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A PROBLEM WITH AN INTEGRAL BOUNDARY CONDITION FOR A TIME FRACTIONAL DIFFUSION EQUATION AND AN INVERSE PROBLEM

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Abstract. For a linear inhomogeneous time fractional diffusion equation on bounded cylindrical domain the problem with an integral boundary condition is studied. The inverse problem for the restoration of the whole right-hand side of the equation is also studied. The conditions of the solvability and the unique solvability of these problems are founded.

1. Introduction

The conditions of classical solvability of a time fractional Cauchy problem and the first boundary value problem to a time fractional diffusion equation were obtained, for example, in [2, 3, 4, 10, 11, 14]. There were proved the existence and uniqueness theorems and the representation of classical solution in terms of the Green vector-function. The inverse problems to such equations ([1, 6, 12, 13, 15] and references therein) have important practical applications. Some studies in inverse problems for restoration of a right-hand side of the equation, or a minor coefficient in the equation, or the initial data of a solution (see, for instance, [1, 8]) use the integral type over-determination conditions.

In the present paper, for a time fractional diffusion equation we study the solvability of a problem with an integral boundary condition. Namely, we study the boundary value problem

$$D_t^{\beta} u - \Delta u = F_0(x,t), \quad (x,t) \in \Omega_0 \times (0,T], \tag{1}$$

$$\int\limits_{\Omega_1} K(x,t,z)u(z,t)dz = F(x,t), \quad (x,t) \in \Omega_1 \times [0,T], \tag{2}$$

$$u(x,0) = F_2(x), \quad x \in \bar{\Omega}_0, \tag{3}$$

with the Caputo fractional derivative (or Caputo-Djrbashian fractional derivative) [2]

$$D^{\beta} v(t) = \frac{1}{\Gamma(1-\beta)} \int\limits_0^t \frac{v'(\tau)}{(t-\tau)^{\beta}} d\tau = \frac{1}{\Gamma(1-\beta)} \left[\frac{d}{dt} \int\limits_0^t \frac{v(\tau)}{(t-\tau)^{\beta}} d\tau - \frac{v(0)}{t^{\beta}} \right]$$

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of order $\beta \in (0,1)$, where Ω_0 is a boundary domain in $\mathbb{R}^n$, $n \geqslant 2$ with a smooth boundary Ω_1, K, F_0, F, F_2 are given functions.

We can consider this problem as the inverse problem for the determination a pair of functions: a classical solution u of the second boundary value problem

$$\frac{\partial u(x,t)}{\partial \nu(x)} = F_1(x,t), \ \ (x,t) \in \Omega_1 \times [0,T], \quad u(x,0) = F_2(x), \ \ x \in \bar{\Omega}_0, \tag{4}$$

for the equation (1) and its unknown boundary values F_1 under integral type over-determination condition (2). Here $\nu(x)$ is the unite vector of the interior normal to the surface Ω_1 in the point $x \in \Omega_1$. So, we can use the Green function's method in study the solvability of these problems. Note that in [13] the unknown boundary condition was founded by using the another over-determination condition.

We also study the inverse problem of finding the solution of the problem (1), (3), (4) and the right-hand side $F_0(x,t)$ of the equation (1) under the over-determination condition

$$\int_{\Omega_1} R(x,t,z)u(z,t)dz = F(x,t), \ \ (x,t) \in \Omega_0 \times [0,T]$$

with given functions R, F_1, F_2, F. Note that the inverse problems for the determination of the right-hand side $F_0 = F_0(x)$ or $F_0 = F_0(t)$ of a such kind of equations were studied (see, for example, [1, 15]).

2. Definitions and auxiliary results

Assume that Ω_1 is the surface of class $C^{1+\gamma}$, $\gamma \in (0,1)$, $Q_i = \Omega_i \times (0,T]$, $i = 0,1$, $Q_2 = \Omega_0$, $C(Q_0)$ $(C(\bar{Q}_0))$ is the space of continuous functions on Q_0 ($\bar{Q}_0$, respectively), $C^{\gamma}(\bar{\Omega}_i)$ is the space of Hölder continuous functions on $\bar{\Omega}_i$, $C^{\gamma}(\bar{Q}_i)$ is the space of Hölder continuous functions in space variables $x \in \bar{\Omega}_i$ for all $t \in [0,T]$ and jointly continuous in $(x,t) \in \bar{Q}_i$, $i = 0,1$, $C_{2,\beta}(Q_0) = \{v \in C(Q_0) \,|\, \Delta v, D_t^{\beta} v \in C(Q_0)\}$,

$$C_{2,\beta}(\bar{Q}_0) = C_{2,\beta}(Q_0) \cap C(\bar{Q}_0), \quad C^1_{2,\beta}(Q_0) = \{v \in C_{2,\beta}(\bar{Q}_0) \,|\, \partial v / \partial \nu \in C(\bar{Q}_1)\},$$

