications

In this Section, we present some applications for the established results above.

EXAMPLE 1. Consider the fractional differential equation

$$\left\{(2+\cos t)\left(D_-^{\alpha}x(t)\right)'\right\}' - \frac{M}{t^2}\int_t^{\infty}(v-t)^{-\alpha}x(v)dv = 0, \quad t \geqslant 1, \tag{3.1}$$

where $\alpha \in (0,1)$, $M > 0$ is a constant.

This corresponds to Eq. (1.1) with $\eta = 1$, $a(t) = 2+\cos t$, $r(t) = 1$, $g(x) = x$, $\frac{x}{g(x)} \geqslant k = 1$, $F(t,G) = \frac{M}{t^2}\int_t^{\infty}(v-t)^{-\alpha}x(v)dv$, $q(t) = \frac{M}{t^2}$. Since $1 \leqslant a(t) \leqslant 3$, we obtain

$$\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\eta}}(s)}ds \geqslant \frac{1}{3}\int_1^{\infty} ds = \infty.$$

In addition, we can get

$$\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right)ds = \int_1^{\infty}\frac{1}{r(s)}ds = \infty, \tag{3.2}$$

$$\begin{aligned}
&\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{1}{\eta}} d\tau\right) d\xi \\
&= \int_{1}^{\infty}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)dsd\tau d\xi = \int_{1}^{\infty}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\frac{1}{a(\tau)}\cdot\frac{M}{\tau}d\tau d\xi \\
&\geqslant \frac{M}{3}\int_{1}^{\infty}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\frac{1}{\tau}d\tau d\xi = \infty.
\end{aligned} \tag{3.3}$$

On the other hand, for a sufficiently large T, we have

$$A_1(T,t) = \int_T^t \frac{1}{a^{\frac{1}{\eta}}(s)}ds \geqslant \frac{1}{3}\int_T^t ds \to \infty \ (t \to \infty). \tag{3.4}$$

Thus, we can take $T^* > T$ such that $A_1(T,t) > 1$ for $t \in [T^*,\infty)$. Letting $\phi(s) = s$, $\varphi(s) = 0$ in (2.23), we get

$$\begin{aligned}
&\int_T^{\infty}\left\{\phi(s)q(s) - \frac{[r(s)\phi'(s)]^{\eta+1}}{(\eta+1)^{\eta+1}\left[k\Gamma(1-\alpha)\phi(s)A_1(T,s)\right]^{\eta} r(s)}\right\}ds \\
&= \int_T^{\infty}\frac{1}{s}\left\{M - \frac{1}{4\Gamma(1-\alpha)A_1(T,s)}\right\}ds \\
&= \int_T^{T^*}\frac{1}{s}\left\{M - \frac{1}{4\Gamma(1-\alpha)A_1(T,s)}\right\}ds + \int_{T^*}^{\infty}\frac{1}{s}\left\{M - \frac{1}{4\Gamma(1-\alpha)A_1(T,s)}\right\}ds \\
&\geqslant \int_T^{T^*}\frac{1}{s}\left\{M - \frac{1}{4\Gamma(1-\alpha)A_1(T,s)}\right\}ds + \int_{T^*}^{\infty}\frac{1}{s}\left\{M - \frac{1}{4\Gamma(1-\alpha)}\right\}ds \to \infty
\end{aligned} \tag{3.5}$$

provided $M > \frac{1}{4\Gamma(1-\alpha)}$, then (2.4), (2.5) and (2.23) are satisfied. Therefore, according to Theorem 4, we deduce that every solution of Eq. (3.1) is oscillatory or satisfies $\lim\limits_{t\to\infty} G(t) = 0$ under the condition $M > \frac{1}{4\Gamma(1-\alpha)}$.

EXAMPLE 2. Consider the following equation

$$\left\{t^{-\frac{7}{5}}\left[\left(D_-^{\alpha}x(t)\right)'\right]^{\frac{7}{5}}\right\}' - t^{-\frac{12}{5}}\left(M + e^{\int_t^{\infty}(v-t)^{-\alpha}x(v)dv}\right)\left(\int_t^{\infty}(v-t)^{-\alpha}x(v)dv\right)^{\frac{7}{5}} = 0, \ t \geqslant 1, \tag{3.6}$$

where $\alpha \in (0,1)$, $M > 0$ is a constant.

