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CONTROL CHAOS IN THE FRACTIONAL LORENZ–HAMILTON SYSTEM

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Abstract. In this paper we discuss the dynamical behavior of a family of fractional differential systems associated to Lorenz-Hamilton system. The stability analysis of equilibrium states of the controlled fractional Lorenz-Hamilton system is studied.

1. Introduction

The fractional calculus has been found to be an important tool in various fields, such as mathematics, physics, engineering, biology, chaotic dynamics and other complex dynamical systems [1, 4, 8]. The interest in the study of fractional-order nonlinear systems lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties, which are not taken into account in the classical integer-order models [1, 2].

The paper is structured as follows. In Section 2 we introduce the fractional Lorenz-Hamilton system (2.4) associated to Lorenz-Hamilton system [5, 6]. The problem of the existence and uniqueness of solution for the fractional system (2.4) is discussed. Section 3 is devoted to studying of the stability of equilibrium points for the fractional system (2.4). Also, the unstable equilibrium states of this system can be controlled via fractional stability theory. In Section 4, the numerical integration for the controlled fractional Lorenz-Hamilton system (4.1) are given.

2. On fractional Lorenz-Hamilton system

There are many definitions of fractional derivatives. One of the more common definitions is the Caputo definition of fractional derivatives. Let $f \in C^{\infty}(\mathbf{R})$ and $\alpha \in \mathbf{R}$, $\alpha > 0$. The α-order Caputo differential operator [4], is described by

$$D_t^{\alpha} f(t) = J^{m-\alpha} f^{(m)}(t), \quad \alpha > 0, \tag{2.1}$$

where $f^{(m)}(t)$ represents the m-order derivative of the function f, $m \in \mathbf{N}^*$ is an integer such that $m-1 \leqslant \alpha \leqslant m$ and J^{β} is the β-order Riemann-Liouville integral operator [8], which is expressed as follows

$$J^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds, \quad \beta > 0, \tag{2.2}$$

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where Γ is the Euler Gamma function. If $\alpha = 1$, then $D_t^\alpha f(t) = \frac{df}{dt}$.

In this paper we suppose that $\alpha \in (0,1]$.

We consider the following differential system of Maxwell-Bloch type on $\mathbf{R}^3$:

$$\dot{x}^1(t) = \frac{1}{2}x^2(t),\ \ \dot{x}^2(t) = -x^1(t)x^3(t),\ \ \dot{x}^3(t) = x^1(t)x^2(t), \tag{2.3}$$

where $\dot{x}^i = \dfrac{dx^i(t)}{dt}$, $i = \overline{1,3}$ and t is the time.

The dynamical system (2.2) is called the *Lorenz-Hamilton system* [5].

The *fractional Lorenz-Hamilton system* associated to Hamilton-Poisson system (2.3) is defined by the following set of fractional differential equations:

$$D_t^\alpha x^1(t) = \frac{1}{2}x^2(t), \quad D_t^\alpha x^2(t) = -x^1(t)x^3(t), \quad D_t^\alpha x^3(t) = x^1(t)x^2(t), \quad \alpha \in (0,1). \tag{2.4}$$

The initial value problem of the fractional Lorenz-Hamilton system (2.4) can be represented in the following matrix form:

$$D_t^\alpha x(t) = Ax(t) + x^1(t)A_1x(t), \qquad x(0) = x_0, \tag{2.5}$$

where $0 < \alpha < 1$, $x(t) = (x^1(t), x^2(t), x^3(t))^T$, $t \in (0,\tau)$ and

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

PROPOSITION 2.1. *The initial value problem of the fractional Lorenz-Hamilton system* (2.5) *has a unique solution.*

Proof. Let $f(x(t)) = Ax(t) + x^1(t)A_1x(t)$. It is obviously continuous and bounded on $D = \{x \in \mathbf{R}^3 | x^1 \in [x_0^1 - \delta, x_0^1 + \delta]\}$ for any $\delta > 0$. We have $f(x(t)) - f(x_1(t)) = A(x(t) - x_1(t)) + y(t)$, where $g(t) = x^1(t)A_1x(t) - x_1^1(t)A_1x_1(t)$. Then

(a) $|f(x(t)) - f(x_1(t))| \leqslant \|A\| \cdot |x(t) - x_1(t)| + |g(t)|$, where $\|\cdot\|$ and $|\cdot|$ denote matrix norm and vector norm respectively.

