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ON PARTIAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

DHARMENDRA KUMAR SINGH

Abstract. In this paper we will be explain how to solve the Cauchy problems to some differential equations with partial fractional derivatives having variable coefficients by homotopy perturbation method. The solutions constructed in Mittag-Leffler function.

1. Introduction

Fractional order integrals and derivatives

The class of fractional operator equations of various types play very important role not only in mathematics but also in physics, control systems, dynamical systems and engineering. Naturally, such equations required to be solved. There are numerous books focused in this direction, that is concerning the linear and nonlinear problems involving different types of fractional derivatives as well as integral.

The Riemman-Liouville fractional integral operator $\alpha > 0$ is defined as [1]

$$\left(I_x^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt. \tag{1}$$

The fractional derivative of order α is defined, for a function $f(x)$ by

$$\left(D_x^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t)dt \quad (0 \leqslant \Re(\alpha) < 1). \tag{2}$$

Equation (1) and (2) gave important properties of the fractional differential and integral operators D_x^{α}, I_x^{α} as follows:

If for $\alpha \in [0,1)$ and f is continuous function, then [11, p. 872]

$$\left(DI_x^{\alpha} f\right)(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)} f(0) + \left(I_x^{\alpha} Df\right)(x), \quad D = \frac{d}{dx}, \quad x_+ = \begin{bmatrix} x, x > 0 \\ 0,\ x < 0 \end{bmatrix}. \tag{3}$$

$$I_x^{\alpha} D_x^{\alpha} f(x) = D_x^{\alpha} I_x^{\alpha} f(x) = f(x). \tag{4}$$

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Further, we mention the basic definition of the Caputo fractional-order integration and differentiation, which are used in the upcoming paper and play the most important role in fractional calculus. Caputo definition of fractional-order derivative is defined as [10]

$$D^{\alpha} f(t) = I^{m-\alpha} D^{m} f(x)$$

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha+1} f^{(m)}(t)dt \tag{5}$$

for $m-1 < \alpha \leqslant m$, $m \in N$, $x > 0$.

$$D_t^{\alpha} t^n = \begin{bmatrix} 0, \quad (n \leqslant \alpha - 1,\, n \in N), \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, \quad (n > \alpha - 1,\, n \in N) \end{bmatrix}. \tag{6}$$

Similarly to the integer-order derivative, the Caputo fractional derivative is a linear operation

$$D^{\alpha} \left(\sum_{i=1}^{n} c_i f_i(t) \right) = \sum_{i=1}^{n} c_i D^{\alpha} f_i(t), \tag{7}$$

where $\{c_i\}_{i-1}^{n}$ are constant.

In applications section we assume that Caputo derivative operators (5) are designated by $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$, $\frac{\partial^{\alpha}}{\partial y^{\alpha}}$, $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$, $\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}$, $\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}$, $\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}$.

For instance

$$\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} x_{+}^{2\alpha} = x_{+}^{2\alpha} = \begin{bmatrix} x^{2\alpha},\; x > 0 \\ 0,\; x \leqslant 0 \end{bmatrix}. \tag{8}$$

Basic idea of the homotopy perturbation method

During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic system containing fractional derivatives, such as Adomian's decomposition method [12, 13, 14], Homotopy analysis method [3, 4, 15]. To illustrate the basic idea of HPTM method, Consider the following nonlinear differential equation

$$A(y) - f(r) = 0, \quad r \in \Omega, \tag{9}$$

with the boundary conditions of

$$B(y) = 0, \quad r \in \Gamma, \tag{10}$$

where $A, B, f(r)$ and Γ are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain Ω, respectively.

The operator A can generally be divided into a linear part L and a nonlinear part N. Equation (9) may therefore be written as.

$$L(y) + N(y) - f(r) = 0. \tag{11}$$

By the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(y_0)] + p[A(v) - f(r)] = 0 \tag{12}$$

or

$$H(v,p) = L(v) - L(y_0) + pL(y_0) + p[N(v) - f(r)] = 0, \tag{13}$$

where $p \in [0,1]$ an embedding parameter, while y_0 an initial approximation of (11), which satisfies the boundary conditions. Obviously, from (12) and (13) we will have

$$H(v,0) = L(v) - L(y_0) = 0 \tag{14}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{15}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from y_0 to y(r). In topology, this is called deformation, while $L(v) - L(y_0)$ and $A(v) - f(r)$ are called homotopy. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [5–9]. The homoptopy perturbation method use the homoptopy parameter p as an expanding parameter [5–9] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots. \tag{16}$$

If $p \to 1$, then (16) corresponding to (12) and becomes the approximate solution of the form

$$f = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i. \tag{17}$$

The series (17) convergent for most cases. However the convergent rate depends on the non linear operator $A(v)$. This method has been proved by J. H. He [6, 9].