This corresponds to Eq. (1.1) with $\eta = \frac{7}{5}$, $a(t) = t^{-\frac{7}{5}}$, $r(t) = 1$, $g(x) = x$, $\frac{x}{g(x)} \geqslant k = 1$, $F(t,G) = t^{-\frac{12}{5}}\left(M + e^{G}\right)G^{\frac{7}{5}}$, $\frac{F(t,G)}{G^{\frac{7}{5}}} \geqslant t^{-\frac{12}{5}}\left(M + e^{G}\right) \geqslant Mt^{-\frac{12}{5}}$, $q(t) = Mt^{-\frac{12}{5}}$. Then we have

$$\int_{t_0}^{\infty}\frac{1}{a^{\frac{1}{\eta}}(s)}ds = \int_1^{\infty} sds = \infty, \quad \int_1^{\infty} g^{-1}\left(\frac{1}{r(s)}\right)ds = \infty,$$

$$\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{1}{\eta}} d\tau\right) d\xi$$
$$= \int_{1}^{\infty}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty} q(s)ds\right]^{\frac{5}{7}} d\tau d\xi = \infty. \tag{3.7}$$

On the other hand, for a sufficiently large T, we have

$$A_1(T,t) = \int_T^t \frac{1}{a^{\frac{1}{\eta}}(s)}ds = \int_T^t s ds \to \infty \ (t \to \infty). \tag{3.8}$$

Thus, we can take $T^* > T$ such that $A_1(T,t) > 1$ for $t \in [T^*,\infty)$. Letting $\phi(s) = s^{\frac{7}{5}}$, $\varphi(s) = 0$ and $H(t,s) = \ln\frac{t}{s}$ in (2.35), we get

$$\limsup_{t\to\infty}\frac{1}{\ln t - \ln t_0}\cdot\left\{\int_{t_0}^t \ln\frac{t}{s}\cdot\left\{\phi(s)q(s) - \frac{[r(s)\phi'(s)]^{\eta+1}}{(\eta+1)^{\eta+1}[k\Gamma(1-\alpha)\phi(s)A_1(T,s)]^{\eta} r(s)}\right\} ds\right\}$$

$$= \limsup_{t\to\infty}\frac{1}{\ln t}\cdot\left\{\int_1^t \ln\frac{t}{s}\cdot\left\{M - \left(\frac{7}{12}\right)^{\frac{12}{5}}\frac{1}{[\Gamma(1-\alpha)A_1(T,s)]^{\frac{7}{5}}}\right\}\frac{1}{s}ds\right\}$$

$$= \limsup_{t\to\infty}\frac{1}{\ln t}\cdot\left\{\int_1^{T^*} \ln\frac{t}{s}\cdot\left\{M - \left(\frac{7}{12}\right)^{\frac{12}{5}}\frac{1}{[\Gamma(1-\alpha)A_1(T,s)]^{\frac{7}{5}}}\right\}\frac{1}{s}ds\right.$$
$$\left. + \int_{T^*}^t \ln\frac{t}{s}\cdot\left\{M - \left(\frac{7}{12}\right)^{\frac{12}{5}}\frac{1}{[\Gamma(1-\alpha)A_1(T,s)]^{\frac{7}{5}}}\right\}\frac{1}{s}ds\right\}$$

$$\geqslant \limsup_{t\to\infty}\frac{1}{\ln t}\cdot\left\{\int_1^{T^*} \ln\frac{t}{s}\cdot\left\{M - \left(\frac{7}{12}\right)^{\frac{12}{5}}\frac{1}{[\Gamma(1-\alpha)A_1(T,s)]^{\frac{7}{5}}}\right\}\frac{1}{s}ds\right.$$
$$\left. + \int_{T^*}^t \ln\frac{t}{s}\cdot\left\{M - \left(\frac{7}{12}\right)^{\frac{12}{5}}\frac{1}{[\Gamma(1-\alpha)]^{\frac{7}{5}}}\right\}\frac{1}{s}ds\right\} = \infty \tag{3.9}$$

provided $M > \left(\frac{7}{12}\right)^{\frac{12}{5}}\frac{1}{[\Gamma(1-\alpha)]^{\frac{7}{5}}}$, then (2.4), (2.5) and (2.44) are satisfied. Therefore, according to Corollary 1, we deduce that every solution of Eq. (3.6) is oscillatory or satisfies $\lim_{t\to\infty} G(t) = 0$ under the condition $M > \left(\frac{7}{12}\right)^{\frac{12}{5}}\frac{1}{[\Gamma(1-\alpha)]^{\frac{7}{5}}}$.

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