A PROBLEM WITH AN INTEGRAL BOUNDARY CONDITION FOR A TIME FRACTIONAL DIFFUSION EQUATION AND AN INVERSE PROBLEM

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Abstract. For a linear inhomogeneous time fractional diffusion equation on bounded cylindrical domain the problem with an integral boundary condition is studied. The inverse problem for the restoration of the whole right-hand side of the equation is also studied. The conditions of the solvability and the unique solvability of these problems are founded.

1. Introduction

The conditions of classical solvability of a time fractional Cauchy problem and the first boundary value problem to a time fractional diffusion equation were obtained, for example, in [2, 3, 4, 10, 11, 14]. There were proved the existence and uniqueness theorems and the representation of classical solution in terms of the Green vector-function. The inverse problems to such equations ([1, 6, 12, 13, 15] and references therein) have important practical applications. Some studies in inverse problems for restoration of a right-hand side of the equation, or a minor coefficient in the equation, or the initial data of a solution (see, for instance, [1, 8]) use the integral type over-determination conditions.

In the present paper, for a time fractional diffusion equation we study the solvability of a problem with an integral boundary condition. Namely, we study the boundary value problem

\[ D^\beta_t u - \Delta u = F_0(x,t), \quad (x,t) \in \Omega_0 \times (0,T), \]
\[ \int_{\Omega_1} K(x,t,z)u(z,t)dz = F(x,t), \quad (x,t) \in \Omega_1 \times [0,T], \]
\[ u(x,0) = F_2(x), \quad x \in \Omega_0, \]

with the Caputo fractional derivative (or Caputo-Djrbashian fractional derivative) [2]

\[ D^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{v'(\tau)}{(t-\tau)^\beta} d\tau = \frac{1}{\Gamma(1-\beta)} \left[ \frac{d}{dt} \int_0^t \frac{v(\tau)}{(t-\tau)^\beta} d\tau - \frac{v(0)}{t^\beta} \right]. \]


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of order $\beta \in (0, 1)$, where $\Omega_0$ is a boundary domain in $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary $\Omega_1$, $K$, $F_0$, $F$, $F_2$ are given functions.

We can consider this problem as the inverse problem for the determination a pair of functions: a classical solution $u$ of the second boundary value problem

$$
\frac{\partial u(x,t)}{\partial \nu(x)} = F_1(x,t), \quad (x,t) \in \Omega_1 \times [0,T], \quad u(x,0) = F_2(x), \quad x \in \overline{\Omega}_0, \quad (4)
$$

for the equation (1) and its unknown boundary values $F_1$ under integral type over-determination condition (2). Here $\nu(x)$ is the unite vector of the interior normal to the surface $\Omega_1$ in the point $x \in \Omega_1$. So, we can use the Green function’s method in study the solvability of these problems. Note that in [13] the unknown boundary condition was founded by using the another over-determination condition.

We also study the inverse problem of finding the solution of the problem (1), (3), (4) and the right-hand side $F_0(x,t)$ of the equation (1) under the over-determination condition

$$
\int_{\Omega_i} R(x,t,z)u(z,t)dz = F(x,t), \quad (x,t) \in \Omega_0 \times [0,T]
$$

with given functions $R$, $F_1$, $F_2$, $F$. Note that the inverse problems for the determination of the right-hand side $F_0 = F_0(x)$ or $F_0 = F_0(t)$ of a such kind of equations were studied (see, for example, [1, 15]).

2. Definitions and auxiliary results

Assume that $\Omega_1$ is the surface of class $C^{1+\gamma}$, $\gamma \in (0, 1)$, $Q_i = \Omega_i \times (0,T]$, $i = 0, 1$, $Q_2 = \Omega_0$, $C(Q_0)$ ($C(\overline{Q}_0)$) is the space of continuous functions on $Q_0$ ($\overline{Q}_0$, respectively), $C^\gamma(\overline{Q}_i)$ is the space of Hölder continuous functions on $\overline{Q}_i$, $C^\gamma(\overline{Q}_i)$ is the space of Hölder continuous functions in space variables $x \in \overline{Q}_i$ for all $t \in [0,T]$ and jointly continuous in $(x,t) \in \overline{Q}_i$, $i = 0, 1$, $C_{2,\beta}(Q_0) = \{ v \in C(Q_0) \mid \Delta v, D_v v \in C(Q_0) \}$, $C_{2,\beta}(\overline{Q}_0) = C_{2,\beta}(Q_0) \cap C(\overline{Q}_0)$, $C_{2,\beta}^1(\overline{Q}_0) = \{ v \in C_{2,\beta}(\overline{Q}_0) \mid \partial v/\partial v \in C(\overline{Q}_1) \}$, $\mathscr{D}(\mathbb{R})$ is the space of indefinitely differentiable functions compactly supported in $\mathbb{R}$, $\mathscr{D}'(\mathbb{R})$ is the space of linear continuous functionals (distributions) over $\mathscr{D}(\mathbb{R})$ and $\mathscr{D}_+(\mathbb{R}) = \{ f \in \mathscr{D}'(\mathbb{R}) : f = 0 \text{ for } t < 0 \}$.