Mittag-Leffler function

In fractional calculus it is well know the importance that have the Mittag-Leffler function [2] defined by the series

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in C,\ \Re(\alpha) > 0) \tag{18}$$

which contain as particular case the exponential function e^z and admits a first generalization given by the two parameter Mittag-Leffler function [1, p. 40] defined by

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in C,\ \Re(\alpha) > 0, Re(\beta) > 0). \tag{19}$$

2. Applications

In this section, we apply the homotopy perturbation method (HPTM) and the fractional calculus method for solving various types of partial fractional differential equations with variables coefficients.

EXAMPLE 1.

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0, \quad t > 0 \tag{20}$$

with the initial condition $u(x,0) = x_{+}^{2\alpha}$ and $0 < \alpha < 1$.

By using homotopy perturbation method presented in section 1,

$$u(x,t) = u_0 + pu_1 + p^2 u_2 + \ldots = \sum_{n=0}^{\infty} p^n u_n(x,t). \tag{21}$$

Let $L(u) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$, $N(u) = -\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}}$, $L(y_0) = \frac{\partial^{\alpha} y_0}{\partial t^{\alpha}}$ and $y_0 = x_{+}^{2\alpha}$, then Equation (20) reduces to

$$\sum_{n=0}^{\infty} p^n \frac{\partial^{\alpha} u_n(x,t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha} y_0}{\partial t^{\alpha}} + p \frac{\partial^{\alpha} y_0}{\partial t^{\alpha}} - p \left(\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \sum_{n=0}^{\infty} p^n \frac{\partial^{2\alpha} u_n(x,t)}{\partial x^{2\alpha}} \right) = 0. \tag{22}$$

Equating the coefficients of like power of p, we have

$$p^0 : \frac{\partial^{\alpha} u_0(x,t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha} y_0}{\partial t^{\alpha}}, \quad y_0 = x_{+}^{2\alpha},$$

$$p^1 : \frac{\partial^{\alpha} u_1(x,t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha} y_0}{\partial t^{\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}}, \quad u_1(x,0) = 0,$$

$$p^2 : \frac{\partial^{\alpha} u_2(x,t)}{\partial t^{\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_1(x,t)}{\partial x^{2\alpha}}, \quad u_2(x,0) = 0,$$

$$\ldots$$

$$p^n : \frac{\partial^{\alpha} u_n(x,t)}{\partial t^{\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_{n-1}(x,t)}{\partial x^{2\alpha}}, \quad u_n(x,0) = 0.$$

The following solutions are obtained from solving the above problems

$$p^0 : u_0(x,t) = y_0 = x_{+}^{2\alpha}$$

$$p^1 : \frac{\partial^{\alpha} u_1(x,t)}{\partial t^{\alpha}} = \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}}, \quad u(x,0) = 0.$$

Operating I_t^{α} (the operator with respect to time) on both side of the above expression

$$I_t^{\alpha} \left[\frac{\partial^{\alpha} u_1(x,t)}{\partial t^{\alpha}} \right] = I_t^{\alpha} \left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}} \right].$$

Here $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the Caputo or Riemann-Liouville fraction derivative operator, using equation (4) in the left side of the above expression.

$$u_1(x,t) = I_t^\alpha \left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}} \right].$$

Now using (1) in the right side of the above equation, we have

$$u_1(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \left(\frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}} \right) d\tau.$$

Using (6) in the above integral, we have

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x_+^{2\alpha} d\tau$$

$$u_1(x,t) = x_+^{2\alpha} \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Similarly

$$u_2(x,t) = t^{2\alpha} \frac{x_+^{2\alpha}}{\Gamma(2\alpha+1)}, \ldots, u_n(x,t) = t^{n\alpha} \frac{x_+^{2\alpha}}{\Gamma(n\alpha+1)}.$$

With (21) the series solution is given by

$$u(x,t) = \left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \ldots \right) x_+^{2\alpha}.$$

$$u(x,t) = x_+^{2\alpha} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = x_+^{2\alpha} E_\alpha(t^\alpha). \tag{23}$$

Note that the solution (23) of the problem (20) was obtained by J. H. He [9] by homotopy method.

