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FRACTIONAL CALCULUS OF VARIATIONS WITH A GENERALIZED FRACTIONAL DERIVATIVE

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Abstract. In this paper, we introduce a generalization of the Hilfer-Prabhakar derivative and obtain the Euler-Lagrange equations and Hamiltonian formulation with respect to this fractional derivative in the theory of fractional calculus of variations. Also, we get a sufficient condition for optimality.

1. Introduction

With the extensions of the theory of fractional calculus and fractional differential equations, the theory of calculus of variations with fractional derivatives (fractional calculus of variations) was also developed. In year 1996, Riewe surveyed the calculus of variations with fractional derivatives for describing nonconservative systems in mechanics [26, 27]. Later, other researchers presented some results on applications of fractional calculus of variations in optimal control theory, robotics, biology, chemistry and economics [11, 18, 19, 20, 21]. For example, Almeida et al. expressed a fractional equation of motion of a vibrating string [8]. Bastos et al. introduced discrete-time and difference variational problems and discussed on necessary optimal conditions for these problems.

Most of these results have been developed on finding critical points of functionals by expressing the necessary conditions (establishing the Euler-Lagrange equation) and the main difference of them is the type of fractional derivatives. In this sense, the combined Caputo derivatives [22] and Hilfer derivative [1] have been incorporated. Also, a few works were devoted to get sufficient conditions for fractional calculus of variations. For example, Almeida and Torres discussed on sufficient conditions of fractional calculus of variations [7, 9].

In this paper, we intend to introduce the problem of fractional calculus of variations with the generalized Hilfer-Prabhakar derivative introduced by Garra et al. [14]. Some of the applications of this fractional derivative in mathematical physics and probability have been mentioned in [14] and [23]. We obtain the associated necessary and sufficient conditions for global extremum of functionals with respect to this type of fractional derivative.

The paper is organized as follows: in Section 2 after recalling some preliminaries, we define a generalized form of Hilfer-Prabhakar derivative made by a specific linear

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combination of the left-sided and right-sided Hilfer-Prabhakar derivatives. We also present an integration by parts formula for this fractional derivative.

In Section 3, we focus on the Euler-Lagrange equation. First we consider boundary conditions which enable us to use the fundamental lemma of calculus of variations and then we get the necessary conditions for optimality of two important types of functionals.

In Section 4, we present the Hamiltonian form of Euler-Lagrange equations, which are used more in physics and mechanics. In Section 5, we state the necessary definitions and theorems in convex analysis and optimization to get a relationship between the Hessian matrix and sufficient condition of functional containing multiple functions. In Section 6, we present two illustrative examples.

2. Preliminaries

DEFINITION 2.1. Let $f(x)\in L^1[a,b]$, where $-\infty\leqslant a<x<b\leqslant\infty$, be locally integrable real-valued function, and $\mu\in(0,1)$. The left-sided and right-sided Riemann-Liouville fractional integrals of order μ are defined as

$$(\mathscr{I}^{\mu}_{a^+}f)(x)=\frac{1}{\Gamma(\mu)}\int_a^x(x-s)^{\mu-1}f(s)ds,\quad (\mathscr{I}^{\mu}_{b^-}f)(x)=\frac{1}{\Gamma(\mu)}\int_x^b(s-x)^{\mu-1}f(s)ds. \tag{2.1}$$

Also, the Riesz fractional integral of order μ is defined as

$$({}^R_a\mathscr{I}^{\mu}_b f)(x)=\frac{1}{2}\Big((\mathscr{I}^{\mu}_{a^+}f)(x)+(\mathscr{I}^{\mu}_{b^-}f)(x)\Big),\quad x\in[a,b]. \tag{2.2}$$

DEFINITION 2.2. Let $f(x)\in L^1[a,b]$, $-\infty\leqslant a<x<b\leqslant\infty$, $\mu\in(0,1)$, $D=\frac{d}{dx}$, and $(\mathscr{I}^{1-\mu}_{a^+}f)(x)$, $(\mathscr{I}^{1-\mu}_{b^-}f)(x)\in W^{1,1}[a,b]$, where $W^{n,1}[a,b]$ is the Sobolev space defined as

$$W^{n,1}[a,b]=\{f(x)\in L^1[a,b]:\frac{d^n}{dx^n}f(x)\in L^1[a,b]\},\quad n\in\mathbb{N}. \tag{2.3}$$

The left-sided and right-sided Riemann-Liouville fractional derivatives of order μ are defined as

$$(\mathscr{D}^{\mu}_{a^+}f)(x)=D\,(\mathscr{I}^{1-\mu}_{a^+}f)(x),\quad x>a,\quad (\mathscr{D}^{\mu}_{b^-}f)(x)=(-D)(\mathscr{I}^{1-\mu}_{b^-}f)(x),\quad x<b. \tag{2.4}$$

Also, the Riesz fractional derivative of order μ is defined as

$$({}^R_a\mathscr{D}^{\mu}_b f)(x)=\frac{1}{2}\Big((\mathscr{D}^{\mu}_{a^+}f)(x)-(\mathscr{D}^{\mu}_{b^-}f)(x)\Big),\quad x\in[a,b]. \tag{2.5}$$

DEFINITION 2.3. Let $f(x)\in AC([a,b])$, $-\infty\leqslant a<x<b\leqslant\infty$, the space of real-valued functions $f(x)$ which have absolutely continuous functions, and $\mu\in(0,1)$. The left-sided and right-sided Caputo fractional derivatives of order μ are defined as

$$({}^C\mathscr{D}^{\mu}_{a^+}f)(x)=(\mathscr{I}^{1-\mu}_{a^+}Df)(x),\quad x>a,\quad ({}^C\mathscr{D}^{\mu}_{b^-}f)(x)=-(\mathscr{I}^{1-\mu}_{b^-}Df)(x),\quad x<b. \tag{2.6}$$