It is easy to see that $g(t) = (x^1(t) - x_1^1(t))A_1x(t) + x_1^1(t)A_1(x(t) - x_1(t))$. Then

$$|g(t)| \leqslant |(x^1(t) - x_1^1(t))A_1x(t)| + |x_1^1(t)A_1(x(t) - x_1(t))|.$$

We have $|g(t)| \leqslant \|A_1\|(|x(t)| \cdot |x^1(t) - x_1^1(t))| + |x_1^1(t)| \cdot |x(t) - x_1(t))|$ and using the inequality $|x^1(t) - x_1^1(t))| \leqslant |x(t) - x_1(t))|$ one obtains

(b) $|y(t)| \leqslant \|A_1\|(|x(t)| + |x_1^1(t)|)|x(t) - x_1(t))|$.

According to (b), the relation (a) becomes

$$|f(x(t)) - f(x_1(t))| \leqslant (\|A\| + \|A_1\|(|x(t)| + |x_1^1(t)|)|x(t) - x_1(t)|.$$

Replacing $\|A\| = \frac{1}{2}$, $\|A_1\| = \sqrt{2}$, from the above we deduce that

(c) $|f(x(t)) - f(x_1(t))| \leqslant L|x(t) - x_1(t)|$, where $L = \frac{1}{2} + \sqrt{2}(2|x_0| + \delta) > 0$.

The inequality (c) shows that $f(x(t))$ satisfies a Lipschitz condition. Based on the results of Theorems 1 and 2 in [2], we can conclude that the initial value problem of the system (2.5) has a unique solution. $\square$

Solving the system of equations $D_t^\alpha x^i(t)=0,\ i=\overline{1,3}$, we find the equilibrium points of the fractional Lorenz-Hamilton system (2.4) as follows:

$$e_0=(0,0,0),\quad e_1^m=(m,0,0)\ \text{ and }\ e_3^m=(0,0,m)\ \text{ for all }\ m\in\mathbf{R},\ m\neq 0$$

REMARK 2.1. The Poisson geometry of the Lorenz-Hamilton system (2.3) has been solved in [6].

3. Stability study of fractional Lorenz-Hamilton system

We start with the stability analysis of equilibrium points for the fractional system (2.4). The Jacobian matrix of the system (2.4) is

$$J(x)=\begin{pmatrix} 0 & \frac{1}{2} & 0\\ -x^3 & 0 & -x^1\\ x^2 & x^1 & 0\end{pmatrix}.$$

For the study of stability of the system (2.4) we shall use the following proposition.

PROPOSITION 3.1. ([7]) *Let x_e be an equilibrium point of fractional system (2.4) and $J(x_e)$ be the Jacobian matrix $J(x)$ evaluated at x_e. The point x_e is locally asymptotically stable, iff all eigenvalues $\lambda(J(x_e))$ of the matrix $J(x_e)$ satisfy the condition:*

$$|\arg(\lambda(J(x_e)))|>\frac{\alpha\pi}{2}.$$

PROPOSITION 3.2. *The equilibrium points e_0,e_1^m $(m\neq 0)$ and e_3^m $(m\neq 0)$ are unstable for all $\alpha\in(0,1)$.*

Proof. (1) Suppose that $m\in\mathbf{R}$. The characteristic polynomial of the matrix $J(e_1^m)=\begin{pmatrix} 0 & \frac{1}{2} & 0\\ 0 & 0 & -m\\ 0 & m & 0\end{pmatrix}$ is $p_{J(e_1^m)}(\lambda)=\det(J(e_1^m)-\lambda I)=-\lambda(\lambda^2+m^2)$. Then the characteristic roots of $J(e_1^m)$ are $\lambda_1=0$ and $\lambda_{2,3}=\pm|m|i$. Since $|\arg(\lambda_1)|=0<\frac{\pi}{2}\alpha$ for all $\alpha\in(0,1)$, by Proposition 3.1, it follows that e_0 and e_1^m are unstable.