EXAMPLE 2. Consider the partial fractional differential equation with variable coefficients

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} - \frac{y^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial x^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial y^{2\alpha}} = 0, \quad t > 0 \tag{24}$$

with the initial condition $u(x,y,0) = y_+^{2\alpha}$ and $0 < \alpha < 1$.

Applying the homotopy perturbation method, we get

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} - \frac{\partial^\alpha y_0}{\partial t^\alpha} + p \frac{\partial^\alpha y_0}{\partial t^\alpha} - p \left(\frac{y^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial x^{2\alpha}} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial y^{2\alpha}} \right) = 0. \tag{25}$$

Equating the coefficients of like power of p, we have

$$p^0 : \frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} - \frac{\partial^\alpha y_0}{\partial t^\alpha}, \quad y_0 = y_+^{2\alpha},$$

$$p^1 : \frac{\partial^\alpha u_1(x,y,t)}{\partial t^\alpha} + \frac{\partial^\alpha y_0}{\partial t^\alpha} - \frac{y^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha} u_0(x,y,t)}{\partial x^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha} u_0(x,y,t)}{\partial y^{2\alpha}},$$
$$u_1(x,y,0) = 0,$$

$$p^2 : \frac{\partial^\alpha u_2(x,y,t)}{\partial t^\alpha} - \frac{y^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha} u_1(x,y,t)}{\partial x^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha} u_1(x,y,t)}{\partial y^{2\alpha}},$$
$$u_2(x,y,0) = 0,$$

...

$$p^n : \frac{\partial^\alpha u_n(x,y,t)}{\partial t^\alpha} - \frac{y^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha} u_{n-1}(x,y,t)}{\partial x^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha} u_{n-1}(x,y,t)}{\partial y^{2\alpha}},$$
$$u_n(x,y,0) = 0.$$

By choosing $u_0(x,y,t) = y_0$, and using equation (1), (4) and (6) for solving the above problems, we have

$$u_1(x,y,t) = t^\alpha \frac{x^{2\alpha}}{\Gamma(\alpha+1)}, \quad u_2(x,y,t) = t^{2\alpha}\frac{y_+^{2\alpha}}{\Gamma(2\alpha+1)}, \quad u_3(x,y,t) = t^{3\alpha}\frac{x^{2\alpha}}{\Gamma(3\alpha+1)},$$
$$..., u_{2n}(x,y,t) = t^{2n\alpha}\frac{y_+^{2\alpha}}{\Gamma(2n\alpha+1)}, \quad u_{2n+1}(x,y,t) = t^{(2n+1)\alpha}\frac{x^{2\alpha}}{\Gamma((2n+1)\alpha+1)}. \tag{26}$$

The series solution is given by

$$u(x,y,t) = \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + ... + \frac{t^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} + ...\right)x^{2\alpha}$$
$$+\left(1 + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + ... + \frac{t^{2n\alpha}}{\Gamma(2n\alpha+1)} + ...\right)y_+^{2\alpha}.$$
$$= x^{2\alpha}\sum_{n=0}^{\infty}\frac{t^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} + y_+^{2\alpha}\sum_{n=0}^{\infty}\frac{t^{2n\alpha}}{\Gamma(2n\alpha+1)} \tag{27}$$
$$u(x,y,t) = (x^2 t)^\alpha E_{2\alpha,\alpha+1}(t^{2\alpha}) + y_+^{2\alpha}E_{2\alpha}(t^{2\alpha}). \tag{28}$$

Note that the solution (28) of the problem (24)obtained by J. H. He [9] by homotopy method.