Also, the Riesz-Caputo fractional derivative of order μ is defined as

$$({}^{RC}_{\ a}\mathscr{D}^{\mu}_{b} f)(x) = \frac{1}{2}\left(({}^{C}\mathscr{D}^{\mu}_{a^+} f)(x) - ({}^{C}\mathscr{D}^{\mu}_{b^-} f)(x)\right), \quad x \in [a,b]. \tag{2.7}$$

DEFINITION 2.4. Let $\mu \in (0,1)$, $\nu \in [0,1]$, $f(x) \in L^1[a,b]$, $-\infty \leqslant a < x < b \leqslant \infty$, and $(\mathscr{I}^{(1-\nu)(1-\mu)}_{a^+} f)(x)$, $(\mathscr{I}^{(1-\nu)(1-\mu)}_{b^-} f)(x) \in AC[a,b]$. The left-sided and right-sided Hilfer fractional derivatives $(\mathscr{D}^{\mu,\nu}_{a^+} f)(x)$ and $(\mathscr{D}^{\mu,\nu}_{b^-} f)(x)$ of order μ and type ν, are defined by [17]

$$(\mathscr{D}^{\mu,\nu}_{a^+} f)(x) = \left(\mathscr{I}^{(1-\nu)(1-\mu)}_{a^+}\,(D)\,\mathscr{I}^{\nu(1-\mu)}_{a^+} f\right)(x), \quad a < x < b, \tag{2.8}$$

$$(\mathscr{D}^{\mu,\nu}_{b^-} f)(x) = \left(\mathscr{I}^{(1-\nu)(1-\mu)}_{b^-}\,(-D)\,\mathscr{I}^{\nu(1-\mu)}_{b^-} f\right)(x), \quad a < x < b. \tag{2.9}$$

For more information of this type of fractional derivative, see [16].

DEFINITION 2.5. The generalized Mittag-Leffler function is defined as [24]

$$E^{\gamma}_{\rho,\mu}(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)\Gamma(\rho n+\mu)} \frac{x^n}{n!}, \quad \Re(\rho) > 0,\ \Re(\mu) > 0,\ \gamma \in \mathbb{C}. \tag{2.10}$$

DEFINITION 2.6. Let $f \in L^1[a,b]$, $-\infty \leqslant a < x < b \leqslant \infty$. The left-sided and right-sided Prabhakar integrals are defined as

$$(\mathbf{E}^{\gamma}_{\rho,\mu,w,a^+} f)(x) = \int_a^x (x-s)^{\mu-1}\, E^{\gamma}_{\rho,\mu}(w(x-s)^{\rho})\, f(s)ds, \quad w \in \mathbb{C}, \tag{2.11}$$

$$(\mathbf{E}^{\gamma}_{\rho,\mu,w,b^-} f)(x) = \int_x^b (s-x)^{\mu-1}\, E^{\gamma}_{\rho,\mu}(w(s-x)^{\rho})\, f(s)ds, \quad w \in \mathbb{C}. \tag{2.12}$$

For more information about the applications of these integrals, see [28].

DEFINITION 2.7. We define $HP[a,b]$, $-\infty \leqslant a < x < b \leqslant \infty$, the space of real-valued functions $f(x) \in L^1[a,b]$ which

$$(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),w,a^+} f)(x) \quad \text{and} \quad (\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),w,b^-} f)(x) \in AC[a,b].$$

The left-sided and right-sided Hilfer-Prabhakar derivatives of f are defined as [14]

$$(\mathscr{D}^{\gamma,\mu,\nu}_{\rho,w,a^+} f)(x) = \left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),w,a^+}\,(D)\,\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),w,a^+} f\right)(x), \quad x \in (a,b), \tag{2.13}$$

$$(\mathscr{D}^{\gamma,\mu,\nu}_{\rho,w,b^-} f)(x) = \left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),w,b^-}\,(-D)\,\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),w,b^-} f\right)(x), \quad x \in (a,b), \tag{2.14}$$

where $\gamma, w \in \mathbb{R}$, $\rho > 0$, $\mu \in (0,1)$, and $\nu \in [0,1]$.

PROPERTY 2.8. For $f,\ g \in HP[a,b]$, the following relation holds for the Prabhakar integrals

$$\int_a^b f(x)\ (\mathbf{E}^{\gamma}_{\rho,\mu,w,a^+}g)(x)dx = \int_a^b g(x)\ (\mathbf{E}^{\gamma}_{\rho,\mu,w,b^-}f)(x)dx. \tag{2.15}$$

Proof. The result is obtained by applying the relations (2.11) and (2.12), and changing the order of integration. □

PROPERTY 2.9. Let $f,\ g \in HP[a,b]$. The following integration by parts formula holds

$$\int_a^b f(x)\ (\mathscr{D}^{\gamma,\mu,\nu}_{\rho,w,a^+}g)(x)dx = \int_a^b g(x)\ (\mathscr{D}^{\gamma,\mu,1-\nu}_{\rho,w,b^-}f)(x)dx + \left[(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),w,b^-}f)(x)\ (\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),w,a^+}g)(x)\right]\Bigg|_a^b. \tag{2.16}$$

Proof. The result is obtained by applying (2.14) and (2.16), and changing the order of integration. □

DEFINITION 2.10. Let $f \in HP[a,b]$. For $\theta \in [0,1]$, we define the generalized fractional Hilfer-Prabhakar derivative (GFHPD) of $f(t)$ as

$$(\mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}f)(x) = \theta(\mathscr{D}^{\gamma,\mu,\nu}_{\rho,w,a^+}f)(x) - (1-\theta)(\mathscr{D}^{\gamma,\mu,\nu}_{\rho,w,b^-}f)(x). \tag{2.17}$$