(2) Suppose that $m\in\mathbf{R},m\neq 0$. The characteristic polynomial of the matrix $J(e_3^m)=\begin{pmatrix} 0 & \frac{1}{2} & 0\\ -m & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ is $p_{J(e_3^m)}(\lambda)=\det(J(e_3^m)-\lambda I)=-\lambda(\lambda^2+\frac{m}{2})$ with characteristic roots $\lambda_1=0,\ \lambda_{2,3}=\pm\frac{1}{2}\sqrt{-2m}$ if $m<0$ and $\lambda_1=0,\ \lambda_{2,3}=\pm\frac{i}{2}\sqrt{2m}$ if $m>0$. Applying again Proposition 3.1, one obtains the required result. $\square$

In the case when x_e is a unstable equilibrium state of the fractional system (2.4), we associate to (2.4) a new fractional system as follows.

The *controlled fractional Lorenz-Hamilton system* associated to (2.4) is defined by:

$$\begin{cases} D_t^\alpha x^1 = \frac{1}{2}x^2 - k_1(x^1 - x_e^1) \\ D_t^\alpha x^2 = -x^1x^3 - k_2(x^2 - x_e^2), \\ D_t^\alpha x^3 = x^1x^2 - k_3(x^3 - x_e^3), \end{cases} \qquad \alpha \in (0,1), \tag{3.1}$$

where x_e represents an equilibrium point of (2.4) and $k_i \in \mathbf{R}$, $i = \overline{1,3}$ are constants.

If one selects the parameters k_i, $i = \overline{1,3}$ which then make the eigenvalues of the linearized equation of the controlled system (3.1) satisfy the condition from Proposition 3.1, then the trajectories of (3.1) asymptotically approaches the unstable equilibrium state x_e in the sense that $\lim_{t\to\infty} \|x(t) - x_e\| = 0$, where $\|\cdot\|$ is the Euclidean norm.

The Jacobian matrix of the controlled fractional system (3.1) is

$$J(x,k) = \begin{pmatrix} -k_1 & \frac{1}{2} & 0 \\ -x^3 & -k_2 & -x^1 \\ x^2 & x^1 & -k_3 \end{pmatrix}.$$

Let us we study the problem of stabilizing of the fractional system (2.4) at the equilibrium points e_0, e_1^m $(m \neq 0)$ and e_3^m $(m \neq 0)$.

PROPOSITION 3.3. *The equilibrium state* e_0 *of the controlled fractional system* (3.1) *is locally asymptotically stable for* $k_i > 0$, $i = \overline{1,3}$ *and* $\alpha \in (0,1)$.

Proof. The characteristic polynomial of the Jacobian matrix $J(e_0,k)$ is $p_{J(e_0,k)}(\lambda) = -(\lambda + k_1)(\lambda + k_2)(\lambda + k_3)$ with characteristic roots $\lambda_i = -k_i$ for $i = \overline{1,3}$. Since $|arg(\lambda_i| = \pi > \dfrac{\alpha\pi}{2}$ for $i = \overline{1,3}$, by Proposition 3.1, it follows that e_0 is locally asymptotically stable. □

PROPOSITION 3.4. (i) *If* $k_1 > 0$, $k_2 + k_3 > 0$, *then* e_1^m *is asymptotically stable for all* $m \in \mathbf{R}$, $m \neq 0$ *and* $\alpha \in (0,1)$;

(ii) *If* $k_1 > 0$, $k_2 + k_3 < 0$, *then* e_1^m *is asymptotically stable for all*

$$m \in \Big(-\infty, -\frac{|k_2 - k_3|}{2}\Big) \cup \Big(\frac{|k_2 - k_3|}{2}, \infty\Big) \text{ and } 0 < \alpha < \frac{2}{\pi}\arctan\frac{\sqrt{-(k_2 - k_3)^2 + 4m^2}}{|k_2 + k_3|}.$$

Proof. The Jacobian matrix of (3.1) at e_1^m is $J(e_1^m,k) = \begin{pmatrix} -k_1 & \frac{1}{2} & 0 \\ 0 & -k_2 & -m \\ 0 & m & -k_3 \end{pmatrix}$ whose characteristic polynomial $p_{J(e_1^m,k)}(\lambda) = \det(J(e_1^m,k) - \lambda I)$ is

$$p_{J(e_1^m,k)}(\lambda) = -(\lambda + k_1)[\lambda^2 + (k_2 + k_3)\lambda + k_2k_3 + m^2].$$

The roots of the characteristic equation are

$$\lambda_1 = -k_1, \quad \lambda_{2,3} = \frac{-(k_2+k_3) \pm \sqrt{(k_2-k_3)^2 - 4m^2}}{2}.$$

Denote $\Delta = (k_2-k_3)^2 - 4m^2$ and $m_1 = -\frac{|k_2-k_3|}{2}$, $m_2 = \frac{|k_2-k_3|}{2}$.