EXAMPLE 3. Consider the partial fractional differential equations with variable coefficients

$$\frac{\partial^\alpha u(x,y,z,t)}{\partial t^\alpha} - (xyz)_+^{4\alpha}$$
$$-\frac{\Gamma(2\alpha+1)}{3\Gamma(4\alpha+1)}\left(x_+^{2\alpha}\theta(y)\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial x^{2\alpha}} + y^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial y^{2\alpha}} + z^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial z^{2\alpha}}\right) = 0,$$
$$t > 0 \tag{29}$$

with the initial condition $u(x,y,z,0)=0$ and $0<\alpha<1$, $\theta(y)=\left[\begin{array}{l}1, y>0\\ 0, y<0.\end{array}\right.$

Apply the homotopy perturbation method, we have

$$\frac{\partial^{\alpha}u(x,y,z,t)}{\partial t^{\alpha}}-\frac{\partial^{\alpha}y_0}{\partial t^{\alpha}}+p\frac{\partial^{\alpha}y_0}{\partial t^{\alpha}}-p(xyz)_+^{4\alpha}-\frac{\Gamma(2\alpha+1)}{3\Gamma(4\alpha+1)}$$

$$\times p\left(x_+^{2\alpha}\theta(y)\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial x^{2\alpha}}+y^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial y^{2\alpha}}+z^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial z^{2\alpha}}\right)=0. \quad (30)$$

Equating the coefficient of like power of p, we get

$$p^0:\frac{\partial^{\alpha}u_0(x,y,z,t)}{\partial t^{\alpha}}-\frac{\partial^{\alpha}y_0}{\partial t^{\alpha}},\quad y_0=0,$$

$$p^1:\frac{\partial^{\alpha}u_1(x,y,z,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha}y_0}{\partial t^{\alpha}}-(xyz)_+^{4\alpha}-\frac{\Gamma(2\alpha+1)}{3\Gamma(4\alpha+1)}$$

$$\times\left(x_+^{2\alpha}\theta(y)\frac{\partial^{2\alpha}u_0(x,y,z,t)}{\partial x}+y^{2\alpha}\frac{\partial^{2\alpha}u_0(x,y,z,t)}{\partial y^{2\alpha}}+z^{2\alpha}\frac{\partial^{2\alpha}u_0(x,y,z,t)}{\partial z^{2\alpha}}\right)=0,$$

$$u_1(x,y,z,0)=0,$$

$$p^2:\frac{\partial^{\alpha}u_2(x,y,z,t)}{\partial t^{\alpha}}-\frac{\Gamma(2\alpha+1)}{3\Gamma(4\alpha+1)}$$

$$\times\left(x_+^{2\alpha}\theta(y)\frac{\partial^{2\alpha}u_1(x,y,z,t)}{\partial x^{2\alpha}}+y^{2\alpha}\frac{\partial^{2\alpha}u_1(x,y,z,t)}{\partial y^{2\alpha}}+z^{2\alpha}\frac{\partial^{2\alpha}u_1(x,y,z,t)}{\partial z^{2\alpha}}\right)=0,$$

$$u_2(x,y,z,0)=0,$$

$$\ldots$$

$$p^n:\frac{\partial^{\alpha}u_n(x,y,z,t)}{\partial t^{\alpha}}-\frac{\Gamma(2\alpha+1)}{3\Gamma(4\alpha+1)}$$

$$\times\left(x_+^{2\alpha}\theta(y)\frac{\partial^{2\alpha}u_{n-1}(x,y,z,t)}{\partial x^{2\alpha}}+y^{2\alpha}\frac{\partial^{2\alpha}u_{n-1}(x,y,z,t)}{\partial y^{2\alpha}}+z^{2\alpha}\frac{\partial^{2\alpha}u_{n-1}(x,y,z,t)}{\partial z^{2\alpha}}\right)=0,$$

$$u_n(x,y,z,0)=0.$$

Choosing $u_0(x,y,z,t)=y_0$, and solving above problems by using equations (1), (4) and (6)

$$u_1(x,y,z,t)=(x_+^{4\alpha}y_+^{4\alpha}z_+^{4\alpha})\frac{t^{\alpha}}{\Gamma(\alpha+1)},\quad u_2(x,y,z,t)=(x_+^{4\alpha}y_+^{4\alpha}z_+^{4\alpha})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,y,z,t)=(x_+^{4\alpha}y_+^{4\alpha}z_+^{4\alpha})\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\ldots,\quad u_n(x,y,z,t)=(x_+^{4\alpha}y_+^{4\alpha}z_+^{4\alpha})\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (31)$$

The series solution is given by

$$u(x,y,z,t) = \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \ldots\right) x_+^{4\alpha} y_+^{4\alpha} z_+^{4\alpha}$$
$$= x_+^{4\alpha} y_+^{4\alpha} z_+^{4\alpha} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - 1$$

$$u(x,y,z,t) = x_+^{4\alpha} y_+^{4\alpha} z_+^{4\alpha} E_{\alpha}(t^{\alpha}) - 1 \tag{32}$$

Note that the solution (32) of the problem (29) was obtained by J. H. He [9] by homotopy method.