We denote by $f*g$ the convolution of the distributions $f$ and $g$, use the function $f_\lambda \in \mathscr{D}_+(\mathbb{R})$:

$$
f_\lambda(t) = \frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)} \quad \text{for } \lambda > 0 \quad \text{and} \quad f_{\lambda}(t) = f_{1+\lambda}(t) \text{ for } \lambda \leq 0,
$$

where $\Gamma(z)$ is the Gamma-function, $\theta(t)$ is the Heaviside function and derivative understood as a derivative in $\mathscr{D}'(\mathbb{R})$. Note that

$$
f_\lambda * f_\mu = f_{\lambda+\mu}.
$$
Recall that the Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta > 0$ is defined as
\[ v^{(\beta)}(t) = f^{(-\beta)}(t) * v(t). \]

If $D^\beta v$ exists then
\[ D^\beta v(t) = v^{(\beta)}(t) - f_{1-\beta}(t)v(0) \quad \text{for} \quad \beta \in (0,1). \]

Let the following assumptions hold:
(A) $F_0 \in C^\gamma(\bar{Q}_0)$, $F_2 \in C^\gamma(\bar{Q}_0)$, supp$F_2 \subset \Omega_0$, $F \in C^\gamma(\bar{Q}_1)$,
(B) $K(x,t) ((x,t) \in \bar{Q}_1, y \in \bar{\Omega}_1)$ is a jointly continuous in all variables function, Hölder continuous in $x,y \in \bar{\Omega}_1$ for all $t \in [0,T]$.

**Definition 1.** A function $u \in C_{2,\beta}(\bar{Q}_0)$ satisfying the equation (1) on $Q_0$ and the conditions (2), (3) is called a solution of the problem (1)–(3).

Definition 1 and the assumption (A) about $F_2$ imply the compatibility condition
\[ F(x,0) = 0, \quad x \in \bar{\Omega}_1. \quad (5) \]

**Definition 2.** A vector-function $(G_0(x,t,y,\tau), G_1(x,t,y,\tau), G_2(x,t,y))$ such that under rather regular $F_0, F_1, F_2$ the function
\[ u(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau)F_0(y,\tau)dy + \int_0^t d\tau \int_{\Omega_1} G_1(x,t,y,\tau)F_1(y,\tau)dy + \int_{\Omega_0} G_2(x,t,y)F_2(y)dy, \quad (x,t) \in Q_0 \quad (6) \]
is a classical (from $C^1_{2,\beta}(Q_0)$) solution of the problem (1), (4) is called a Green vector-function of this second boundary-value problem.

The existence of a unique Green vector-function of the problem (1), (4) may be prove as in [9] for the first boundary value problem.

The results of [4, 14] and the Levi method (see [4, 5]) imply the following estimates in the case $n \geq 3$:
\[ G_j(x,t,y,\tau) \leq \frac{C}{(t-\tau)|x-y|^{n-2}}, \quad j = 0, 1, \]
\[ G_2(x,t,y) \leq \frac{C}{t^\beta|x-y|^{n-2}}, \quad \beta > 0, \quad |x-y|^2 < 4(t-\tau)^\beta, \]
\[ G_j(x,t,y,\tau) \leq \frac{C(t-\tau)^{-\beta}}{|x-y|^n} \cdot \left( \frac{|x-y|^2}{4(t-\tau)^\beta} \right)^{1+\frac{n}{2(\beta-1)}} e^{-c \left( \frac{|x-y|^2}{4(t-\tau)^\beta} \right)^{\frac{1}{\beta-1}}} \quad \text{for} \quad j = 0, 1, \]
\[ G_2(x,t,y) \leq \frac{C}{|x-y|^n} \cdot \left( \frac{|x-y|^2}{4t^\beta} \right)^{\frac{n}{2(\beta-1)}} e^{-c \left( \frac{|x-y|^2}{4t^\beta} \right)^{\frac{1}{\beta-1}}} \quad \text{for} \quad |x-y|^2 > 4(t-\tau)^\beta \]