$(\mathbf{1})$ We have $\Delta < 0$ iff $m \in (-\infty, m_1) \cup (m_2, \infty)$. Then $\lambda_{2,3} = \frac{-(k_2+k_3) \pm i\sqrt{-\Delta}}{2}$.

$(1a)$ We suppose that $k_1 > 0$ and $k_2+k_3 > 0$. In this case we have $\lambda_1 < 0$ and $\mathrm{Re}\,(\lambda_{2,3}) < 0$. Since $|\arg(\lambda_1)| = \pi$ and $|\arg(\lambda_{2,3})| = \pi > \frac{\pi\alpha}{2}$ for all $\alpha \in (0,1)$, by Proposition 3.1, it implies that e_1^m is locally asymptotically stable.

$(1b)$ For $k_1 > 0$ and $k_2+k_3 < 0$, we have $\lambda_1 < 0$ and $\mathrm{Re}\,(\lambda_{2,3}) > 0$. Applying Proposition 3.1, e_1^m is locally asymptotically stable, for

$$0 < \alpha < \frac{2}{\pi} \arctan \frac{\sqrt{-(k_2-k_3)^2 + 4m^2}}{|k_2+k_3|}.$$

Therefore, the assertion (ii) holds.

$(\mathbf{2})$ We have $\Delta = 0$ for $m = \pm\frac{|k_2-k_3|}{2}$. The eigenvalues of the matrix $J(e_1^m, k)$ are $\lambda_1 = -k_1$, $\lambda_{2,3} = \frac{-(k_2+k_3)}{2}$. If $k_1 > 0$ and $k_2+k_3 > 0$, then $\lambda_i < 0$ for $i = \overline{1,3}$. Since the eigenvalues are all negative, it follows that e_1^m is asymptotically stable.

If $k_1 > 0$ and $k_2+k_3 < 0$, then $\lambda_1 < 0$ and $\lambda_{2,3} > 0$ and e_1^m is unstable.

$(\mathbf{3})$ We have $\Delta > 0$ iff $m \in (m_1, m_2)$. Then $\lambda_{2,3} = \frac{-(k_2+k_3) \pm \sqrt{\Delta}}{2}$.

$(3a)$ We suppose that $k_1 > 0$ and $k_2+k_3 > 0$. Then $\lambda_i < 0$ for $i = \overline{1,3}$. Since the eigenvalues are all negative, it follows that e_1^m is asymptotically stable.

$(3b)$ We suppose that $k_1 > 0$ and $k_2+k_3 < 0$. In this case, $J(e_1^m, k)$ has at least a positive eigenvalue and so e_1^m is unstable.

Therefore, according to $(1a), (2)$ and $(3a)$ it follows that the assertion (i) holds. □

EXAMPLE 3.1. By choosing the control parameters $k_i, i = \overline{1,3}$ that satisfy one condition from Proposition 3.4, then the trajectories of the controlled fractional system (3.1) are driven to the unstable equilibrium point e_1^m $(m \neq 0)$. For example, if we select $k_1 > 0$, $k_2 = k_3 = \ell < 0$, then the condition (ii) of Proposition 3.4 is achieved. This implies that, the trajectories of the system (3.1) converge to $e_1 = (m,0,0)$ and $\alpha \in \left(0, \frac{2}{\pi}\arctan\left|\frac{m}{\ell}\right|\right)$. For example, substituting $k_1 = 1$, $k_2 = k_3 = -1$ in (3.1) we obtains that the controlled fractional system is asymptotically stable at $e_1 = (\sqrt{3}, 0, 0)$ for $\alpha \in (0, \frac{2}{3})$, although the real part of the eigenvalues $\lambda_{2,3} = 1 \pm i\sqrt{3}$ is positive.