EXAMPLE 4. Consider the partial fractional differential equations with variable coefficients form

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0, \quad t > 0 \tag{33}$$

with the initial conditions $u(x,0) = 0$ and $\frac{\partial u(x,o)}{\partial t} = x_+^{2\alpha}$ and $\frac{1}{2} < \alpha < 1$.

By applying the homotopy perturbation method, we have

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}} + p\frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}} - p\left(\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}}\right) = 0. \tag{34}$$

Equating the coefficient of various power of p, we get

$$p^0 : \frac{\partial^{2\alpha} u_0(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}}, \quad y_0 = t x_+^{2\alpha}, \quad \frac{\partial u_0(x,0)}{\partial t} = x_+^{2\alpha},$$

$$p^1 : \frac{\partial^{2\alpha} u_1(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}}, \quad u_1(x,0) = 0, \quad \frac{\partial u_1(x,0)}{\partial t} = 0,$$

$$p^2 : \frac{\partial^{2\alpha} u_2(x,t)}{\partial t^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_1(x,t)}{\partial x^{2\alpha}}, \quad u_2(x,0) = 0, \quad \frac{\partial u_2(x,0)}{\partial t} = 0,$$
$$\ldots$$
$$p^n : \frac{\partial^{2\alpha} u_n(x,t)}{\partial t^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha} u_{n-1}(x,t)}{\partial x^{2\alpha}}, \quad u_n(x,0) = 0, \quad \frac{\partial u_n(x,0)}{\partial t} = 0.$$

By choosing $u_0(x,t) = y_0$, and using equation (1), (4) and (6) for solving problems, we obtain

$$u_1(x,t) = t^{2\alpha+1} \frac{x_+^{2\alpha}}{\Gamma(2\alpha+2)}, \quad u_2(x,t) = t^{4\alpha+1} \frac{x_+^{2\alpha}}{\Gamma(4\alpha+2)}, \ldots,$$
$$u_n(x,t) = t^{(2n\alpha+1)} \frac{x_+^{2\alpha}}{\Gamma(2n\alpha+2)}. \tag{35}$$

The series solution is given by

$$u(x,t) = x_+^{2\alpha} \sum_{n=0}^{\infty} \frac{t^{2n\alpha+1}}{\Gamma(2n\alpha+2)}. \tag{36}$$

$$u(x,t) = x_+^{2\alpha} t E_{2\alpha,2}(t^{2\alpha}). \tag{37}$$

The solution (37) of the problem (33) was obtained by J. H. He [9] by homotopy method.

EXAMPLE 5. Consider the partial fractional differential equation with variable coefficients form

$$\frac{\partial^{2\alpha} u(x,y,t)}{\partial t^{2\alpha}} - \frac{x^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial x^{2\alpha}} - \frac{y^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial y^{2\alpha}} = 0, \quad t > 0 \tag{38}$$

with the initial conditions $u(x,y,0) = x_+^{4\alpha}$ and $\frac{\partial u(x,y,0)}{\partial t} = y_+^{4\alpha}$, $\frac{1}{2} < \alpha < 1$.

Apply the homotopy perturbation method we have

$$\frac{\partial^{2\alpha} u(x,y,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}} + p\frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}}$$
$$-p\left(\frac{x^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial x^{2\alpha}} + \frac{y^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u(x,y,t)}{\partial y^{2\alpha}}\right) = 0. \tag{39}$$