where $C, c$ are positive constants, and similar estimates in the case $n = 2$. Note that the character features in $t$ is accurate. The Green vector-function owns Hölder properties.
THEOREM 1. Assume that \( F_0 \in C^\nu(\Omega_0), \) \( F_1 \in C^\nu(\Omega_1), \) \( F_2 \in C^\nu(\Omega_0), \) supp\( F_2 \subset \Omega_0. \) Then there exists a unique solution \( u \in C^1_{2, \beta}(\Omega_0) \) of the second boundary value problem (1), (4), and it is defined by the formula (6).

The theorem is proving as the corresponding result in [5, 5.3].

3. The existence and uniqueness theorems

Substituting the right-hand side of (6) in the condition (2) we obtain

\[
\int_0^t d\tau \int_{\Omega_1} K(x, t, z) dz \int_{\Omega_0} G_0(z, t, y) F_0(y, \tau) dy
\]

\[
+ \int_0^t d\tau \int_{\Omega_1} K(x, t, z) dz \int_{\Omega_1} G_1(z, t, y, \tau) F_1(y, \tau) dy
\]

\[
+ \int_{\Omega_1} K(x, t, z) dz \int_{\Omega_0} G_1(z, t, y) F_2(y) dy = F(x, t), \quad (x, t) \in \Omega_1,
\]

that is

\[
\sum_{j=0}^1 \int_0^t d\tau \int_{\Omega_j} K_j(x, t, y, \tau) F_j(y, \tau) dy + \int_{\Omega_0} K_2(x, y, t) F_2(y) dy = F(x, t), \quad (x, t) \in \Omega_1,
\]

where

\[
K_j(x, t, y, \tau) = \int_{\Omega_1} K(x, t, z) G_j(z, t, y, \tau) dz, \quad j = 0, 1,
\]

\[
K_2(x, t, y) = \int_{\Omega_1} K(x, t, z) G_2(z, t, y) dz.
\]

Using the above estimates of components of the Green vector-function we find

\[
|K_j(x, t, y, \tau)|
\]

\[
\leq \int_{\{z \in \Omega_1 : |y-z| < 2(t-\tau)^{\beta/2}\}} |K(x, t, z)G_j(z, t, y, \tau)| dx
\]

\[
+ \int_{\{z \in \Omega_1 : |y-z| > 2(t-\tau)^{\beta/2}\}} |K(x, t, z)G_j(z, t, y, \tau)| dx
\]

\[
\leq C_1 \left[ \int_{\{z \in \Omega_1 : |y-z| < 2(t-\tau)^{\beta/2}\}} \frac{dz}{(t-\tau)|y-z|^{n-2}} \right]
\]

\[
+ \int_{\{z \in \Omega_1 : |y-z| > 2(t-\tau)^{\beta/2}\}} \frac{(t-\tau)^{\beta-1}}{|z-y|^n} \left( \frac{|z-y|^2}{4(t-\tau)^\beta} \right) e^{-c \frac{(|z-y|^2}{4(t-\tau)^\beta}} \right) \frac{1}{2} \frac{1}{\frac{(z-y)^2}{4(t-\tau)^\beta}} dz
\]
\[ \leq C_2 \left[ \frac{1}{t-\tau} \int_0^{2(t-\tau)^{\beta/2}} dr + (t-\tau)^{-1} \int_0^{\frac{\text{diam} \Omega_1}{2(t-\tau)^{\beta/2}}} r^{\frac{n-\beta}{2(2-\beta)}} e^{-c \left( \frac{r^2}{4(t-\tau)^{\beta}} \right)} \, dr \right] \]

\[ \leq C_3 (t-\tau)^{\beta/2} \left[ 1 + \int_1^{\infty} \eta^{\frac{n-\beta}{2}} e^{-c \eta} \, d\eta \right] \leq \hat{k} (t-\tau)^{\beta/2} - 1 \]

\[ = \hat{k} \Gamma \left( \frac{\beta}{2} \right) f_{\frac{\beta}{2}} (t-\tau), \quad x \in \Omega_1, \ y \in \Omega_j, \ 0 \leq \tau < t \leq T, \ j = 0, 1, \]