PROPOSITION 3.5. (i) *If* $k_1 + k_2 > 0$, $k_3 > 0$, *then* e_3^m *is asymptotically stable for all* $m \in (-2k_1k_2, \infty) \setminus \{0\}$ *and* $\alpha \in (0,1)$;

(ii) *If* $k_1+k_2<0,\ k_3>0$, *then* e_3^m *is asymptotically stable for all*

$$m\in\Big(\frac{(k_1-k_2)^2}{2},\infty\Big)\quad \textit{and}\quad 0<\alpha<\frac{2}{\pi}\arctan\frac{\sqrt{-(k_1-k_2)^2+2m}}{|k_1+k_2|}.$$

Proof. The Jacobian matrix of (3.1) at e_3^m is $J(e_1^m,k)=\begin{pmatrix}-k_1 & \frac{1}{2} & 0\\ -m & -k_2 & 0\\ 0 & 0 & -k_3\end{pmatrix}$ whose characteristic polynomial $p_{J(e_3^m,k)}(\lambda)=\det(J(e_3^m,k)-\lambda I)$ is

$$p_{J(e_3^m,k)}(\lambda)=-(\lambda+k_3)\Big[\lambda^2+(k_1+k_2)\lambda+k_1k_2+\frac{m}{2}\Big].$$

The roots of the characteristic equation are $\lambda_{1,2}=\dfrac{-(k_1+k_2)\pm\sqrt{\Delta}}{2}$, $\lambda_3=-k_3$, with $\Delta=(k_1-k_2)^2-2m$.

$(\mathbf{1})$ We have $\Delta<0$ iff $m\in\Big(\dfrac{(k_1-k_2)^2}{2},\infty\Big)$. Then $\lambda_{1,2}=\dfrac{-(k_1+k_2)\pm i\sqrt{-\Delta}}{2}$.

$(1a)$ We suppose that $k_1+k_2>0$ and $k_3>0$. In this case we have $\lambda_3<0$ and $\mathrm{Re}(\lambda_{1,2})<0$. Since $|\arg(\lambda_3)|=\pi$ and $|\arg(\lambda_{1,2})|=\pi>\dfrac{\pi\alpha}{2}$ for all $\alpha\in(0,1)$, by Proposition 2.1(i), it implies that e_3^m is asymptotically stable.

$(1b)$ For $k_1+k_2<0$ and $k_3>0$, we have $\lambda_3<0$ and $\mathrm{Re}(\lambda_{1,2})>0$. Applying Proposition 3.1, e_3^m is asymptotically stable, for $0<\alpha<\dfrac{2}{\pi}\arctan\dfrac{\sqrt{-(k_1-k_2)^2+2m}}{|k_1+k_2|}$. Therefore, the assertion (ii) holds.

$(\mathbf{2})$ We have $\Delta=0$ for $m=\dfrac{(k_1-k_2)^2}{2}$. The eigenvalues of the matrix $J(e_3^m,k)$ are $\lambda_{1,2}=\dfrac{-(k_1+k_2)}{2}$, $\lambda_3=-k_3$. If $k_1+k_2>0$ and $k_3>0$, then $\lambda_i<0$ for $i=\overline{1,3}$. Since the eigenvalues are all negative, it follows that e_3^m is asymptotically stable.

If $k_1+k_2<0$ and $k_3>0$, then $\lambda_{1,2}>0$ and $\lambda_3<0$ and e_3^m is unstable.

$(\mathbf{3})$ We have $\Delta>0$ iff $m\in\Big(-\infty,\dfrac{(k_1-k_2)^2}{2}\Big)$. Then $\lambda_{1,2}=\dfrac{-(k_1+k_2)\pm\sqrt{\Delta}}{2}$.

$(3a)$ We suppose that $k_1+k_2>0$ and $k_3>0$. Then $\lambda_2<0$ and $\lambda_3<0$. Also, we have $\lambda_1=\dfrac{-(k_1+k_2)+\sqrt{\Delta}}{2}<0$ iff $m>-2k_1k_2$. In this case, the eigenvalues are all negative and so e_3^m is asymptotically stable for $m\in\Big(-2k_1k_2,\dfrac{(k_1-k_2)^2}{2}\Big)$, $m\neq 0$.