REMARK 2.11. Let $f \in HP[a,b]$. The relationship between GFHPD fractional derivative and Hilfer fractional derivative is given as follows

$$(\mathscr{D}^{0,\mu,\nu,1}_{\rho,w}f)(x) = (\mathscr{D}^{0,\mu,\nu}_{\rho,w,a^+}f)(x) = (\mathscr{D}^{\mu,\nu}_{a^+}f)(x), \tag{2.18}$$

$$(\mathscr{D}^{0,\mu,\nu,0}_{\rho,w}f)(x) = -(\mathscr{D}^{0,\mu,\nu}_{\rho,w,b^-}f)(x) = -(\mathscr{D}^{\mu,\nu}_{b^-}f)(x). \tag{2.19}$$

REMARK 2.12. Let $f \in HP[a,b]$. If we consider some special values for the parameters of GFHPD (2.17), we can obtain various fractional derivatives as follows

$$(\mathscr{D}^{0,\mu,1,1}_{\rho,w}f)(x) = (\mathscr{D}^{\mu,1}_{a^+}f)(x) = (\mathscr{D}^{\mu}_{a^+}f)(x), \tag{2.20}$$

$$(\mathscr{D}^{0,\mu,0,1}_{\rho,w}f)(x) = (\mathscr{D}^{\mu,0}_{a^+}f(x) = ({}^{C}\mathscr{D}^{\mu}_{a^+}f)(x), \tag{2.21}$$

$$(\mathscr{D}^{0,\mu,1,0}_{\rho,w}f)(x) = -(\mathscr{D}^{\mu,1}_{b^-}f)(x) = -(\mathscr{D}^{\mu}_{b^-}f)(x), \tag{2.22}$$

$$(\mathscr{D}^{0,\mu,0,0}_{\rho,w}f)(x) = -(\mathscr{D}^{\mu,0}_{b^-}f)(x) = -({}^{C}\mathscr{D}^{\mu}_{b^-}f)(x), \tag{2.23}$$

$$(\mathscr{D}^{0,\mu,1,\frac{1}{2}}_{\rho,w}f)(x) = \frac{1}{2}\Big((\mathscr{D}^{\mu,1}_{a^+}f)(x) - \mathscr{D}^{\mu,1}_{b^-}f)(x)\Big) = ({}^{R}_{a}\mathscr{D}^{\mu}_{b}f)(x), \tag{2.24}$$

$$(\mathscr{D}^{0,\mu,0,\frac{1}{2}}_{\rho,w}f)(x) = \frac{1}{2}\Big(({}^{C}\mathscr{D}^{\mu,0}_{a^+}f)(x) - ({}^{C}\mathscr{D}^{\mu,0}_{b^-}f)(x)\Big) = ({}^{RC}_{a}\mathscr{D}^{\mu}_{b}f)(x). \tag{2.25}$$

THEOREM 2.13. *Let $f,g \in HP[a,b]$. The following integration by parts formula holds for the GFHPD derivatives*

$$\int_a^b f(x)\,(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} g)(x)dx = -\int_a^b g(x)\,(\mathscr{D}_{\rho,w}^{\gamma,\mu,1-\nu,1-\theta} f)(x)dx$$
$$+\theta\big[(\mathbf{E}_{\rho,(1-\nu)(1-\mu),w,b^-}^{-\gamma(1-\nu)} f)(x)(\mathbf{E}_{\rho,\nu(1-\mu),w,a^+}^{-\gamma\nu} g)(x)\big]\Big|_a^b$$
$$+(1-\theta)\big[(\mathbf{E}_{\rho,(1-\nu)(1-\mu),w,a^+}^{-\gamma(1-\nu)} f)(x)(\mathbf{E}_{\rho,\nu(1-\mu),w,b^-}^{-\gamma\nu} g)(x)\big]\Big|_a^b. \tag{2.26}$$

Proof. Using (2.16) and (2.17), the relation (2.26) can easily be obtained. □

COROLLARY 2.14. *Let $f,g \in HP[a,b]$. By applying Remark 2.11 for the different values of parameters γ and θ, the following integration by parts formulas hold*

$$\int_a^b f(x)(\mathscr{D}_{a^+}^{\mu,\nu} g)(x)dx = \int_a^b g(x)(\mathscr{D}_{b^-}^{\mu,1-\nu} f)(x)dx + \big[(\mathscr{I}_{b^-}^{(1-\nu)(1-\mu)} f)(x)(\mathscr{I}_{a^+}^{\nu(1-\mu)} g)(x)\big]\Big|_a^b, \tag{2.27}$$

$$\int_a^b f(x)\,({}_a^{RC}\mathscr{D}_b^{\mu} g)(x)dx = -\int_a^b g(x)\,({}_a^{R}\mathscr{D}_b^{\mu} f)(x)dx + \big[g(x)\,({}_a^{R}\mathscr{I}_b^{1-\mu} f)(x)\big]\Big|_a^b, \tag{2.28}$$

$$\int_a^b f(x)\,({}_a^{C}\mathscr{D}_x^{\mu} g)(x)dx = \int_a^b g(x)\,({}_x\mathscr{D}_b^{\mu} f)(x)dx + \big[g(x)\,({}_x\mathscr{I}_b^{1-\mu} f)(x)\big]\Big|_a^b. \tag{2.29}$$

3. Euler-Lagrange equations with GFHPD

In this section, we intend to get the necessary conditions for optimality of the fractional calculus of variations problem. We know that the key of obtaining Euler-Lagrange equation for a functional is the fractional integration by parts formula announced in the previous section. For obtaining the associated Euler-Lagrange equation some definitions are in order.

DEFINITION 3.1. Let $r,k \in \mathbb{N}$, $[\mathbf{a},\mathbf{b}] = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_{r+1},b_{r+1}] \subseteq \mathbb{R}^{r+1}$ and $\mathbf{u}(x) = (u_1(x),\cdots,u_r(x)) \in \mathbb{R}^r$ be a vector function. We say $\mathbf{F}(x,\mathbf{u}(x)) \in C^k[\mathbf{a},\mathbf{b}]$ if $\mathbf{F}$ is k-times continuously differentiable with respect to all arguments.