and similarly,

\[ |K_2(x,t,y)| \leq C_4 \left[ \int_{\{z \in \Omega_1 : |y-z| < 2\beta/2\}} \frac{dz}{t^{\beta}|y-z|^{n-2}} + \int_{\{z \in \Omega_1 : |y-z| > 2\beta/2\}} \frac{1}{|z-y|^n} \left( \frac{|z-y|^2}{4t^\beta} \right)^{\frac{n-\beta}{2(2-\beta)}} e^{-c \left( \frac{|z-y|^2}{4t^\beta} \right)} \frac{1}{2^\beta} \, dz \right] \]

\[ \leq C_5 \left[ \frac{1}{t^\beta} \int_0^{2\beta/2} dr + t^{-\frac{3\beta}{2(2-\beta)}} \int_0^{2\beta/2} r^{\frac{n-\beta}{2(2-\beta)}} e^{-c \left( \frac{r^2}{4t^\beta} \right)} \frac{1}{2^\beta} \, dr \right] \]

\[ \leq \hat{k} t^{\frac{-\beta}{2}} = \hat{k} \Gamma \left( 1 - \frac{\beta}{2} \right) f_{\frac{\beta}{2}} (t), \quad x \in \Omega_1, \ y \in \Omega_0, \ t \in (0,T]. \]

Hereinafter, \( C_i, \hat{k}, \hat{k}_i, \ i \in \mathbb{Z}_+ \) are positive constants.

Note that

\[ f_{\frac{\beta}{2}} (t) \ast \int_0^t K_j(x,t,y,\tau) F_j(y,\tau) \, d\tau = \int_0^t f_{\frac{\beta}{2}} (t-s) ds \int_0^s K_j(x,s,y,\tau) F_j(y,\tau) \, d\tau \]

\[ = \int_0^t \left( \int_0^t f_{\frac{\beta}{2}} (t-s) K_j(x,s,y,\tau) \, d\tau \right) F_j(y,\tau) \, d\tau \]

\[ = \int_0^t \left( \int_0^{t-\tau} f_{\frac{\beta}{2}} (s) K_j(x,t-s,y,\tau) \, d\tau \right) F_j(y,\tau) \, d\tau \]

\[ = \int_0^t \left( f_{\frac{\beta}{2}} (s) \ast K_j(x,s,y,\tau) \right) \bigg|_{s=t-\tau} F_j(y,\tau) \, d\tau \]

\[ = \int_0^t f_{\frac{\beta}{2}} (t-\tau) \ast K_j(x,t-\tau,y,\tau) F_j(y,\tau) \, d\tau. \]

Denote

\[ \mathcal{R}_j(x,t,y,\tau) = f_{\frac{\beta}{2}} (t-\tau) \ast K_j(x,t-\tau,y,\tau), \quad j = 0, 1. \]
We have
\[ |\mathcal{K}(x,t,y)| \leq \hat{k} \Gamma\left(\frac{\beta}{2}\right) f_{1-\frac{\beta}{\tau}}(t-\tau) \ast f_{\frac{\beta}{\tau}}(t-\tau) = \hat{k} \Gamma\left(\frac{\beta}{2}\right) \left( f_{1-\frac{\beta}{\tau}} \ast f_{\frac{\beta}{\tau}} \right)(t-\tau) = \hat{k} \Gamma\left(\frac{\beta}{2}\right) f_1(t-\tau) = \hat{k}_1 \theta(t-\tau). \]

Since the character features in the variable \( t \) is accurate then
\[ \mathcal{K}(x,t,y,\tau) \neq 0, \ (x,t) \in Q_1, \ (y,\tau) \in Q_j, \ j = 0, 1 \]
and are the continuous functions. Similarly, taking the equality
\[ f_{1-\frac{\beta}{\tau}}(t) \ast f_{1-\frac{\beta}{\tau}}(t) = f_{2-\beta}(t) \]
into account, we obtain that the function
\[ \mathcal{R}_2(x,t,y) := f_{1-\frac{\beta}{\tau}}(t) \ast K_2(x,t,y) \]
is continuous on \( Q_1 \times \Omega_0 \) and has the estimate
\[ |\mathcal{R}_2(x,t,y)| \leq \hat{k}_2 t^{1-\beta}, \ (x,t) \in Q_1, \ y \in \Omega_0. \]