$\mathscr{D}(\mathbb{R})$ is the space of indefinitely differentiable functions compactly supported in $\mathbb{R}$, $\mathscr{D}'(\mathbb{R})$ is the space of linear continuous functionals (distributions) over $\mathscr{D}(\mathbb{R})$ and $\mathscr{D}'_+(\mathbb{R}) = \{f \in \mathscr{D}'(\mathbb{R}) : f = 0 \text{ for } t < 0\}$.

We denote by $f * g$ the convolution of the distributions f and g, use the function $f_\lambda \in \mathscr{D}'_+(\mathbb{R})$:

$$f_\lambda(t) = \frac{\theta(t) t^{\lambda-1}}{\Gamma(\lambda)} \quad \text{for } \lambda > 0 \quad \text{and} \quad f_\lambda(t) = f'_{1+\lambda}(t) \text{ for } \lambda \leqslant 0,$$

where $\Gamma(z)$ is the Gamma-function, $\theta(t)$ is the Heaviside function and derivative understood as a derivative in $\mathscr{D}'(\mathbb{R})$. Note that

$$f_\lambda * f_\mu = f_{\lambda+\mu}.$$

Recall that the Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta > 0$ is defined as

$$v^{(\beta)}(t) = f_{-\beta}(t) * v(t).$$

If $D^{\beta}v$ exists then

$$D^{\beta}v(t) = v^{(\beta)}(t) - f_{1-\beta}(t)v(0) \quad \text{for} \quad \beta \in (0,1).$$

Let the following assumptions hold:

(A) $F_0 \in C^{\gamma}(\bar{Q}_0)$, $F_2 \in C^{\gamma}(\bar{\Omega}_0)$, $suppF_2 \subset \Omega_0$, $F \in C^{\gamma}(\bar{Q}_1)$,

(B) $K(x,t,y)$ $((x,t) \in \bar{Q}_1, y \in \bar{\Omega}_1)$ is a jointly continuous in all variables function, Hölder continuous in $x,y \in \bar{\Omega}_1$ for all $t \in [0,T]$.

DEFINITION 1. A function $u \in C_{2,\beta}(\bar{Q}_0)$ satisfying the equation (1) on Q_0 and the conditions (2), (3) is called a solution of the problem (1)–(3).

Definition 1 and the assumption (A) about F_2 imply the compatibility condition

$$F(x,0) = 0, \ \ x \in \bar{\Omega}_1. \tag{5}$$

DEFINITION 2. A vector-function $(G_0(x,t,y,\tau), G_1(x,t,y,\tau), G_2(x,t,y))$ such that under rather regular F_0, F_1, F_2 the function

$$u(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau)F_0(y,\tau)dy + \int_0^t d\tau \int_{\Omega_1} G_1(x,t,y,\tau)F_1(y,\tau)dy$$
$$+ \int_{\Omega_0} G_2(x,t,y)F_2(y)dy, \quad (x,t) \in Q_0 \tag{6}$$

is a classical (from $C^1_{2,\beta}(Q_0)$) solution of the problem (1), (4) is called a Green vector-function of this second boundary-value problem.

The existence of a unique Green vector-function of the problem (1), (4) may be prove as in [9] for the first boundary value problem.

The results of [4, 14] and the Levi method (see [4, 5]) imply the following estimates in the case $n \geqslant 3$:

$$G_j(x,t,y,\tau) \leqslant \frac{C}{(t-\tau)|x-y|^{n-2}}, \quad j = 0,1,$$

$$G_2(x,t,y) \leqslant \frac{C}{t^{\beta}|x-y|^{n-2}}, \quad |x-y|^2 < 4(t-\tau)^{\beta},$$

$$G_j(x,t,y,\tau) \leqslant \frac{C(t-\tau)^{\beta-1}}{|x-y|^n} \cdot \left(\frac{|x-y|^2}{4(t-\tau)^{\beta}}\right)^{1+\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|x-y|^2}{4(t-\tau)^{\beta}}\right)^{\frac{1}{2-\beta}}}, \quad j = 0,1,$$

$$G_2(x,t,y) \leqslant \frac{C}{|x-y|^n} \cdot \left(\frac{|x-y|^2}{4t^{\beta}}\right)^{\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|x-y|^2}{4t^{\beta}}\right)^{\frac{1}{2-\beta}}}, \quad |x-y|^2 > 4(t-\tau)^{\beta}$$

where C, c are positive constants, and similar estimates in the case $n = 2$. Note that the character features in t is accurate. The Green vector-function owns Hölder properties.