$(3b)$ We suppose that $k_1+k_2<0$ and $k_3>0$. In this case, $J(e_3^m,k)$ has at least a positive eigenvalue and so e_3^m is unstable.

According to $(1a)$, (2) and $(3a)$, it follows that the assertion (ii) holds. □

EXAMPLE 3.2. By choosing the control parameters $k_i,\ i=\overline{1,3}$ that satisfy one condition from Proposition 3.5, then the trajectories of the controlled fractional system (3.1) are driven to the unstable equilibrium point e_3^m $(m\neq 0)$. For example, we select

$k_1 = 2$, $k_2 = -1$, $k_3 = 3$, then the stability condition (i) of Proposition 3.5 is achieved. This implies that, the trajectories of the system (2.4) converge to the equilibrium point $e_3 = (0,0,m)$ when $m \in (4,\infty)$ and $\alpha \in (0,1)$. For example, the controlled fractional system is asymptotically stable at $e_3 = (0,0,4.5)$ for $\alpha \in (0,1)$.

4. Numerical integration of the fractional system (4.1)

In this section we discuss the numerical solution of controlled fractional Lorenz-Hamilton system (3.1).

Consider the fractional differential equations

$$\begin{cases} D_t^\alpha x^i(t) = g_i(x^1(t),x^2(t),x^3(t)), & t\in(0,\tau),\ \alpha\in(0,1) \\ x(0) = (x_0^1,x_0^2,x_0^3) & \end{cases} \tag{4.1}$$

where $g_1(t) = \frac{1}{2}x^2(t) - k_1(x^1(t) - x_e^1)$, $g_2(t) = -x^1(t)x^3(t) - k_2(x^2(t) - x_e^2)$ and $g_3(t) = x^1(t)x^2(t) - k_3(x^3(t) - x_e^3)$.

Since $g(t) = (g_1(t), g_2(t), g_3(t))$ is continuous, the initial value problem (4.1) is equivalent to the nonlinear Volterra integral equation ([3]), which is given as follows:

$$x^i(t) = x_0^i + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_i(x^1(s),x^2(s),x^3(s))ds, \qquad i = \overline{1,3}. \tag{4.2}$$

Diethelm et al. have given a predictor-corrector scheme [3], based on the Adams-Bashforth-Moulton algorithm to integrate the equation (4.2). We apply this scheme to the controlled fractional system (4.1). For this, let $h = \dfrac{\tau}{N}$, $t_n = nh$ for $n = 0,1,\ldots,N$. We use the following notations: $x^i[n] = x^i(nh)$, $i = \overline{1,3}$ and $x_p^i[n] = x_p^i(nh)$, $i = \overline{1,3}$.

The controlled fractional system (4.1) can be discretized as follows:

$$\begin{cases} x^1[n+1] = x_0^1 + \dfrac{h^\alpha}{\Gamma(\alpha+2)}\Big(\sum\limits_{j=0}^{n} a[j,n+1]\Big(\dfrac{1}{2}x^2[j] - k_1(x^1[j] - x_e^1)\Big) \\ \qquad + \Big(\dfrac{1}{2}x_p^2[n+1] - k_1(x_p^1[n+1] - x_e^1)\Big)\Big), \\ x^2[n+1] = x_0^2 + \dfrac{h^\alpha}{\Gamma(\alpha+2)}\Big(\sum\limits_{j=0}^{n} a[j,n+1](-x^1[j]x^3[j] - k_2(x^2[j] - x_e^2)) \\ \qquad + (-x_p^1[n+1]x_p^3[n+1] - k_2(x_p^2[n+1] - x_e^2))\Big), \\ x^3[n+1] = x_0^3 + \dfrac{h^\alpha}{\Gamma(\alpha+2)}\Big(\sum\limits_{j=0}^{n} a[j,n+1](x^1[j]x^2[j] - k_3(x^3[j] - x_e^3)) \\ \qquad + (x_p^1[n+1]x_p^2[n+1] - k_3(x_p^3[n+1] - x_e^3))\Big), \end{cases} \tag{4.3}$$