Thus, from (7) we obtain the linear integral Volterra equation of the first type
\[ \int_0^t d\tau \int_{Q_1} \mathcal{R}_1(x,t,y,\tau) F_1(y,\tau) dy = h(x,t), \quad (x,t) \in Q_1 \tag{8} \]
relatively unknown \( F_1 \) where
\[ h(x,t) = f_{1-\frac{\beta}{\tau}}(t) \ast F(x,t) - \int_0^t d\tau \int_{\Omega_0} \mathcal{R}_0(x,t,y,\tau) F_0(y,\tau) dy \]
\[ - \int_{\Omega_0} \mathcal{R}_2(x,t,y) F_2(y) dy, \quad (x,t) \in Q_1. \]

This integral equation has the jointly continuous kernel \( \mathcal{R}_1(x,t,y,\tau) \), Hölder continuous in \( x,y \in \Omega_1 \) for all \( t, \tau \in [0,T] \), and \( h \in C^\gamma(Q_1), \ h(x,0) = 0. \)

Conversely, if \( F_1 \in C^\gamma(Q_1) \) is a solution of the equation (8), which is equivalent to the equation (7), then by Theorem 1 the function (6) is a solution (from \( C^1_{2,\beta}(Q_0) \)) of the problem (1), (4) and satisfies the condition (2).

We obtain the following result.

**Theorem 2.** Assume that (A), (B) and (5) hold, there exists a solution \( F_1 \in C^\gamma(Q_1) \) of the equation (8). Then there exists a solution \( u \in C^1_{2,\beta}(Q_0) \) of the problem (1)–(3). It is defined by (6).
THEOREM 3. At terms of uniqueness of a solution of the equation (8) a solution $u \in C_{2, \beta}^1(Q_0)$ of the problem (1)–(3) is unique.

Proof. Take two solutions $u_1, u_2 \in C_{2, \beta}^1(Q_0)$ of the problem (1)–(3). Putting $u = u_1 - u_2$ we obtain

$$D_\beta^t u = \Delta u, \quad (x, t) \in Q_0,$$

$$\int_{\Omega_1} K(x, t, y) u(y, t) dy = 0, \quad (x, t) \in \tilde{Q}_1,$$

$$u(x, 0) = 0, \quad x \in \tilde{\Omega}_0.$$  \hspace{1cm} (9)

By Theorem 1, for the solution $u \in C_{2, \beta}^1(Q_0)$ of the second boundary value problem

$$D_\beta^t u = \Delta u, \quad (x, t) \in Q_0,$$

$$\frac{\partial u(x, t)}{\partial \nu(x)} = F_1(x, t), \quad (x, t) \in \tilde{Q}_1, \quad u(x, 0) = 0, \quad x \in \tilde{\Omega}_0$$

with some unknown $F_1 \in C'(\tilde{Q}_1)$ we have

$$u(x, t) = \int_0^t d\tau \int_{\Omega_1} G_1(x, t, y, \tau) F_1(y, \tau) dy, \quad (x, t) \in \tilde{\Omega}_0.$$ \hspace{1cm} (10)

As it has been shown before, using (9), we obtain the first type linear Volterra integral equation

$$\int_0^t d\tau \int_{\Omega_1} \mathcal{R}_1(x, t, y, \tau) F_1(y, \tau) dy = 0, \quad (x, t) \in \tilde{Q}_1,$$

relatively unknown $F_1$. By Theorem’s assumption,

$$F_1(y, \tau) = 0, \quad (y, \tau) \in \tilde{Q}_1.$$ \hspace{1cm} (11)

Then (10) implies that $u = 0$ on $\tilde{\Omega}_0$. \hfill $\square$

4. The restoration of a whole right-hand side of equation

We study the inverse problem

$$\frac{\partial u(x, t)}{\partial \nu(x)} = F_1(x, t), \quad (x, t) \in \tilde{\Omega}_1,$$ \hspace{1cm} (12)

$$u(x, 0) = F_2(x), \quad x \in \tilde{\Omega}_0,$$ \hspace{1cm} (13)

$$\int_{\Omega_1} R(x, t, z) u(z, t) dz = F(x, t), \quad (x, t) \in \tilde{\Omega}_0$$ \hspace{1cm} (14)
for the equation (1) under the following assumption

(C) $F_1 \in C^\gamma(\overline{Q}_1), F_2 \in C^\gamma(\overline{Q}_0), \text{ supp} F_2 \subset \Omega_0, F \in C^\gamma(\overline{Q}_0), R(x,t,y) \ (x,t) \in \overline{Q}_0, y \in \overline{Q}_1$ is a a jointly continuous in all variables function, Hölder continuous in $x \in \overline{Q}_0, y \in \overline{Q}_1$ for all $t \in [0,T]$.