THEOREM 1. *Assume that* $F_0 \in C^{\gamma}(\bar{Q}_0)$, $F_1 \in C^{\gamma}(\bar{Q}_1)$, $F_2 \in C^{\gamma}(\bar{\Omega}_0)$, $suppF_2 \subset \Omega_0$. *Then there exists a unique solution* $u \in C^1_{2,\beta}(Q_0)$ *of the second boundary value problem (1), (4), and it is defined by the formula (6).*

The theorem is proving as the corresponding result in [5, 5.3].

3. The existence and uniqueness theorems

Substituting the right-hand side of (6) in the condition (2) we obtain

$$\int_0^t d\tau \int_{\Omega_1} K(x,t,z)dz \int_{\Omega_0} G_0(z,t,y,\tau)F_0(y,\tau)dy$$
$$+\int_0^t d\tau \int_{\Omega_1} K(x,t,z)dz \int_{\Omega_1} G_1(z,t,y,\tau)F_1(y,\tau)dy$$
$$+\int_{\Omega_1} K(x,t,z)dz \int_{\Omega_0} G_1(z,t,y)F_2(y)dy = F(x,t), \quad (x,t) \in \bar{Q}_1,$$

that is

$$\sum_{j=0}^{1} \int_0^t d\tau \int_{\Omega_j} K_j(x,t,y,\tau)F_j(y,\tau)dy + \int_{\Omega_0} K_2(x,y,t)F_2(y)dy = F(x,t), \quad (x,t) \in \bar{Q}_1, \quad (7)$$

where

$$K_j(x,t,y,\tau) = \int_{\Omega_1} K(x,t,z)G_j(z,t,y,\tau)dz, \quad j = 0,1,$$
$$K_2(x,t,y) = \int_{\Omega_1} K(x,t,z)G_2(z,t,y)dz.$$

Using the above estimates of components of the Green vector-function we find

$$|K_j(x,t,y,\tau)|$$
$$\leqslant \int_{\{z\in\Omega_1:|y-z|<2(t-\tau)^{\beta/2}\}} \left|K(x,t,z)G_j(z,t,y,\tau)\right|dx$$
$$+ \int_{\{z\in\Omega_1:|y-z|>2(t-\tau)^{\beta/2}\}} \left|K(x,t,z)G_j(z,t,y,\tau)\right|dx$$
$$\leqslant C_1\Big[\int_{\{z\in\Omega_1:|y-z|<2(t-\tau)^{\beta/2}\}} \frac{dz}{(t-\tau)|y-z|^{n-2}}dz$$
$$+ \int_{\{z\in\Omega_1:|y-z|>2(t-\tau)^{\beta/2}\}} \frac{(t-\tau)^{\beta-1}}{|z-y|^n}\left(\frac{|z-y|^2}{4(t-\tau)^{\beta}}\right)^{1+\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|z-y|^2}{4(t-\tau)^{\beta}}\right)^{\frac{1}{2-\beta}}} dz\Big]$$

$$\leqslant C_2\Big[\frac{1}{t-\tau}\int\limits_0^{2(t-\tau)^{\beta/2}} dr + (t-\tau)^{-1-\frac{n\beta}{2(2-\beta)}}\int\limits_{2(t-\tau)^{\beta/2}}^{diam\Omega_1} r^{\frac{n}{2-\beta}} e^{-c\left(\frac{r^2}{4(t-\tau)^\beta}\right)^{\frac{1}{2-\beta}}} dr\Big]$$

$$\leqslant C_3(t-\tau)^{\frac{\beta}{2}-1}\Big[1+\int\limits_1^\infty \eta^{\frac{n-\beta}{2}} e^{-c\eta} d\eta\Big] \leqslant \hat{k}(t-\tau)^{\frac{\beta}{2}-1}$$

$$= \hat{k}\Gamma\Big(\frac{\beta}{2}\Big) f_{\frac{\beta}{2}}(t-\tau), \quad x\in\bar{\Omega}_1,\ y\in\bar{\Omega}_j,\ 0\leqslant\tau<t\leqslant T,\ j=0,1,$$

and similarly,

$$\Big|K_2(x,t,y)\Big| \leqslant C_4\Big[\int\limits_{\{z\in\Omega_1:|y-z|<2t^{\beta/2}\}} \frac{dz}{t^\beta |y-z|^{n-2}} dz$$