Equating the coefficient of like power of p, we have

$$p^0 : \frac{\partial^{2\alpha} u_0(x,y,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}}, \quad y_0 = x_+^{4\alpha} + t y_+^{4\alpha}, \quad \frac{\partial u_0(x,y,0)}{\partial t} = y_+^{4\alpha},$$

$$p^1 : \frac{\partial^{2\alpha} u_1(x,y,t)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha} y_0}{\partial t^{2\alpha}} - \frac{x^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u_0(x,y,t)}{\partial x^{2\alpha}} - \frac{y^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u_0(x,y,t)}{\partial y^{2\alpha}} = 0,$$

$$u_1(x,y,0) = 0, \quad \frac{\partial u_1(x,y,0)}{\partial t} = 0,$$

$$p^2 : \frac{\partial^{2\alpha} u_2(x,y,t)}{\partial t^{2\alpha}} - \frac{x^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u_1(x,y,t)}{\partial x^{2\alpha}} - \frac{y^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u_1(x,y,t)}{\partial y^{2\alpha}} = 0,$$

$$u_2(x,y,0) = 0, \quad \frac{\partial u_2(x,y,0)}{\partial t} = 0,$$

$$\dots$$

$$p^n : \frac{\partial^{2\alpha} u_n(x,y,t)}{\partial t^{2\alpha}} - \frac{x^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u_{n-1}(x,y,t)}{\partial x^{2\alpha}} - \frac{y^{2\alpha}\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \frac{\partial^{2\alpha} u_{n-1}(x,y,t)}{\partial y^{2\alpha}},$$

$$u_n(x,y,0) = 0, \quad \frac{\partial u_n(x,y,0)}{\partial t} = 0.$$

By choosing $u_0(x,y,t) = y_0$, and solving above problems with the help of equations (1), (4) and (6), we get

$$\begin{aligned} u_1(x,y,t) &= t^{2\alpha}\frac{x_+^{4\alpha}}{\Gamma(2\alpha+1)} + t^{2\alpha+1}\frac{y_+^{4\alpha}}{\Gamma(2\alpha+2)}, \\ u_2(x,y,t) &= t^{4\alpha}\frac{x_+^{4\alpha}}{\Gamma(4\alpha+1)} + t^{4\alpha+1}\frac{y_+^{4\alpha}}{\Gamma(4\alpha+2)}, \\ u_3(x,y,t) &= t^{6\alpha}\frac{x_+^{4\alpha}}{\Gamma(6\alpha+1)} + t^{6\alpha+1}\frac{y_+^{4\alpha}}{\Gamma(6\alpha+2)}, \ldots, \\ u_{2n}(x,y,t) &= t^{2n\alpha}\frac{x_+^{4\alpha}}{\Gamma(2n\alpha+1)} + t^{(2n\alpha+1)}\frac{y_+^{4\alpha}}{\Gamma(2n\alpha+2)}. \end{aligned} \tag{40}$$

The series solution is given by

$$u(x,y,t) = x_+^{4\alpha}\sum_{n=0}^{\infty}\frac{(t^{2\alpha})^n}{\Gamma(2n\alpha+1)} + y_+^{4\alpha}t\sum_{n=0}^{\infty}\frac{(t^{2\alpha})^n}{\Gamma(2n\alpha+2)}$$

$$u(x,y,t) = x_+^{4\alpha}E_{2\alpha}(t^{2\alpha}) + (y_+^{4\alpha}t)E_{2\alpha,2}(t^{2\alpha}). \tag{41}$$

Note that the solution(41) of the problem (38) was obtained by J. H. He [9] by homotopy method.

EXAMPLE 6. Consider the partial fractional differential equation with variable coefficients

$$\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial t^{2\alpha}} - (x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha}) - \frac{1}{\Gamma(2\alpha+1)} \times \left(x^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial x^{2\alpha}} + y^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial y^{2\alpha}} + z^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial z^{2\alpha}}\right) = 0, \quad t>0 \tag{42}$$

with the initial conditions $u(x,y,z,0) = 0$, $\frac{\partial u(x,y,z,0)}{\partial t} = x_+^{2\alpha} + y_+^{2\alpha} - z_+^{2\alpha}$ and $\frac{1}{2} < \alpha < 1$.