DEFINITION 3.2. The vector function $\mathbf{u}$ is a local extremum of functional $\mathbf{J}[\mathbf{u}]$ if for any $\mathbf{u}^*$ there exists $\delta \in (0,\infty)$ such that $\|\mathbf{u}-\mathbf{u}^*\| < \delta$ and $\mathbf{J}[\mathbf{u}^*]-\mathbf{J}[\mathbf{u}] > 0$ or $\mathbf{J}[\mathbf{u}^*]-\mathbf{J}[\mathbf{u}] < 0$. Also, we say that the vector function $\mathbf{u}$ is a global extremum of functional $\mathbf{J}[\mathbf{u}]$ if for $(x,\mathbf{u}^*) \in D_{\mathbf{F}}$, $\mathbf{J}[\mathbf{u}^*]-\mathbf{J}[\mathbf{u}] > 0$ or $\mathbf{J}[\mathbf{u}^*]-\mathbf{J}[\mathbf{u}] < 0$.

3.1. A functional containing multiple fractional derivatives

PROBLEM 3.3. Given $y^*(x) \in HP[a,b]$, the space of continuously differential functions, find the function $y(x)$ such that the functional

$$\mathbf{J}[y] = \int_a^b \mathbf{F}\Big(x, y, \mathscr{D}_{\rho_1,w_1}^{\gamma_1,\mu_1,\nu_1,\theta_1} y, \cdots, \mathscr{D}_{\rho_n,w_n}^{\gamma_n,\mu_n,\nu_n,\theta_n} y\Big)\, dx, \tag{3.1}$$

with given boundary conditions attains the optimal value. The function F has continuously derivatives with respect to all components and $y(x)$, $\dfrac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \in HP[a,b]$, for all $j = 1,...,n$.

To respond to this problem, let $\varepsilon \in \mathbb{R}$ and define a family of curves $y^*(x) = y(x) + \varepsilon\eta(x) \in C^1[a,b]$ which satisfy equation (3.1) and given boundary conditions. To derive the necessary conditions of extremum, we compute [15]

$$\frac{\partial \mathbf{J}(\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial \mathbf{J}(y^*(x))}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0, \tag{3.2}$$

and since the fractional operator has linear property, we get

$$0 = \int_a^b \left(\frac{\partial \mathbf{F}}{\partial y}\eta + \sum_{j=1}^n \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} \eta \right) dx, \tag{3.3}$$

or equivalently by using (2.26), we have

$$\begin{aligned}
0 = {} & \int_a^b \left(\frac{\partial \mathbf{F}}{\partial y} - \sum_{j=1}^n \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,1-\nu_j,1-\theta_j} \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \right) \eta\, dx \\
& + \sum_{j=1}^n \Bigg[\theta_j \left(\mathbf{E}_{\rho_j,(1-\nu_j)(1-\mu_j),w_j,b^-}^{-\gamma_j(1-\nu_j)} \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \right) \left(\mathbf{E}_{\rho_j,\nu_j(1-\mu_j),w_j,a^+}^{-\gamma_j\nu_j} \eta \right) \\
& +(1-\theta_j) \left(\mathbf{E}_{\rho_j,(1-\nu_j)(1-\mu_j),w_j,a^+}^{-\gamma_j(1-\nu_j)} \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \right) \left(\mathbf{E}_{\rho_j,\nu_j(1-\mu_j),w_j,b^-}^{-\gamma_j\nu_j} \eta \right) \Bigg] \Bigg|_a^b .
\end{aligned} \tag{3.4}$$

DEFINITION 3.4. We consider the boundary conditions of functional (3.1) with respect to the associated terminal conditions of (3.4), that are

$$\begin{aligned}
& \Bigg[\sum_{j=1}^n \theta_j \left(\mathbf{E}_{\rho_j,(1-\nu_j)(1-\mu_j),w_j,b^-}^{-\gamma_j(1-\nu_j)} \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \right) \left(\mathbf{E}_{\rho_j,\nu_j(1-\mu_j),w_j,a^+}^{-\gamma_j\nu_j} \eta \right) \\
& +(1-\theta_j) \left(\mathbf{E}_{\rho_j,(1-\nu_j)(1-\mu_j),w_j,a^+}^{-\gamma_j(1-\nu_j)} \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \right) \left(\mathbf{E}_{\rho_j,\nu_j(1-\mu_j),w_j,b^-}^{-\gamma_j\nu_j} \eta \right) \Bigg] \Bigg|_a^b = 0.
\end{aligned} \tag{3.5}$$

It should be noted that depending on the values of parameters of (3.1), the boundary condition may be changed and we can consider different cases such that the equation (3.5) holds.

THEOREM 3.5. *Let $y(x) \in HP[a,b]$ be a solution of Problem 3.3 with the boundary conditions, then $y(x)$ must satisfy the following fractional Euler-Lagrange equation*

$$\frac{\partial \mathbf{F}}{\partial y} - \sum_{j=1}^{n} \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,1-\nu_j,1-\theta_j} \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} = 0, \quad \forall x \in [a,b]. \tag{3.6}$$

The function F has continuously derivatives with respect to all components and $\dfrac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho_j,w_j}^{\gamma_j,\mu_j,\nu_j,\theta_j} y} \in HP[a,b]$, for all $j = 1,...,n$.

Proof. The boundary conditions in Definition 3.4 lead to the terminal condition (3.5). Since $\eta(x) \in C^1[a,b]$ is an arbitrary function, we can use the fundamental lemma in calculus of variations to get the relation (3.6). □

REMARK 3.6. Some special cases for the Euler-Lagrange equation (3.6):

1. In the case $\gamma = 0,\ \theta = 0$ or $\gamma = 0,\ \theta = 1$, we get the Euler-Lagrange equations of Hilfer multiple fractional derivatives [1].

2. In the case $\gamma = 0,\ \nu = 0$ and $\theta = 1$, we get the Euler-Lagrange equations of Caputo multiple fractional derivatives [2], [7].

3. In the case $\gamma = 0,\ \nu = 1$ and $\theta = 1$, we get the Euler-Lagrange equations of Riemman-Lioville multiple fractional derivatives [5], [10].