$$+ \int\limits_{\{z\in\Omega_1:|y-z|>2t^{\beta/2}\}} \frac{1}{|z-y|^n}\Big(\frac{|z-y|^2}{4t^\beta}\Big)^{\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|z-y|^2}{4t^\beta}\right)^{\frac{1}{2-\beta}}} dz\Big]$$

$$\leqslant C_5\Big[\frac{1}{t^\beta}\int\limits_0^{2t^{\beta/2}} dr + t^{-\frac{n\beta}{2(2-\beta)}}\int\limits_{2t^{\beta/2}}^{diam\Omega_1} r^{\frac{n}{2-\beta}-2} e^{-c\left(\frac{r^2}{4t^\beta}\right)^{\frac{1}{2-\beta}}} dr\Big]$$

$$\leqslant \hat{k}t^{-\frac{\beta}{2}} = \hat{k}\Gamma\Big(1-\frac{\beta}{2}\Big) f_{1-\frac{\beta}{2}}(t), \quad x\in\bar{\Omega}_1,\ \ y\in\bar{\Omega}_0,\ \ t\in(0,T].$$

Hereinafter C_i, $\hat{k}$, $\hat{k}_i$, $i\in\mathbb{Z}_+$ are positive constants.

Note that

$$f_{1-\frac{\beta}{2}}(t) * \int\limits_0^t K_j(x,t,y,\tau)F_j(y,\tau)d\tau = \int\limits_0^t f_{1-\frac{\beta}{2}}(t-s)ds\int\limits_0^s K_j(x,s,y,\tau)F_j(y,\tau)d\tau$$

$$= \int\limits_0^t\Big(\int\limits_\tau^t f_{1-\frac{\beta}{2}}(t-s)K_j(x,s,y,\tau)ds\Big)F_j(y,\tau)d\tau$$

$$= \int\limits_0^t\Big(\int\limits_0^{t-\tau} f_{1-\frac{\beta}{2}}(s)K_j(x,t-s,y,\tau)ds\Big)F_j(y,\tau)d\tau$$

$$= \int\limits_0^t\Big(f_{1-\frac{\beta}{2}}(s) * K_j(x,s,y,\tau)\Big)\Big|_{s=t-\tau}F_j(y,\tau)d\tau$$

$$= \int\limits_0^t f_{1-\frac{\beta}{2}}(t-\tau) * K_j(x,t-\tau,y,\tau)F_j(y,\tau)d\tau.$$

Denote

$$\mathscr{R}_j(x,t,y,\tau) = f_{1-\frac{\beta}{2}}(t-\tau) * K_j(x,t-\tau,y,\tau), \quad j=0,1.$$

We have

$$|\mathscr{R}_j(x,t,y,\tau)| \leqslant \hat{k}\Gamma\Big(\frac{\beta}{2}\Big) f_{1-\frac{\beta}{2}}(t-\tau) * f_{\frac{\beta}{2}}(t-\tau)$$
$$= \hat{k}\Gamma\Big(\frac{\beta}{2}\Big)\Big(f_{1-\frac{\beta}{2}} * f_{\frac{\beta}{2}}\Big)(t-\tau) = \hat{k}\Gamma\Big(\frac{\beta}{2}\Big) f_1(t-\tau) = \hat{k}_1\theta(t-\tau).$$

Since the character features in the variable t is accurate then

$$\mathscr{R}_j(x,t,y,\tau) \neq 0, \ \ (x,t) \in \bar{Q}_1, \ \ (y,\tau) \in \bar{Q}_j, \ \ j = 0,1$$

and are the continuous functions. Similarly, taking the equality

$$f_{1-\frac{\beta}{2}}(t) * f_{1-\frac{\beta}{2}}(t) = f_{2-\beta}(t)$$

into account, we obtain that the function

$$\mathscr{R}_2(x,t,y) := f_{1-\frac{\beta}{2}}(t) * K_2(x,t,y)$$

is continuous on $\bar{Q}_1 \times \bar{\Omega}_0$ and has the estimate

$$|\mathscr{R}_2(x,t,y)| \leqslant \hat{k}_2 t^{1-\beta}, \ \ (x,t) \in \bar{Q}_1, \ \ y \in \bar{\Omega}_0.$$

