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REMARK TO HISTORY OF FRACTIONAL DERIVATIVES ON COMPLEX PLANE: SONINE–LETNIKOV AND NISHIMOTO DERIVATIVES

VASILY E. TARASOV

Abstract. An important part of fractional calculus is connected with the fractional-order integro-differentiation in the complex plane, which is generalization of the well-known Cauchy differentiation formula. In some publications this generalization is called the Nishimoto fractional derivatives. In our paper, we emphasize the fact that the generalization of the Cauchy's differentiation formula to non-integer orders has been suggested by Sonine in 1872 more than a hundred years before Nishimoto's works. The Sonine approach has been adjusted by Letnikov in 1873 and then it has been developed by Nekrasov in 1888.

Derivatives and integrals of non-integer orders [1, 2, 3, 4] have a long history [5]–[11] that is connected with the names of famous mathematicians Riemann, Liouville, Riesz, Grünwald, Letnikov, Sonine, Marchaud, Weyl, Riesz, Caputo and others. An application of non-standard properties of fractional-order integro-differentiation [12, 13, 14, 15] allows us to describe local and hereditarity system. Most fractional derivatives and integrals, which are used in applications, are considered on the real axis. An important role in application of fractional calculus can play fractional integrals and derivatives in the complex plane that are the extension of the Cauchy differentiation formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^{n+1}} \tag{1}$$

to non-integer values of n.

In noticeable number of articles, which devoted to fractional derivatives on complex plane, this extension is called the Nishimoto fractional derivatives. Nishimoto proposed an extension of the Cauchy formula (1) to fractional order in the papers [16, 17, 18] in 1976-1977 and the book [19] in 1986 (see also [20, 21, 22]).

These articles do not take into account the fact that the generalization of the Cauchy's differentiation formula (1) to non-integer orders has been suggested by Sonine more than a hundred years ago in the paper [23] in 1872. The Sonine approach has been adjusted by Letnikov [24] in 1873. Then it has been developed by Nekrasov (see Theorem VII [25], page 87) in 1888 (the Nekrasov's contribution has been noted and developed by Osler [26] in 1970). Some historical notes to formula (2) has been given in the book [1] on pages 435 and in Section 22 of [1]. The definition based on

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a generalization of the formula for differentiating the Cauchy type integral has been suggested in the form

$$f^{(\alpha)}(z) := \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^{\alpha+1}}. \tag{2}$$

Therefore the fractional integro-differentiation in the complex plane, which is based on a generalization of the Cauchy's differentiation formula, should be called the Sonine fractional derivative or the Sonine-Letnikov fractional derivative

Note that an application of formula (2) requires precision aimed to single out a branch of the multivalued function $(\xi - z)^{\alpha+1}$. It is usually achieved by means of a cut, which goes from the branching point to infinity or by fixing $arg(\xi - z)$ in some way. Different choices of a cut, which fixes the branch of the multivalued function $(\xi - z)^{\alpha+1}$ in (2), and of the curve C gives different values of $f^{(\alpha)}(z)$ in general [1].

REFERENCES

[1] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Integrals and Derivatives of Fractional Order and Applications*, Nauka i Tehnika, Minsk, 1987; and *Fractional Integrals and Derivatives Theory and Applications*, Gordon and Breach, New York, 1993.

[2] V. KIRYAKOVA, *Generalized Fractional Calculus and Applications*, Longman, Harlow and Wiley, New York, (1994).

[3] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1998.

[4] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006, 353 pages.

[5] A. V. LETNIKOV, *On the historical development of the theory of differentiation with arbitrary index*, Sbornik Mathematics (Matematicheskii Sbornik), vol. **3**, no. 2. (1868) 85–112, (in Russian), URL: `http://mi.mathnet.ru/eng/msb8048` (`http://mi.mathnet.ru/eng/msb/v3/i2/p85`).

[6] B. ROSS, *A brief history and exposition of the fundamental theory of fractional calculus*, in Fractional Calculus and Its Applications, Proceedings of the International Conference Held at the University of New Haven, June 1974, Series: Lecture Notes in Mathematics, vol. **457**, Springer, Berlin, Heidelberg, 1975, pp. 1–36, DOI: 10.1007/BFb0067096.

[7] L. DEBNATH, *A brief historical introduction to fractional calculus*, International Journal of Mathematical Education in Science and Technology, vol. **35**, no. 4, (2004) 487–501.

[8] J. TENREIRO MACHADO, V. KIRYAKOVA, F. MAINARDI, *Recent history of fractional calculus*, Communications in Nonlinear Science and Numerical Simulation, vol. **16**, no. 3, (2011) 1140–1153.