4. In the case $\gamma = 0,\ \nu = 0$ and $\theta = \frac{1}{2}$, we get the Euler-Lagrange equations of Riesz-Caputo multiple fractional derivatives [3], [6], [12].

5. In the case $\gamma = 0,\ \nu = 1$ and $\theta = \frac{1}{2}$, we get the Euler-Lagrange equations of Riesz multiple fractional derivatives [3].

3.2. A functional containing multiple functions

PROBLEM 3.7. For all vector functions $\mathbf{y}^*(x) = (y_1^*(x), \cdots, y_n^*(x)) \in (C^1[a,b])^n$ find the vector function $\mathbf{y}(x) = (y_1(x), \cdots, y_n(x)) \in (C^1[a,b])^n$ such that the functional

$$\mathbf{J}[\mathbf{y}] = \int_a^b \mathbf{F}\Big(x, y_1, \cdots, y_n, \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_1, \cdots, \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_n\Big)\, dx, \tag{3.7}$$

with given boundary conditions attains the optimal value. The function F has continuously derivatives with respect to all components and $y_j^*(x)$, $y_j(x)$, $\dfrac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_j} \in HP[a,b]$, for all $j = 1,...,n$.

Let $\varepsilon \in \mathbb{R}$ and define a family of vector functions $\mathbf{y}^*(x) = \mathbf{y}(x) + \varepsilon\eta(x) \in (C^1[a,b])^n$ that satisfy the relation (3.7) and given boundary conditions where $y_j^*(x) = y_j(x) + \varepsilon\eta_j(x)$, $j = 1, \cdots, n$, is the j-th argument of $\mathbf{y}^*(x)$. Now, by similar discussion to Theorem 3.5 we state the following theorem for the necessary conditions of extremum.

THEOREM 3.8. *Let $\mathbf{y}(x) \in (C^1[a,b])^n$ be a solution of Problem 3.7 with the boundary conditions*

$$\Bigg[\theta\left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),w,b^-}\frac{\partial \mathbf{F}}{\partial \mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_j}\right)\left(\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),w,a^+}\eta_j\right)$$
$$+(1-\theta)\left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),w,a^+}\frac{\partial \mathbf{F}}{\partial \mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_j}\right)\left(\mathbf{E}^{-\gamma\nu}_{\rho,\nu(1-\mu),w,b^-}\eta_j\right)\Bigg]\Bigg|_a^b = 0, \quad j = 1, \cdots, n, \tag{3.8}$$

where F has continuously derivatives with respect to all components and $y_j^*(x)$, $y_j(x)$, $\dfrac{\partial \mathbf{F}}{\partial \mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_j} \in HP[a,b]$, for all $j \in \{1,...,n\}$, then, $\mathbf{y}(x)$ must satisfy the following fractional Euler-Lagrange equations

$$\frac{\partial \mathbf{F}}{\partial y_j} - \mathscr{D}^{\gamma,\mu,1-\nu,1-\theta}_{\rho,w}\frac{\partial \mathbf{F}}{\partial \mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_j} = 0, \quad j = 1, \cdots, n. \tag{3.9}$$

4. Fractional Hamiltonian formulation

In this section, we intend to present the fractional Hamiltonian formulation in terms of the GFHPD. In other words, instead of solving the multiple fractional differential equations (3.9), we will make $2n$ fractional differential equations which are equivalent to a system of Euler-Lagrange equations.

We express $\mathbf{F}\left(x, y_1, \cdots, y_n, \mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_1, \cdots, \mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_n\right) \in C^2[\mathbf{a},\mathbf{b}]$ in terms of a new function $\mathscr{H}$ as

$$\mathscr{H} = -\mathbf{F} + \sum_{j=1}^{n} p_j\,(\mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_j), \tag{4.10}$$

where

$$p_j = \frac{\partial \mathbf{F}}{\partial(\mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_j)}, \quad j = 1, \cdots, n. \tag{4.11}$$

Also, F has twice continuously derivatives with respect to all components and $y_j(x)$, $p_j, \in HP[a,b]$ for all $j \in \{1,...,n\}$, In order to write $(\mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_1), \cdots, (\mathscr{D}^{\gamma,\mu,\nu,\theta}_{\rho,w}y_n)$ as the functions of variables x, $y_1, \cdots, y_n$, we assume that the Jacobian is nonzero, i.e., [15]

$$\frac{\partial(p_1,\cdots,p_n)}{\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_1)\cdots\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_n)} = \det\begin{bmatrix} \frac{\partial^2\mathbf{F}}{\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_1)^2} & \cdots & \frac{\partial^2\mathbf{F}}{\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_1)\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_n)} \\ \vdots & \cdots & \vdots \\ \frac{\partial^2\mathbf{F}}{\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_n)\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_1)} & \cdots & \frac{\partial^2\mathbf{F}}{\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_n)^2} \end{bmatrix} \neq 0. \quad (4.12)$$

The function $\mathscr{H}$ is called the Hamiltonian and the new variables x, y_1, $\cdots$, y_n, p_1, $\cdots$, p_n, $\mathscr{H}$ are called the canonical variables. We present how the Euler-Lagrange equation (3.9) is in terms of the fractional canonical variables. By the relation (4.10), we have

$$\begin{aligned} d\mathscr{H} &= -\frac{\partial\mathbf{F}}{\partial x}\,dx - \sum_{j=1}^{n}\left(\frac{\partial\mathbf{F}}{\partial y_j}\,dy_j + \frac{\partial\mathbf{F}}{\partial(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_j)}\,d(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_j)\right) \\ &\quad + \sum_{j=1}^{n}\left(dp_j\,(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_j) + p_j\,d(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_j)\right), \end{aligned} \quad (4.13)$$

which by using equation (4.11) leads to

$$d\mathscr{H} = -\frac{\partial\mathbf{F}}{\partial x}\,dx + \sum_{j=1}^{n}\left(-\frac{\partial\mathbf{F}}{\partial y_j}\,dy_j + dp_j\,(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_j)\right). \quad (4.14)$$