Thus, from (7) we obtain the linear integral Volterra equation of the first type

$$\int_0^t d\tau \int_{\Omega_1} \mathscr{R}_1(x,t,y,\tau) F_1(y,\tau) dy = h(x,t), \quad (x,t) \in \bar{Q}_1 \tag{8}$$

relatively unknown F_1 where

$$h(x,t) = f_{1-\frac{\beta}{2}}(t) * F(x,t) - \int_0^t d\tau \int_{\Omega_0} \mathscr{R}_0(x,t,y,\tau) F_0(y,\tau) dy$$
$$- \int_{\Omega_0} \mathscr{R}_2(x,t,y) F_2(y) dy, \quad (x,t) \in \bar{Q}_1.$$

This integral equation has the jointly continuous kernel $\mathscr{R}_1(x,t,y,\tau)$, Hölder continuous in $x,y \in \bar{\Omega}_1$ for all $t,\tau \in [0,T]$, and $h \in C^{\gamma}(\bar{Q}_1)$, $h(x,0) = 0$.

Conversely, if $F_1 \in C^{\gamma}(\bar{Q}_1)$ is a solution of the equation (8), which is equivalent to the equation (7), then by Theorem 1 the function (6) is a solution (from $C^1_{2,\beta}(Q_0)$) of the problem (1), (4) and satisfies the condition (2).

We obtain the following result.

THEOREM 2. *Assume that (A), (B) and (5) hold, there exists a solution $F_1 \in C^{\gamma}(\bar{Q}_1)$ of the equation (8). Then there exists a solution $u \in C^1_{2,\beta}(Q_0)$ of the problem (1)–(3). It is defined by (6).*

THEOREM 3. *At terms of uniqueness of a solution of the equation (8) a solution* $u \in C^1_{2,\beta}(Q_0)$ *of the problem (1)–(3) is unique.*

Proof. Take two solutions $u_1, u_2 \in C^1_{2,\beta}(Q_0)$ of the problem (1)–(3). Putting $u = u_1 - u_2$ we obtain

$$D_t^\beta u = \Delta u, \quad (x,t) \in Q_0,$$
$$\int_{\Omega_1} K(x,t,y)u(y,t)dy = 0, \quad (x,t) \in \bar{Q}_1, \tag{9}$$
$$u(x,0) = 0, \quad x \in \bar{\Omega}_0.$$

By Theorem 1, for the solution $u \in C^1_{2,\beta}(Q_0)$ of the second boundary value problem

$$D_t^\beta u = \Delta u, \quad (x,t) \in Q_0,$$
$$\frac{\partial u(x,t)}{\partial \nu(x)} = F_1(x,t), \ \ (x,t) \in \bar{Q}_1, \quad u(x,0) = 0, \quad x \in \bar{\Omega}_0$$

with some unknown $F_1 \in C^\gamma(\bar{Q}_1)$ we have

$$u(x,t) = \int_0^t d\tau \int_{\Omega_1} G_1(x,t,y,\tau)F_1(y,\tau)dy, \quad (x,t) \in \bar{Q}_0. \tag{10}$$

As it has been shown before, using (9), we obtain the first type linear Volterra integral equation

$$\int_0^t d\tau \int_{\Omega_1} \mathscr{R}_1(x,t,y,\tau)F_1(y,\tau)dy = 0, \quad (x,t) \in \bar{Q}_1,$$

relatively unknown F_1. By Theorem's assumption,

$$F_1(y,\tau) = 0, \quad (y,\tau) \in \bar{Q}_1. \tag{11}$$

Then (10) implies that $u = 0$ on $\bar{Q}_0$. $\square$

4. The restoration of a whole right-hand side of equation

We study the inverse problem

$$\frac{\partial u(x,t)}{\partial \nu(x)} = F_1(x,t), \quad (x,t) \in \bar{Q}_1, \tag{12}$$

$$u(x,0) = F_2(x), \ \ x \in \bar{\Omega}_0, \tag{13}$$

$$\int_{\Omega_1} R(x,t,z)u(z,t)dz = F(x,t), \ \ (x,t) \in \bar{Q}_0 \tag{14}$$

for the equation (1) under the following assumption

(C) $F_1 \in C^{\gamma}(\bar{Q}_1)$, $F_2 \in C^{\gamma}(\bar{\Omega}_0)$, $supp F_2 \subset \Omega_0$, $F \in C^{\gamma}(\bar{Q}_0)$, $R(x,t,y)$ ($(x,t) \in \bar{Q}_0, y \in \bar{\Omega}_1$) is a a jointly continuous in all variables function, Hölder continuous in $x \in \bar{\Omega}_0$, $y \in \bar{\Omega}_1$ for all $t \in [0,T]$.

