REMARK TO HISTORY OF FRACTIONAL DERIVATIVES ON COMPLEX PLANE: SONINE–LETNIKOV AND NISHIMOTO DERIVATIVES

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Abstract. An important part of fractional calculus is connected with the fractional-order integro-differentiation in the complex plane, which is a generalization of the well-known Cauchy differentiation formula. In some publications, this generalization is called the Nishimoto fractional derivatives. In our paper, we emphasize the fact that the generalization of the Cauchy’s differentiation formula to non-integer orders has been suggested by Sonine in 1872 more than a hundred years before Nishimoto’s works. The Sonine approach has been adjusted by Letnikov in 1873 and then it has been developed by Nekrasov in 1888.

Derivatives and integrals of non-integer orders [1, 2, 3, 4] have a long history [5]–[11] that is connected with the names of famous mathematicians Riemann, Liouville, Riesz, Grünwald, Letnikov, Sonine, Marchaud, Weyl, Riesz, Caputo and others. An application of non-standard properties of fractional-order integro-differentiation [12, 13, 14, 15] allows us to describe local and hereditarity system. Most fractional derivatives and integrals, which are used in applications, are considered on the real axis. An important role in application of fractional calculus can play fractional integrals and derivatives in the complex plane that are the extension of the Cauchy differentiation formula

\[ f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi \]  

(1)

to non-integer values of \( n \).

In noticeable number of articles, which devoted to fractional derivatives on complex plane, this extension is called the Nishimoto fractional derivatives. Nishimoto proposed an extension of the Cauchy formula (1) to fractional order in the papers [16, 17, 18] in 1976-1977 and the book [19] in 1986 (see also [20, 21, 22]).

These articles do not take into account the fact that the generalization of the Cauchy’s differentiation formula (1) to non-integer orders has been suggested by Sonine more than a hundred years ago in the paper [23] in 1872. The Sonine approach has been adjusted by Letnikov [24] in 1873. Then it has been developed by Nekrasov (see Theorem VII [25], page 87) in 1888 (the Nekrasov’s contribution has been noted and developed by Osler [26] in 1970). Some historical notes to formula (2) has been given in the book [1] on pages 435 and in Section 22 of [1]. The definition based on


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a generalization of the formula for differentiating the Cauchy type integral has been suggested in the form

\[ f^{(\alpha)}(z) := \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{C} \frac{f(\xi)d\xi}{(\xi - z)^{\alpha + 1}}. \]  

(2)

Therefore the fractional integro-differentiation in the complex plane, which is based on a generalization of the Cauchy’s differentiation formula, should be called the Sonine fractional derivative or the Sonine-Letnikov fractional derivative.

Note that an application of formula (2) requires precision aimed to single out a branch of the multivalued function \((\xi - z)^{\alpha + 1}\). It is usually achieved by means of a cut, which goes from the branching point to infinity or by fixing \(\arg(\xi - z)\) in some way. Different choices of a cut, which fixes the branch of the multivalued function \((\xi - z)^{\alpha + 1}\) in (2), and of the curve \(C\) gives different values of \(f^{(\alpha)}(z)\) in general [1].

REFERENCES


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