Now, as $\mathscr{H} = \mathscr{H}(x, y_1, \cdots, y_n, p_1, \cdots, p_n)$ and writing

$$d\mathscr{H} = \frac{\partial\mathscr{H}}{\partial x}dx + \sum_{j=1}^{n}\left(\frac{\partial\mathscr{H}}{\partial y_j}\,dy_j + \frac{\partial\mathscr{H}}{\partial p_j}\,dp_j\right), \quad (4.15)$$

we compare the equation (4.14) with (4.15) and use the equation (3.9) to get the canonical system of Euler-Lagrange equations as

$$\frac{\partial\mathscr{H}}{\partial y_j} = -\mathscr{D}_{\rho,w}^{\gamma,\mu,1-\nu,1-\theta}p_j, \quad \frac{\partial\mathscr{H}}{\partial p_j} = \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_j, \quad j = 1,\cdots,n,. \quad (4.16)$$

and

$$\frac{\partial\mathscr{H}}{\partial x} = -\frac{\partial\mathbf{F}}{\partial x}. \quad (4.17)$$

REMARK 4.1.

1. In the case $\gamma = 0,\ \theta = 0$ or $\gamma = 0,\ \theta = 1$, we get the Hamiltonian formulation of Hilfer fractional derivatives of multiple functions [1].

2. In the case $\gamma = 0$, $\nu = 0$ and $\theta = 1$, we get the Hamiltonian formulation of Caputo multiple fractional derivatives [13].

3. In the case $\gamma = 0$, $\nu = 1$ and $\theta = 1$, we get the Hamiltonian formulation of Riemman-Lioville fractional derivatives of multiple functions [4], [25].

4. In the case $\gamma = 0$, $\nu = 0$ and $\theta = \frac{1}{2}$, we get the Hamiltonian formulation of Riesz-Caputo multiple fractional derivatives [3].

Similarly, it should be noted that the Hamiltonian form of a functional with several fractional derivatives of type GFHPD can be defined.

5. Sufficient condition

In this section, we present a sufficient condition for optimality of functionals including multiple functions.

DEFINITION 5.1. i) For each $x \in [\mathbf{a}, \mathbf{b}]$, we define

$$\Omega_x = \{\mathbf{u}(x) \in \mathbb{R}^{2n} | \big(x, \mathbf{u}(x)\big) \in D_{\mathbf{F}}\}. \tag{5.18}$$

ii) A function $\mathbf{F}$ is concave on convex set Ω_x if and only if

$$\mathbf{F}(x, \mathbf{z}_2) - \mathbf{F}(x, \mathbf{z}_1) \leqslant (\mathbf{z}_2 - \mathbf{z}_1)\nabla\mathbf{F}(x, \mathbf{z}_1), \quad \text{for all } \mathbf{z}_1, \mathbf{z}_2 \in \Omega_x. \tag{5.19}$$

THEOREM 5.2. *Let Ω_x as*

$$\Omega_x = \Big\{ \Big(y_1(x), \cdots, y_n(x), \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_1, \cdots, \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_n\Big) \in \mathbb{R}^{2n} : \\ \Big(x, y_1(x), \cdots, y_n(x), \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_1, \cdots, \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_n\Big) \in D_{\mathbf{F}} \Big\}, \tag{5.20}$$

be a convex set. We suppose that in relation (3.7)*,* $\mathbf{F}$ *is a concave function on the convex set* Ω_x *and* $\mathbf{y}(x)$ *satisfies the fractional Euler-Lagrange equations* (3.9) *with terminal conditions* (3.8)*, then,* $\mathbf{J}(\mathbf{y})$ *has a global maximum at* $\mathbf{y}(x)$*.*

Proof. In Definition 5.1, we set $\mathbf{u}(x) = \big(y_1, \cdots, y_n, \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_1, \cdots, \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_n\big)(x)$, therefore for all $\mathbf{u}(x)$ and $\widehat{\mathbf{u}}(x) \in \Omega_x$, we can deduce from (5.19)

$$\mathrm{F}(x, \widehat{\mathbf{u}}(x)) - \mathrm{F}(x, \mathbf{u}(x)) \leqslant \sum_{i=1}^{n} \left[\left((\widehat{y}_i - y_i) \frac{\partial \mathbf{F}}{\partial y_i} + \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} (\widehat{y}_i - y_i) \right) \right]. \tag{5.21}$$

Now, we integrate the relation (5.21) over interval $[a,b]$ to get

$$\begin{aligned}\mathbf{J}(\widehat{\mathbf{y}})-\mathbf{J}(\mathbf{y}) &= \int_a^b \left[\mathrm{F}(x,\widehat{\mathbf{u}}(x))-\mathrm{F}(x,\mathbf{u}(x))\right]dx\\ &\leqslant \int_a^b \sum_{i=1}^n \left[\frac{\partial \mathbf{F}}{\partial y_j} - \mathscr{D}_{\rho,w}^{\gamma,\mu,1-\nu,1-\theta}\ \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_j}\right](\widehat{y}_i - y_i)dx\\ &+\sum_{j=1}^n \left[\theta\left(\mathbf{E}_{\rho,(1-\nu)(1-\mu),w,b^-}^{-\gamma(1-\nu)}\ \frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_j}\right)\left(\mathbf{E}_{\rho,\nu(1-\mu),w,a^+}^{-\gamma\nu}(\widehat{y}_i - y_i)\right)\right.\\ &\left.+(1-\theta)\left(\mathbf{E}_{\rho,(1-\nu)(1-\mu),w,a^+}^{-\gamma(1-\nu)}\frac{\partial \mathbf{F}}{\partial \mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y_j}\right)\left(\mathbf{E}_{\rho,\nu(1-\mu),w,b^-}^{-\gamma\nu}(\widehat{y}_i - y_i)\right)\right]\Bigg|_a^b.\end{aligned} \tag{5.22}$$