This problem consists in finding a pair of functions

$$(u,F_0) \in \mathscr{M}(Q_0) = \mathscr{M} := C^1_{2,\beta}(Q_0) \times C^{\gamma}(\bar{Q}_0)$$

satisfying the equation (1) and the above conditions (12)–(14).

The problem's definition and the assumption (C) about F_2 imply the compatibility condition

$$F(x,0) = 0, \;\; x \in \bar{\Omega}_0. \tag{15}$$

THEOREM 4. *Assume that (C) and (15) hold, there exists a solution $F_0 \in C^{\gamma}(\bar{Q}_0)$ of the integral equation*

$$\int_0^t d\tau \int_{\Omega_0} R_0(x,t,y,\tau) F_0(y,\tau) dy = r_0(x,t), \quad (x,t) \in \bar{Q}_0, \tag{16}$$

with

$$r_0(x,t) = f_{1-\frac{\beta}{2}}(t) * F(x,t) - \int_0^t d\tau \int_{\Omega_1} R_1(x,t,y,\tau) F_1(y,\tau) dy - \int_{\Omega_0} R_2(x,t,y) F_2(y) dy,$$

$$R_j(x,t,y,\tau) = f_{1-\frac{\beta}{2}}(t-\tau) * \int_{\Omega_1} R(x,t-\tau,z) G_j(z,t-\tau,y,\tau) dz, \;\; (y,\tau) \in Q_j, \;\; j = 0,1,$$

$$R_2(x,t,y) = f_{1-\frac{\beta}{2}}(t) * \int_{\Omega_1} R(x,t,z) G_2(z,t,y) dz, \;\; y \in \Omega_0, \;\; (x,t) \in \bar{Q}_0.$$

Then there exists a solution $(u,F_0) \in \mathscr{M}(Q_0)$ of the problem (1), (12)–(14) where u is defined by (6). At terms of uniqueness of a solution of the equation (16) a solution $(u,F_0) \in \mathscr{M}(Q_0)$ of the problem (1), (12)–(14) is unique.

Proof. Substituting the right-hand side of (6) in the condition (14) and using the previous reasoning we obtain the linear integral Volterra equation (16) of the first type relatively unknown F_0. This integral equation has a jointly continuous kernel $R_0(x,t,y,\tau)$, Hölder continuous in $x,y \in \bar{\Omega}_0$ for all $t,\tau \in [0,T]$, $r_0 \in C^{\gamma}(\bar{Q}_0)$ and $r_0(x,0) = 0$, $x \in \bar{\Omega}_0$. Conversely, if $F_0 \in C^{\gamma}(\bar{Q}_0)$ is a solution of the equation (16), then the function (6) is a solution (from $C^1_{2,\beta}(Q_0)$) of the problem (1), (12)–(14).

Take two solutions (u_1,F_0^1), (u_2,F_0^2) of the problem (1), (12)–(14). Putting $u = u_1 - u_2$, $F_0 = F_0^1 - F_0^2$ we obtain

$$D_t^{\beta} u = \Delta u + F_0, \quad (x,t) \in Q_0,$$

$$\frac{\partial u(x,t)}{\partial \nu(x)} = 0, \quad (x,t) \in \bar{Q}_1,$$

$$u(x,0) = 0, \quad x \in \bar{\Omega}_0,$$

$$\int_{\Omega_1} R(x,t,y)u(x,s)ds = 0, \quad (x,t) \in \bar{Q}_0. \tag{17}$$

By Theorem 1, the solution $u(x,t)$ of obtained direct second boundary value problem has the representation

$$u(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau)F_0(y,\tau)dy, \quad (x,t) \in \bar{Q}_0. \tag{18}$$

Substitute it in (17). As it has been shown before, we obtain the first type linear Volterra integral equation

$$\int_0^t d\tau \int_{\Omega_0} R_0(x,t,y,\tau)F_0(y,\tau)dy = 0, \quad (x,t) \in \bar{Q}_0$$

relatively unknown F_0. By Theorem's assumption,

$$F_0(y,\tau) = 0, \quad (y,\tau) \in \bar{Q}_0.$$

Then (18) implies that $u = 0$ on $\bar{Q}_0$. $\square$

5. Problems with other conditions

5.1. In the same way we can study the boundary value problem

$$D_t^\beta u - \Delta u = F_0(x,t), \ \ (x,t) \in Q_0, \tag{19}$$

$$\int_{\Omega_0} \mathscr{K}(x,t,z)u(z,t)dz = F(x,t), \ \ (x,t) \in \bar{Q}_1, \tag{20}$$

$$u(x,0) = F_2(x), \ \ x \in \bar{\Omega}_0, \tag{21}$$

where $\mathscr{K}$, F_0, F, F_2 are given functions.