[9] J. A. TENREIRO MACHADO, A. M. GALHANO, J. J. TRUJILLO, *Science metrics on fractional calculus development since 1966*, Fractional Calculus and Applied Analysis, vol. **16**, no. 2, (2013) 479–500.

[10] J. A. TENREIRO MACHADO, A. M. S. F. GALHANO, J. J. TRUJILLO, *On development of fractional calculus during the last fifty years*, Scientometrics. vol. **98**, no. 1, (2014) 577–582.

[11] D. VALERIO, J. MACHADO, V. KIRYAKOVA, *Some pioneers of the applications of fractional calculus*, Fractional Calculus and Applied Analysis. vol. **17**, no. 2, (2014) 552–578. DOI: 10.2478/s13540-014-0185-1

[12] V. E. TARASOV, *No violation of the Leibniz rule. No fractional derivative*, Communications in Nonlinear Science and Numerical Simulation. vol. **18**, no. 11, (2013) 2945–2948.

[13] M. D. ORTIGUEIRA, J. A. TENREIRO MACHADO, *What is a fractional derivative?*, Journal of Computational Physics. vol. **293**, (2015) 4–13.

[14] V. E. TARASOV, *On chain rule for fractional derivatives*, Communications in Nonlinear Science and Numerical Simulation. vol. **30**, no. 1–3, (2016) 1–4.

[15] V. E. TARASOV, *Leibniz rule and fractional derivatives of power functions*, Journal of Computational and Nonlinear Dynamics. vol. **11**, no. 3, (2016) 031014.

[16] K. NISHIMOTO, *Fractional derivative and integral, Part I*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama), vol. **17**, (1976) 11–19.

[17] K. NISHIMOTO, *Nishimoto's fractional diff erintegration and the solution of Legendre's differential equation*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama). vol. **17**, (1976) 21–25.

[18] K. NISHIMOTO, *Osler's cut and Nishimoto's cut*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama). vol. **18**, (1977) 9–13.

[19] K. NISHIMOTO, *Fractional Calculus: Integrations and Differentiations of Arbitrary Order*, vol. **1**, Descartes Press, Koriyama, Japan, 1984.

[20] K. NISHIMOTO, *An Essence of Nishimoto's Fractional Calculus, Calculus of the 21st Century: Integrations and Differentiations of Arbitrary Order*, Descartes Press, Koriyama, 1991.

[21] S. OWA, K. NISHIMOTO, *On a fractional differintegral equation*, The Journal of the College of Engineering, Series B, (College of Engineering, Nihon University, Koriyama). vol. **24**, (1983) 67–72.

[22] H. M. SRIVASTAVA, S. OWA, K. NISHIMOTO, *Some fractional differintegral equations*, Journal of Mathematical Analysis and Applications, vol. **106**, no. 2, (1985) 360–366.

[23] N. YA. SONINE, *On differentiation with an arbitrary index*, (Sur la differentiation a indice quelconque), Sbornik Mathematics (Matematicheskii Sbornik), vol. **6**, no. 1, (1872) 1–36, (in Russian), URL: `http://mi.mathnet.ru/eng/msb7015` (`http://mi.mathnet.ru/eng/msb/v6/i1/p1`).

[24] A. V. LETNIKOV, *On explanation of the main propositions of differentiation theory with an arbitrary index (concerning one article)*, (Note relative a l'explication des principes fondamentaux de la theorie de la differentiation a indice quelconque (A propos d'un memoire)), Sbornik Mathematics (Matematicheskii Sbornik), vol. **6**, no. 4, (1873) 413–445, (in Russian), URL: `http://mi.mathnet.ru/eng/msb7039` (`http://mi.mathnet.ru/eng/msb/v6/i4/p413`).

[25] P. A. NEKRASOV, *General differentiation*, (Sur la differentiation generale), Sbornik Mathematics (Matematicheskii Sbornik), vol. **14**, no. 1, (1888) p. 45–168, (in Russian), URL: `http://mi.mathnet.ru/eng/msb7192` (`http://mi.mathnet.ru/eng/msb/v14/i1/p45`).

[26] T. J. OSLER, *Leibniz rule for fractional derivatives generalized and an application to infinite series*, SIAM Journal on Applied Mathematics, vol. **18**, no. 3, (1970) 658–674.

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Vasily E. Tarasov
Skobeltsyn Institute of Nuclear Physics
Lomonosov Moscow State University
Moscow 119991, Russia
e-mail: `tarasov@theory.sinp.msu.ru`