This problem consists in finding a pair of functions

$$(u,F_0) \in \mathcal{M}(Q_0) = \mathcal{M} := C_{2,\beta}^1(Q_0) \times C^\gamma(\overline{Q}_0)$$

satisfying the equation (1) and the above conditions (12)–(14).

The problem’s definition and the assumption (C) about $F_2$ imply the compatibility condition

$$F(x,0) = 0, \ x \in \overline{Q}_0.$$  \hspace{1cm} (15)

**THEOREM 4.** Assume that (C) and (15) hold, there exists a solution $F_0 \in C^\gamma(\overline{Q}_0)$ of the integral equation

$$\int_0^t d\tau \int_{\Omega_0} R_0(x,t,y,\tau) F_0(y,\tau) dy = r_0(x,t), \ (x,t) \in \overline{Q}_0,$$  \hspace{1cm} (16)

with

$$r_0(x,t) = \int_{\overline{Q}_1} f_1(x,t) * F(x,t) - \int_0^t d\tau \int_{\Omega_1} R_1(x,t,y,\tau) F_1(y,\tau) dy - \int_{\Omega_0} R_2(x,t,y) F_2(y) dy,$$

$$R_j(x,t,y,\tau) = \int_{\Omega_1} R(x,t-\tau,z) G_j(z,t-\tau,y,\tau) dz, \ (y,\tau) \in Q_j, \ j = 0,1,$$

$$R_2(x,t,y) = \int_{\Omega_1} R(x,t,z) G_2(z,t,y) dz, \ y \in \Omega_0, \ (x,t) \in \overline{Q}_0.$$

Then there exists a solution $(u,F_0) \in \mathcal{M}(Q_0)$ of the problem (1), (12)–(14) where $u$ is defined by (6). At terms of uniqueness of a solution of the equation (16) a solution $(u,F_0) \in \mathcal{M}(Q_0)$ of the problem (1), (12)–(14) is unique.

**Proof.** Substituting the right-hand side of (6) in the condition (14) and using the previous reasoning we obtain the linear integral Volterra equation (16) of the first type relatively unknown $F_0$. This integral equation has a jointly continuous kernel $R_0(x,t,y,\tau)$, Hölder continuous in $x,y \in \overline{Q}_0$ for all $t,\tau \in [0,T]$, $r_0 \in C^\gamma(\overline{Q}_0)$ and $r_0(x,0) = 0, \ x \in \overline{Q}_0$. Conversely, if $F_0 \in C^\gamma(\overline{Q}_0)$ is a solution of the equation (16), then the function (6) is a solution (from $C_{2,\beta}^1(Q_0)$) of the problem (1), (12)–(14).

Take two solutions $(u_1,F_0^1), (u_2,F_0^2)$ of the problem (1), (12)–(14). Putting $u = u_1 - u_2$, $F_0 = F_0^1 - F_0^2$ we obtain

$$D^\beta_t u = \Delta u + F_0, \ (x,t) \in Q_0,$$
\[
\frac{\partial u(x,t)}{\partial \nu(x)} = 0, \quad (x,t) \in \bar{Q}_1,
\]

\[
u(x,0) = 0, \quad x \in \bar{\Omega}_0,
\]

\[
\int_{\Omega_1} R(x,t,y)u(x,s)ds = 0, \quad (x,t) \in \bar{Q}_0.
\] (17)

By Theorem 1, the solution \(u(x,t)\) of obtained direct second boundary value problem has the representation

\[
u(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y)F_0(y,\tau)dy, \quad (x,t) \in \bar{Q}_0.
\] (18)

Substitute it in (17). As it has been shown before, we obtain the first type linear Volterra integral equation

\[
\int_0^t d\tau \int_{\Omega_0} R_0(x,t,y)F_0(y,\tau)dy = 0, \quad (x,t) \in \bar{Q}_0
\]

relatively unknown \(F_0\). By Theorem’s assumption,

\[
F_0(y,\tau) = 0, \quad (y,\tau) \in \bar{Q}_0.
\]

Then (18) implies that \(u = 0\) on \(\bar{Q}_0\). □

5. Problems with other conditions

5.1. In the same way we can study the boundary value problem

\[
D^\beta_t u - \Delta u = F_0(x,t), \quad (x,t) \in Q_0,
\] (19)

\[
\int_{\Omega_0} \mathcal{H}(x,t,z)u(z,t)dz = F(x,t), \quad (x,t) \in \bar{Q}_1,
\] (20)

\[
u(x,0) = F_2(x), \quad x \in \bar{\Omega}_0,
\] (21)

where \(\mathcal{H}, F_0, F, F_2\) are given functions.