At this point, by using the boundary conditions, replacing η_j by $(\widehat{y}_j - y_j)$ and applying the relations (3.8) and (3.9), the right hand side of relation (5.22) vanishes. Therefore, we get

$$\mathbf{J}(\widehat{\mathbf{y}}) \leqslant \mathbf{J}(\mathbf{y}), \tag{5.23}$$

which completes the proof. □

6. Illustrative examples

EXAMPLE 6.1. We intend to find the maximizer $y \in C^1[0,1]$ for the following functional with the associated boundary conditions

$$\mathbf{J}(y) = -\frac{1}{2}\int_0^1 (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y)^2(x)dx, \tag{6.24}$$

where $y(x)$, and $(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y) \in HP[0,1]$. By using the relation (3.6), we get the Euler-Lagrange equation as

$$\mathscr{D}_{\rho,w}^{\gamma,\mu,1-\nu,1-\theta}\ (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta} y) = 0,\ \ \forall\, x \in [0,1]. \tag{6.25}$$

- In the special case $\gamma = 0$ and $\theta = 1$, we have

$$\mathbf{J}(y) = -\frac{1}{2}\int_0^1 (\mathscr{D}_{0+}^{\mu,\nu} y)^2(x)dx,$$

  and by using (3.5), we know that the natural boundary conditions for this functional is

$$\left[\left(\mathscr{I}_{1^-}^{(1-\nu)(1-\mu)}\frac{\partial \mathbf{F}}{\partial \mathscr{D}_{0^+}^{\mu,\nu} y}\right)\left(\mathscr{I}_{0^+}^{\nu(1-\mu)}\eta\right)\right]_0^1 = 0. \tag{6.26}$$

Also, the Euler-Lagrange equation (6.25) is reduced to $\mathscr{D}_{1^-}^{\mu,(1-\nu)}\,(\mathscr{D}_{0^+}^{\mu,\nu}y) = 0$ and the solution of this fractional differential equation for $\alpha > \frac{1}{2}$ is [1]

$$y(x) = (2\alpha-1)\frac{\Gamma(\alpha)}{\Gamma(\mu)}\int_0^x \frac{d\varsigma}{(1-\varsigma)^{(1-\alpha)}(x-\varsigma)^{1-\mu}}, \quad \alpha > \frac{1}{2}, \tag{6.27}$$

where $\alpha = \mu + \nu - \mu\nu$.

- In the special case $\gamma = 0$, $\theta = 1$ and $\nu = 0$, we have

$$\mathbf{J}(y) = -\frac{1}{2}\int_0^1 ({}^{C}\mathscr{D}_{0+}^{\mu}y)^2(x)dx,$$

and by using (3.5), we know that the natural boundary conditions for this functional is

$$\left[\left(\mathscr{I}_{1^-}^{(1-\mu)}\frac{\partial \mathbf{F}}{\partial\, \mathscr{D}_{0^+}^{\mu,\nu}y}\right)\eta\right]\Big|_0^1 = 0. \tag{6.28}$$

We consider the boundary conditions for $\mathbf{J}(y)$ as

$$y(0) = 0 \quad \text{and} \quad y(1) = 1, \tag{6.29}$$

and since a family of functions $y(x) + \eta(x) \in C^1$ must satisfy these boundary conditions, we have $\eta(0) = 0$ and $\eta(1) = 0$. Therefore, the terminal condition (6.28) is established.

Also, the Euler-Lagrange equation (6.25) is reduced to $\mathscr{D}_{1^-}^{\mu}\,({}^{C}\mathscr{D}_{0+}^{\mu}y) = 0$ and the solution of this fractional differential equation for $\mu > \frac{1}{2}$ is [1]

$$y(x) = (2\mu-1)\int_0^x \frac{d\varsigma}{[(1-\varsigma)(x-\varsigma)]^{1-\mu}}, \quad \mu > \frac{1}{2}. \tag{6.30}$$

In this sense, we will discuss on the sufficient conditions of maximizer solutions. For this purpose, we set $u_1(x) = y(x)$ and $u_2(x) = (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y)(x)$ in (5.18) and choose $\Big(y_1(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_1)(x)\Big)$ and $\Big(y_2(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_2)(x)\Big) \in \Omega_x$, where

$$\Omega_x = \left\{\Big(y(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y)(x)\Big) \in \mathbb{R}^2 : \Big(x, y(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y)(x)\Big) \in D_F\right\}, \tag{6.31}$$

and

$$D_F = \left\{\big(x, y(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y)(x)\big) : y(x) \in HP[0,1]\right\}.$$

Hence we deduce $\Big(x, y_1(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_1)(x)\Big)$ and $\Big(x, y_2(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_2)(x)\Big) \in D_F$ and according to the linearity property of fractional derivative $(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y)(x)$ for any $\lambda \in [0,1]$

$$\begin{aligned}&\lambda\Big(x, y_1(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_1)(x)\Big) + (1-\lambda)\Big(x, y_2(x), (\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}y_2)(x)\Big)\\ &= \Big(x, \lambda y_1(x) + (1-\lambda)y_2(x), \big(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}[\lambda y_1(x) + (1-\lambda)y_2(x)]\big)(x)\Big) \in D_F,\end{aligned}$$

which implies that

$$\Big(\lambda y_1(x)+(1-\lambda)y_2(x),\big(\mathscr{D}_{\rho,w}^{\gamma,\mu,\nu,\theta}[\lambda y_1(x)+(1-\lambda)y_2(x)]\big)(x)\Big)\in\Omega_x,\quad 0\leqslant\lambda\leqslant 1,$$

and Ω_x is convex set. Also, $\mathbf{F}(x,u_1,u_2)=-\frac{1}{2}u_2^2$, $\frac{\partial^2\mathbf{F}}{\partial u_1^2}=0$, $\frac{\partial^2\mathbf{F}}{\partial u_2^2}=-1$ and $\frac{\partial^2\mathbf{F}}{\partial u_1^2}\frac{\partial^2\mathbf{F}}{\partial u_2^2}-\left(\frac{\partial^2\mathbf{F}}{\partial u_1\partial u_2}\right)^2=0$. We conclude that $\mathbf{F}$ is a concave function on the convex set Ω_x and by using Theorem 5.2, we deduce that the solutions (6.27) and (6.30) give the global maximums of related functionals.