This problem is different from the problem (1)–(3) by the condition (20) where integration is on Ω_0. We obtain similar preliminary results.

ASSUMPTION (D). $\mathscr{K}(x,t,y)$ $((x,t) \in \bar{Q}_1, y \in \bar{\Omega}_0)$ is a jointly continuous in all variables function, Hölder continuous in $x \in \bar{\Omega}_1$, $y \in \bar{\Omega}_0$ for all $t \in [0,T]$.

For the solution $u \in C_{2,\beta}(\bar{Q}_0)$ the compatibility condition

$$\int_{\Omega_0} \mathscr{K}(x,0,z)F_2(z)dz = F(x,0), \ \ x \in \bar{\Omega}_1 \tag{22}$$

is necessary.

THEOREM 5. *Assume that (A), (D) and (22) hold, there exists a solution $F_1 \in C^{\gamma}(\bar{Q}_1)$ of the integral equation*

$$\int_0^t d\tau \int_{\Omega_1} \mathscr{P}_1(x,t,y,\tau) F_1(y,\tau)dy = p(x,t), \quad (x,t) \in \bar{Q}_1 \tag{23}$$

with

$$p(x,t) = f_{1-\beta}(t) * F(x,t) - \int_0^t d\tau \int_{\Omega_0} \mathscr{P}_0(x,t,y,\tau)F_0(y,\tau)dy$$
$$- \int_{\Omega_0} \mathscr{P}_2(x,t,y)F_2(y)dy, \quad (x,t) \in \bar{Q}_1,$$
$$\mathscr{P}_j(x,t,y,\tau) = f_{1-\beta}(t-\tau) * \mathscr{K}_j(x,t-\tau,y,\tau), \ (y,\tau) \in \bar{Q}_j, \ j = 0,1,$$
$$\mathscr{P}_2(x,t,y) = f_{1-\beta}(t) * \mathscr{K}_2(x,t,y), \ y \in \bar{\Omega}_0, \ (x,t) \in \bar{Q}_1,$$

and

$$\mathscr{K}_j(x,t,y,\tau) = \int_{\Omega_0} \mathscr{K}(x,t,z)G_j(z,t,y,\tau)dz, \ x \in \bar{\Omega}_1, \ y \in \bar{\Omega}_j, \ 0 \leqslant \tau < t \leqslant T, \ j = 0,1,$$
$$\mathscr{K}_2(x,t,y) = \int_{\Omega_0} \mathscr{K}(x,t,z)G_2(z,t,y)dz, \ x \in \bar{\Omega}_1, \ y \in \bar{\Omega}_0, \ t \in (0,T].$$

Then there exists a solution $u \in C^1_{2,\beta}(Q_0)$ of the problem (19)–(21). It is defined by (6).

Proof. This theorem is proving as Theorems 2 and 3. Using the above estimates of components of the Green vector-function we find

$$|\mathscr{K}_j(x,t,y,\tau)|$$
$$\leqslant C_6 \Big[\int_{\{z \in \Omega_0 : |y-z| < 2(t-\tau)^{\beta/2}\}} \frac{dz}{(t-\tau)|y-z|^{n-2}} dz$$
$$+ \int_{\{z \in \Omega_0 : |y-z| > 2(t-\tau)^{\beta/2}\}} \frac{(t-\tau)^{\beta-1}}{|z-y|^n} \left(\frac{|z-y|^2}{4(t-\tau)^{\beta}} \right)^{1+\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|z-y|^2}{4(t-\tau)^{\beta}}\right)^{\frac{1}{2-\beta}}} dz \Big]$$
$$\leqslant C_7 \Big[\frac{1}{t-\tau} \int_0^{2(t-\tau)^{\beta/2}} r dr + (t-\tau)^{-1-\frac{n\beta}{2(2-\beta)}} \int_{2(t-\tau)^{\beta/2}}^{diam \Omega_0} r^{1+\frac{n}{2-\beta}} e^{-c\left(\frac{r^2}{4(t-\tau)^{\beta}}\right)^{\frac{1}{2-\beta}}} dr \Big]$$
$$\leqslant \hat{k}(t-\tau)^{\beta-1} = \hat{k}\Gamma(\beta) f_{\beta}(t-\tau),$$
$$x \in \bar{\Omega}_1, \ y \in \bar{\Omega}_j, \ 0 \leqslant \tau < t \leqslant T, \ j = 0,1,$$