$$\begin{cases} x_p^1[n+1] = \dfrac{h^\alpha}{\alpha\Gamma(\alpha)}\Big(\sum\limits_{j=0}^{n} b[j,n+1]\Big(\dfrac{1}{2}x^2[j] - k_1(x^1[j]-x_e^1)\Big)\Big), \\ x_p^2[n+1] = \dfrac{h^\alpha}{\alpha\Gamma(\alpha)}\Big(\sum\limits_{j=0}^{n} b[j,n+1](-x^1[j]x^3[j] - k_2(x^2[j]-x_e^2))\Big), \\ x_p^3[n+1] = \dfrac{h^\alpha}{\alpha\Gamma(\alpha)}\Big(\sum\limits_{j=0}^{n} b[j,n+1](x^1[j]x^2[j] - k_3(x^3[j]-x_e^3))\Big), \end{cases} \tag{4.4}$$

where

$$\begin{aligned} a[0,n+1] &= n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, \\ a[j,n+1] &= (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, \quad j=\overline{1,n}, \\ b[j,n+1] &= (n+1-j)^\alpha - (n-j)^\alpha, \quad j=\overline{0,n}. \end{aligned}$$

The above scheme given by the relations (4.3) and (4.4) is called the *predictor-corrector Moulton-Adams algorithm for controlled fractional differential system* (4.1).

The error estimate for the algorithm described by (4.3) and (4.4) is

$$\max_{0\leqslant j\leqslant N}\{\, x^i[j] - x_p^i[j] | i=\overline{1,3}\,\} = O(h^{\alpha+1}).$$

Applying the algorithm (4.3)–(4.4), the system (4.1) is numerically integrated for $\alpha=0.8$, $k_1=2$, $k_2=-1$, $k_3=3$ and $x_e=(0,0,4.5)$ (see Example 3.2).

For this, we consider $h=0.01$, $\varepsilon=0.01$, $N=500$, $t=502$ and the initial conditions $x^1(0)=\varepsilon$, $x^2(0)=\varepsilon$, $x^3(0)=\varepsilon+4.5$.

Using the software Maple 11, the orbits $(n,x^i(n))$, $i=\overline{1,3}$ of system (4.1) are represented in the figures Fig. 1–3.

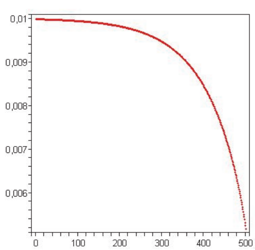

Fig.1. $(n,x^1(n))$

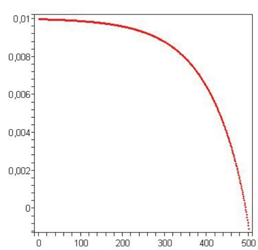

Fig.2. $(n,x^2(n))$

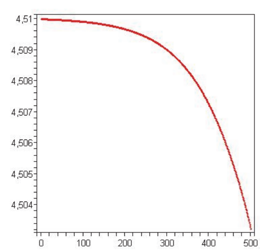

Fig.3. $(n,x^3(n)$

In the coordinate system $Ox^1x^2x^3$, the orbits $(x^1(n),x^2(n)),x^3(n))$ for the solutions of equations (4.1) are represented in the figure Fig. 4.

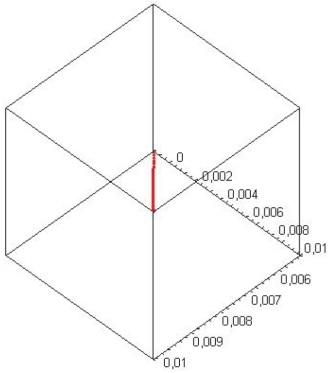

Fig.4. $(x^1(n), x^2(n), x^3(n))$ for $k_1 = 2$, $k_2 = -1$, $k_3 = 3$ and $\alpha = 0.8$.

The numerical simulations show the validity of the theoretical analysis.

CONCLUSIONS. The dynamics of the fractional Lorenz-Hamilton system (2.4) was discussed in this paper. The analysis of the fractional stability of equilibrium states for the controlled fractional Lorenz-Hamilton system (3.1) was studied. Finally, the numerical simulation for the fractional system (4.1) is given.

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