Apply the homotopy perturbation method, we have

$$\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}y_0}{\partial t^{2\alpha}} + p\frac{\partial^{2\alpha}y_0}{\partial t^{2\alpha}} - (x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha}) - \frac{1}{\Gamma(2\alpha+1)}p\left(x^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial x^{2\alpha}} + y^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial y^{2\alpha}} + z^{2\alpha}\frac{\partial^{2\alpha}u(x,y,z,t)}{\partial z^{2\alpha}}\right) = 0. \tag{43}$$

Equating the coefficients of like power of p, we have

$$p^0: \frac{\partial^{2\alpha}u_0(x,y,z,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}y_0}{\partial t^{2\alpha}}, \quad y_0 = t(x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha}), \quad \frac{\partial u_0(x,y,z,0)}{\partial t} = x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha},$$

$$p^1: \frac{\partial^{2\alpha}u_1(x,y,z,t)}{\partial t^{2\alpha}} + \frac{\partial^{2\alpha}y_0}{\partial t^{2\alpha}} - (x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha}) - \frac{1}{\Gamma(2\alpha+1)} \times \left(x^{2\alpha}\frac{\partial^{2\alpha}u_0(x,y,z,t)}{\partial x^{2\alpha}} + y^{2\alpha}\frac{\partial^{2\alpha}u_0(x,y,z,t)}{\partial y^{2\alpha}} + z^{2\alpha}\frac{\partial^{2\alpha}u_0(x,y,z,t)}{\partial z^{2\alpha}}\right),$$

$$u_1(x,y,z,0)=0, \quad \frac{\partial u_1(x,y,z,0)}{\partial t}=0,$$

$$p^2: \frac{\partial^{2\alpha} u_2(x,y,z,t)}{\partial t^{2\alpha}} - \frac{1}{\Gamma(2\alpha+1)}\left(x^{2\alpha}\frac{\partial^{2\alpha} u_1(x,y,z,t)}{\partial x^{2\alpha}} + y^{2\alpha}\frac{\partial^{2\alpha} u_1(x,y,z,t)}{\partial y^{2\alpha}}\right.$$

$$\left. + z^{2\alpha}\frac{\partial^{2\alpha} u_1(x,y,z,t)}{\partial z^{2\alpha}}\right)=0, \qquad u_2(x,y,z,0)=0, \quad \frac{\partial u_2(x,y,z,0)}{\partial t}=0,$$

$$\ldots$$

$$p^n: \frac{\partial^{2\alpha} u_n(x,y,z,t)}{\partial t^{2\alpha}} - \frac{1}{\Gamma(2\alpha+1)}$$

$$\times\left(x^{2\alpha}\frac{\partial^{2\alpha} u_{n-1}(x,y,z,t)}{\partial x^{2\alpha}} + y^{2\alpha}\frac{\partial^{2\alpha} u_{n-1}(x,y,z,t)}{\partial y^{2\alpha}} + z^{2\alpha}\frac{\partial^{2\alpha} u_{n-1}(x,y,z,t)}{\partial z^{2\alpha}}\right)=0,$$

$$u_n(x,y,z,0)=0, \quad \frac{\partial u_n(x,y,z,0)}{\partial t}=0.$$

By choosing $u_0(x,y,z,t)=y_0=t\left(x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha}\right)$, and using equations (1), (4) and (6), we obtain

$$u_1(x,y,z,t)=(x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+(x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha})\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$$u_2(x,y,z,t)=(x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha})\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+(x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha})\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)},$$

$$u_3(x,y,z,t)=(x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha})\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}+(x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha})\frac{t^{6\alpha+1}}{\Gamma(6\alpha+2)},\ldots,$$

$$u_n(x,y,z,t)=(x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha})\frac{t^{2n\alpha}}{\Gamma(2n\alpha+1)}+(x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha})\frac{t^{2n\alpha+1}}{\Gamma(2n\alpha+2)}.$$

$$u(x,y,z,t)=\left(x_+^{2\alpha}+y_+^{2\alpha}-z_+^{2\alpha}\right)tE_{2\alpha,2}(t^{2\alpha})+\left(x_+^{2\alpha}+y_+^{2\alpha}+z_+^{2\alpha}\right)\left[E_{2\alpha}(t^{2\alpha})-1\right]. \quad (44)$$

Note that the solution(44) of the problem (42) was obtained by J. H. He [9] by homotopy method.

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Dharmendra Kumar Singh
Department of Mathematics
University Institute of Engineering and Technology
Chhatrapati Shahu Ji Maharaj University
Kanpur-208024, India
e-mail: drdksinghabp@gmail.com