This problem is different from the problem (1)–(3) by the condition (20) where integration is on \(\Omega_0\). We obtain similar preliminary results.

ASSUMPTION (D). \(\mathcal{H}(x,t,y) ((x,t) \in \bar{Q}_1, y \in \bar{\Omega}_0)\) is a jointly continuous in all variables function, H"older continuous in \(x \in \bar{\Omega}_1, y \in \bar{\Omega}_0\) for all \(t \in [0,T]\).

For the solution \(u \in C_{2,\beta}(\bar{Q}_0)\) the compatibility condition

\[
\int_{\Omega_0} \mathcal{H}(x,0,z)F_2(z)dz = F(x,0), \quad x \in \bar{Q}_1
\] (22)

is necessary.
THEOREM 5. Assume that (A), (D) and (22) hold, there exists a solution $F_1 \in C^1(\bar{Q}_1)$ of the integral equation

$$
\int_0^t d\tau \int_{\bar{Q}_1} \mathcal{P}_1(x,t,y,\tau) F_1(y,\tau) dy = p(x,t), \quad (x,t) \in \bar{Q}_1
$$

with

$$
p(x,t) = f_{1-\beta}(t) * F(x,t) - \int_0^t d\tau \int_{\bar{Q}_0} \mathcal{P}_0(x,t,y,\tau) F_0(y,\tau) dy$$

$$- \int_{\bar{Q}_0} \mathcal{P}_2(x,t,y) F_2(y) dy, \quad (x,t) \in \bar{Q}_1,$$

$$\mathcal{P}_j(x,t,y,\tau) = f_{1-\beta}(t-\tau) * \mathcal{K}_j(x,t-\tau,y,\tau), \quad (y,\tau) \in \bar{Q}_j, \quad j = 0,1,$$

$$\mathcal{K}_j(x,t,y,\tau) = \int_{\bar{Q}_0} \mathcal{K}(x,t,z) G_j(z,t,y,\tau) dz, \quad x \in \bar{Q}_1, \quad y \in \bar{Q}_j, \quad 0 \leq \tau < t \leq T, \quad j = 0,1,$$

$$\mathcal{K}_2(x,t,y) = \int_{\bar{Q}_0} \mathcal{K}(x,t,z) G_2(z,t,y) dz, \quad x \in \bar{Q}_1, \quad y \in \bar{Q}_0, \quad t \in (0,T].$$

Then there exists a solution $u \in C^1_{2,\beta}(Q_0)$ of the problem (19)–(21). It is defined by (6).

Proof. This theorem is proving as Theorems 2 and 3. Using the above estimates of components of the Green vector-function we find

$$\left| \mathcal{K}_j(x,t,y,\tau) \right| \leq C_6 \int_{\Omega_0} \frac{dz}{(t-\tau)|y-z|^{n-2}}$$

$$+ \int_{\bar{Q}_0} \frac{(t-\tau)^{\beta-1}}{|z-y|^n} \left( \frac{|z-y|^2}{4(t-\tau)^\beta} \right)^{1+\frac{n}{2(2-\beta)}} e^{-c \left( \frac{|z-y|^2}{4(t-\tau)^\beta} \right)^{\frac{1}{2(2-\beta)}}} dz$$

$$\leq C_7 \left[ \frac{1}{t-\tau} \int_0^t r dr + (t-\tau)^{-1-\frac{n\beta}{2(2-\beta)}} \int_0^{2diam\Omega_0} \frac{r}{r^{2-\beta}} e^{-c \left( \frac{r^2}{4(t-\tau)^\beta} \right)^{\frac{1}{2(2-\beta)}}} dr \right]$$

$$\leq \hat{k}(t-\tau)^{\beta-1} = \hat{k}(\beta)f_{\beta}(t-\tau),$$

$x \in \bar{Q}_1, \quad y \in \bar{Q}_j, \quad 0 \leq \tau < t \leq T, \quad j = 0,1,$
and similarly,