and similarly,

$$|\mathscr{K}_2(x,t,y)| \leqslant C_8 \Big[\int\limits_{\{z\in\Omega_0:|y-z|<2t^{\beta/2}\}} \frac{dz}{t^{\beta}|y-z|^{n-2}} dz$$

$$+ \int\limits_{\{z\in\Omega_0:|y-z|>2t^{\beta/2}\}} \frac{1}{|z-y|^n} \Big(\frac{|z-y|^2}{4t^{\beta}}\Big)^{\frac{n}{2(2-\beta)}} e^{-c\left(\frac{|z-y|^2}{4t^{\beta}}\right)^{\frac{1}{2-\beta}}} dz\Big]$$

$$\leqslant C_9 \Big[\frac{1}{t^{\beta}} \int\limits_0^{2t^{\beta/2}} r dr + t^{-\frac{n\beta}{2(2-\beta)}} \int\limits_{2t^{\beta/2}}^{diam\Omega_1} r^{\frac{n}{2-\beta}-1} e^{-c\left(\frac{r^2}{4t^{\beta}}\right)^{\frac{1}{2-\beta}}} dr\Big] \leqslant \hat{k}$$

$$x \in \bar{\Omega}_1, \ \ y \in \bar{\Omega}_0, \ \ t \in (0,T].$$

So, the equation (23) has the jointly continuous kernel $\mathscr{P}_1(x,t,y,\tau)$, Hölder continuous in $x,y \in \bar{\Omega}_1$ for all $t,\tau \in [0,T]$, $p \in C^{\gamma}(\bar{Q}_1)$ and $p(x,0)=0$. $\square$

5.2. We also study the inverse problem

$$D_t^{\beta} u - \Delta u = F_0(x,t), \ \ (x,t) \in \Omega_0 \times (0,T], \tag{24}$$

$$\frac{\partial u(x,t)}{\partial \nu(x)} = F_1(x,t), \quad (x,t) \in \Omega_1 \times [0,T], \tag{25}$$

$$u(x,0) = F_2(x), \ \ x \in \bar{\Omega}_0, \tag{26}$$

$$\int\limits_{\Omega_0} P(x,t,z)u(z,t)dz = F(x,t), \ \ (x,t) \in \Omega_0 \times [0,T] \tag{27}$$

with given functions P, F_1, F_2, F.

ASSUMPTION (E). $F_1 \in C^{\gamma}(\bar{Q}_1)$, $F_2 \in C^{\gamma}(\bar{\Omega}_0)$, $suppF_2 \subset \Omega_0$, $F \in C^{\gamma}(\bar{Q}_0)$, $P(x,t,y)$ $((x,t) \in \bar{Q}_0, y \in \bar{\Omega}_0)$ is a a jointly continuous in all variables function, Hölder continuous in $x,y \in \bar{\Omega}_0$ for all $t \in [0,T]$.

This problem consists in finding a pair of functions

$$(u,F_0) \in \mathscr{M}_0(Q_0) = \mathscr{M}_0 := C^1_{2,\beta}(Q_0) \times C^{\gamma}(\bar{Q}_0)$$

satisfying the equation (24) and the conditions (25)–(27).

The compatibility condition

$$\int\limits_{\Omega_0} P(x,0,z)F_2(z)dz = F(x,0), \ \ x \in \bar{\Omega}_0 \tag{28}$$

is necessary.

THEOREM 6. *Assume that (E) holds, there exists a solution $F_0 \in C^{\gamma}(\bar{Q}_0)$ of the integral equation*

$$\int_0^t d\tau \int_{\Omega_0} P_0(x,t,y,\tau)F_0(y,\tau)dy = p_0(x,t), \quad (x,t) \in \bar{Q}_0,$$

with

$$p_0(x,t) = f_{1-\beta}(t) * F(x,t) - \int_0^t d\tau \int_{\Omega_1} P_1(x,t,y,\tau)F_1(y,\tau)dy - \int_{\Omega_0} P_2(x,t,y)F_2(y)dy,$$

$$P_j(x,t,y,\tau) = f_{1-\beta}(t-\tau) * \int_{\Omega_0} P(x,t-\tau,z)G_j(z,t-\tau,y,\tau)dz, \ \ (y,\tau) \in \bar{Q}_j, \ \ j = 0,1,$$

$$P_2(x,t,y) = f_{1-\beta}(t) * \int_{\Omega_0} P(x,t,z)G_2(z,t,y)dz, \ \ y \in \bar{\Omega}_0, \ \ (x,t) \in \bar{Q}_0.$$

Then there exists a solution $(u,F_0) \in \mathscr{M}_0(Q_0)$ of the problem (24)–(27), u is defined by (6).

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