EXAMPLE 6.2. We intend to find the maximizer $y(x)=(y_1(x),y_2(x))\in(AC[0,1])^2$ for the following functional

$$\mathbf{J}(\mathbf{y})=\int_0^1[-y_1^2(x)-3y_2^2(x)-2({}_0^{RC}\mathscr{D}_1^{\mu}y_1)^2(x)-({}_0^{RC}\mathscr{D}_1^{\mu}y_2)^2(x)]dx, \tag{6.32}$$

with boundary conditions

$$\mathbf{y}(0)=(0,1)\quad\text{and}\quad\mathbf{y}(1)=(1,0), \tag{6.33}$$

where $y_1(x)$, $y_2(x)$, $({}_0^{RC}\mathscr{D}_1^{\mu}y_1)(x)$, $({}_0^{RC}\mathscr{D}_1^{\mu}y_1)(x)\in C^2[0,1]$ and $\mu\in(0,1)$.

In this case, using (3.8) for $\mathbf{F}=-u_1^2-3u_2^2-2u_3^2-u_4^2$, $u_1(x)=y_1(x)$, $u_2(x)=y_2(x)$, $u_3(x)=({}_0^{RC}\mathscr{D}_1^{\mu}y_1)(x)$ and $u_4(x)=({}_0^{RC}\mathscr{D}_1^{\mu}y_2)(x)$, the terminal condition is reduced to

$$\left[\left(\mathscr{I}_{1^-}^{1-\mu}\frac{\partial\mathbf{F}}{\partial\,{}_0^{RC}\mathscr{D}_1^{\mu}y_j}+\mathscr{I}_{0^+}^{1-\mu}\frac{\partial\mathbf{F}}{\partial\,{}_0^{RC}\mathscr{D}_1^{\mu}y_j}\right)\eta_j\right]_0^1=0,\quad j=1,2. \tag{6.34}$$

Since, a family of functions $y_1(x)+\eta_1(x)\in C^1$ and $y_2(x)+\eta_2(x)\in C^1$ must satisfy the boundary condition (6.33), we have

$$\eta_1(x)=0\text{ and }\eta_2(x)=0\quad\text{for } x=0,1. \tag{6.35}$$

Thus, the terminal condition (6.34) and consequently the boundary conditions (6.33) established for functional (6.32). On the other hand, we know

$$\frac{\partial\mathbf{F}}{{}_0^{RC}\mathscr{D}_1^{\mu}y_2}=-{}_0^{RC}\mathscr{D}_1^{\mu}y_2,\quad\text{and}\quad\frac{\partial\mathbf{F}}{{}_0^{RC}\mathscr{D}_1^{\mu}y_1}=-4\,{}_0^{RC}\mathscr{D}_1^{\mu}y_1, \tag{6.36}$$

which by applying the relations (3.9) and (6.36), we get the Euler-Lagrange equations for (6.32) as

$$\begin{cases} y_1+2\,{}_0^{R}\mathscr{D}_1^{\mu}\big({}_0^{RC}\mathscr{D}_1^{\mu}y_1\big)=0,\ \ \forall x\in[0,1],\\ 3y_2+{}_0^{R}\mathscr{D}_1^{\mu}\big({}_0^{RC}\mathscr{D}_1^{\mu}y_2\big)=0,\ \ \forall x\in[0,1]. \end{cases} \tag{6.37}$$

In this case, by using the relations (5.18) we define the set Ω_x as

$$\Omega_x = \Big\{ \Big(y_1(x), y_2(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_1)(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_2)(x) \Big) \in \mathbb{R}^4 : \Big(x, y_1(x), y_2(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_1)(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_2)(x) \Big) \in D_F \Big\} \tag{6.38}$$

where

$$D_F = \Big(x, y_1(x), y_2(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_1)(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_2)(x) \Big) : \ y_1(x),\ y_2(x) \in AC[0,1] \Big\}.$$

Since, the fractional derivative $({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y)(x)$ has the linearity property, thus for any $\lambda \in [0,1]$ and

$$\Big(x, y_1(x), y_2(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_1)(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_2)(x) \Big), \quad \Big(x, z_1(x), z_2(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} z_1)(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} z_2)(x) \Big) \in D_F,$$

we have

$$\lambda \Big(x, y_1(x), y_2(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_1)(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_2)(x) \Big) + (1-\lambda) \Big(x, z_1(x), z_2(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} z_1)(x), ({}^{RC}_{0}\mathscr{D}^{\mu}_{1} z_2)(x) \Big) \in D_F$$

which implies that Ω_x is a convex set. Also, the Hessian matrix $H(\mathbf{F})$

$$H(\mathbf{F}) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{4\times 4} \tag{6.39}$$

is negative definite. We conclude that $\mathbf{F}$ is a concave function and hence based on Theorem 5.2, the solutions of Euler-Lagrange equations (6.37) with the boundary conditions (6.33) give the sufficient conditions for the maximum. In other words, any $\mathbf{y} = (y_1, y_2)$ which satisfy the equations

$$\begin{cases} y_1 + 2\, {}^{R}_{0}\mathscr{D}^{\mu}_{1} \left({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_1 \right) = 0, \quad \forall x \in [0,1], \\ 3y_2 + {}^{R}_{0}\mathscr{D}^{\mu}_{1} \left({}^{RC}_{0}\mathscr{D}^{\mu}_{1} y_2 \right) = 0, \quad \forall x \in [0,1], \\ y_1(0) = y_2(1) = 0, \quad y_1(1) = y_2(0) = 1, \end{cases} \tag{6.40}$$

gives the global maximum of functional $\mathbf{J}(\mathbf{y})$ in (6.32).

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