\[ |\mathcal{K}_2(x,t,y)| \leq C_9 \left[ \int_{\{z \in \Omega_0 : |y-z| < 2t^{\beta/2}\}} \frac{dz}{t^{\beta} |y-z|^{n-2}} dz \right. \\
+ \left. \int_{\{z \in \Omega_0 : |y-z| > 2t^{\beta/2}\}} \frac{1}{|z-y|^n} \left( \frac{|z-y|^2}{4t^\beta} \right)^{\frac{\alpha}{2(2\beta-\nu)}} e^{-c \left( \frac{|z-y|^2}{4t^\beta} \right)^{\frac{1-\beta}{2}}} dz \right] \]

\[ \leq C_9 \left[ \frac{1}{t^{\beta}} \int_0^{2t^{\beta/2}} r dr + r^{-\frac{n\beta}{2(2\beta-\nu)}} \int_{2t^{\beta/2}}^{\text{diam} \Omega_1} r^{\frac{n}{2(2\beta-\nu)}-1} e^{-c \left( \frac{r^2}{4t^\beta} \right)^{2-\beta}} dr \right] \leq \hat{k} \]

\[ x \in \tilde{\Omega}_1, \ y \in \tilde{\Omega}_0, \ t \in (0, T]. \]

So, the equation (23) has the jointly continuous kernel \( \mathcal{P}_1(x,t,y,\tau) \), Hölder continuous in \( x,y \in \tilde{\Omega}_1 \) for all \( t, \tau \in [0, T] \), \( p \in C^\gamma(\tilde{\Omega}_1) \) and \( p(x,0) = 0 \). \( \square \)

5.2. We also study the inverse problem

\[ D_1^\beta u - \Delta u = F_0(x,t), \ (x,t) \in \Omega_0 \times (0, T], \]  
\[ \frac{\partial u(x,t)}{\partial \nu(x)} = F_1(x,t), \ (x,t) \in \Omega_1 \times [0, T], \]  
\[ u(x,0) = F_2(x), \ x \in \tilde{\Omega}_0, \]  
\[ \int_{\Omega_0} P(x,t,z) u(z,t) dz = F(x,t), \ (x,t) \in \Omega_0 \times [0, T] \]

with given functions \( P, F_1, F_2, F \).

**Assumption (E).** \( F_1 \in C^\gamma(\tilde{\Omega}_1), \ F_2 \in C^\gamma(\tilde{\Omega}_0), \ \text{supp} F_2 \subset \Omega_0, \ F \in C^\gamma(\tilde{\Omega}_0), \)
\( P(x,t,y) ( (x,t) \in \tilde{\Omega}_0, y \in \tilde{\Omega}_0 ) \) is a a jointly continuous in all variables function, Hölder continuous in \( x,y \in \tilde{\Omega}_0 \) for all \( t \in [0, T] \).

This problem consists in finding a pair of functions

\[ (u,F_0) \in \mathcal{M}_0(Q_0) = \mathcal{M}_0 := C^{1}_{2,\beta}(Q_0) \times C^\gamma(\tilde{\Omega}_0) \]

satisfying the equation (24) and the conditions (25)–(27).

The compatibility condition

\[ \int_{\Omega_0} P(x,0,z) F_2(z) dz = F(x,0), \ x \in \tilde{\Omega}_0 \]

is necessary.
THEOREM 6. Assume that (E) holds, there exists a solution $F_0 \in C^0(\bar{Q}_0)$ of the integral equation

$$\int_0^t d\tau \int_{\Omega_0} P_0(x,t,y,\tau)F_0(y,\tau)dy = p_0(x,t), \quad (x,t) \in \bar{Q}_0,$$

with

$$p_0(x,t) = f_{1-\beta}(t) * F(x,t) - \int_0^t d\tau \int_{\Omega_1} P_1(x,t,y,\tau)F_1(y,\tau)dy - \int_{\Omega_0} P_2(x,t,y)F_2(y)dy,$$

$$P_j(x,t,y,\tau) = f_{1-\beta}(t-\tau) \ast \int_{\Omega_0} P(x,t-\tau,z)G_j(z,t-\tau,y,\tau)dz, \quad (y,\tau) \in \bar{Q}_j, \quad j = 0, 1,$$

$$P_0(x,t,y) = f_{1-\beta}(t) \ast \int_{\Omega_0} P(x,t,z)G_0(z,t,y)dz, \quad y \in \bar{Q}_0, \quad (x,t) \in \bar{Q}_0.$$

Then there exists a solution $(u,F_0) \in \mathcal{M}_0(Q_0)$ of the problem (24)–(27), $u$ is defined by (6).

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