Fractional
Differential
Calculus
Volume 6, Number 2 (2016), 201–217 doi:10.7153/fdc-06-13

THE NEHARI MANIFOLD FOR A SINGULAR ELLIPTIC EQUATION INVOLVING THE FRACTIONAL LAPLACE OPERATOR

ABDELJABBAR GHANMI AND KAMEL SAOUDI

(*Communicated by M. Kirane*)

Abstract. In this work we study the following singular problem involving the fractional Laplace operator:

$$(\mathrm{P}_\lambda)\begin{cases}\mathscr{L}u=\frac{a(x)}{u^\gamma}+\lambda f(x,u) & \text{in } \Omega;\\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$, $N\geqslant 2$ be a bounded smooth domain, $a\in C(\Omega)$, λ is a positive parameter and $0<\gamma<1$, $2<r<2_s^*$ where $2_s^*=\frac{N2}{N-2s}$. Under appropriate assumptions on the function K and the function f and we employ the method of Nehari manifold in order to show the existence of $T_{r,\gamma}$ such that for all $\lambda\in(0,T_{r,\gamma})$, problem (P_λ) has at least two solutions.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N\geqslant 2$ and $2_s^*=\frac{2N}{N-2s}$. The purpose of this work is to study the existence of multiple solutions of the singular elliptic problem involving the non-local operator:

$$(\mathrm{P}_\lambda)\begin{cases}\mathscr{L}u=\frac{a(x)}{u^\gamma}+\lambda f(x,u) & \text{in } \Omega;\\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where $a\in C(\Omega)$, λ is a positive parameter, $0<\gamma<1$, $2<r<2_s^*$ and the linear non-local operator $\mathscr{L}$ is given by

$$\mathscr{L}u(x)=\frac{1}{2}\int_{\mathbb{R}^N}(u(x+y)+u(x-y)-2u(x))K(y)dy.$$

For $s\in(0,1)$, we introduce the fractional Sobolev space

$$H^s(\Omega)=\left\{u\in L^2(\Omega):\frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2s}{2}}}\in L^2(\Omega\times\Omega)\right\},$$

with the Gagliardo norm

$$\|u\|_{H^s(\Omega)}=|u|_2+\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}.$$

Mathematics subject classification (2010): 34B15, 37C25, 35R20.

Keywords and phrases: Fractional Laplace operator, the method of Nehari manifold, singular elliptic equation, multiple positive solutions.

Paper FDC-06-13

Denote
$$\mathscr{Q}=\mathbb{R}^{2N}\setminus\big((\mathbb{R}^N\setminus\Omega)\times(\mathbb{R}^N\setminus\Omega)\big)$$
and we define the space
$$X\overset{\text{def}}{=}\Big\{u:\mathbb{R}^N\to\mathbb{R}\,\text{Lebesgue measurable}:\ u\backslash_{\Omega}\in L^2(\Omega)$$
$$\text{and}\ \ (u(x)-u(y))\sqrt{K(x-y)}\in L^2(\mathscr{Q})\Big\}$$
with the norm
$$\|u\|_X=|u|_2+\left(\int_{\mathscr{Q}}|u(x)-u(y)|^2K(x-y)dxdy\right)^{\frac{1}{2}}.$$
Through this paper we consider the space
$$E=\{u\in X: u=0\ a.e.\ \text{in}\ \mathbb{R}^n\setminus\Omega\},$$
with the norm
$$\|u\|=\left(\int_{\mathscr{Q}}|u(x)-u(y)|^2K(x-y)dxdy\right)^{\frac{1}{2}}.$$
Note that the space E was often called X_0 in the previous literature, see e.g. [18, 19].

We stress that $(E,\|.\|)$ is a Hilbert space and the embedding $E\hookrightarrow L^{2^*}(\Omega)$ is continuous (for detail see [18]). Moreover, $C_0^2(\Omega)\subset E$, $X\subset H^s(\Omega)$ and $E\subset H^s(\mathbb{R}^N)$ (for detail see [19]).

Associated to the problem (P_λ) we define the functional $J_\lambda: E\to\mathbb{R}$ given by
$$J_\lambda(u)=\frac{1}{2}||u||^2-\frac{1}{1-\gamma}\int_\Omega a(x)|u(x)|^{1-\gamma}\mathrm{d}x-\frac{\lambda}{r}\int_\Omega F(x,|u(x)|)\,\mathrm{d}x,\ \ u\in E$$
where $F(x,s):=\int_0^s f(x,t)dt$. We say that $u\in E$ is a weak solution of problem (P_λ) if for every $v\in E$ we have:
$$\int_{\mathbb{R}^{2N}}(u(x)-u(y))((v(x)-v(y))K(x-y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_\Omega a(x)u(x)^{-\gamma}v(x)\,\mathrm{d}x+\lambda\int_\Omega f(x,u(x))v(x)\,\mathrm{d}x. \tag{1.1}$$
Note that u is a positive solution of problem (P_λ), if u is positive and verifies the equation
$$\frac{1}{2}||u||^2-\frac{1}{1-\gamma}\int_\Omega a(x)u(x)^{1-\gamma}\mathrm{d}x-\frac{\lambda}{r}\int_\Omega F(x,u(x))\,\mathrm{d}x=0.$$

Before giving our main results, let us briefly recall literature concerning related nonlinear equations involving fractional powers of the Laplace operator. Problem involving fractional Laplace operator has been given considerable attention since they are a arises in many physical phenomena, in probability and also in finance for more details see for instance [3, 11] and references therein. Meanwhile, elliptic equations involving integral fractional Laplace operator has been investigated in quite a large number of papers, such as [7, 9, 18, 19] and references therein. Some other results dealing with the existence of solutions concerning Dirichlet problem involving the spectral fractional laplacian has been treated in [4, 6, 21] and references therein. Note that, these two fractional operators (i.e. the 'integral' one and the 'spectral' one) are different. We refer the interested reader to [20] for a careful comparison of theses two operator.

Problems (P_λ) have been also studied with different elliptic operators. For Laplace operator and purely singular case, they have been studied by Crandall-Rabinowitz-Tartar [10]. After this paper, many authors have considered the problem above for Laplacian, p-Laplacian operators or fractional p-laplacian, using the technique used in [10] or a combination of this approach with the Nehari's and Perron's methods, among others, we would like to mention Coclite-Palmieri [8], Giacomoni-Saoudi [14] and references therein in the case of the Laplacian equation, In Giacomoni-Schindler-Takáč [15], the case of the p-Laplacian equation is considered and the corresponding quasilinear and singular N-Laplacian equation is considered in Saoudi-Kratou [17]. Ghanmi-Saoudi [13] proved the multiplicity of solutions of the quasi-linear singular $p-$ fractional equation using the method of the Nehari manifold.

The main goal of this paper is to show how the usual variational techniques can be extended to deal with singular fractional Laplace problem with boundary conditions. Hence, To obtain multiple (at least two distinct, positive) solutions of problem (P_λ), we combine some well-known fibering maps (i.e., maps of the form $t \to J_\lambda(tu)$, see (Alves-El Hamidi [1], Brown-Zhang [5]) and by minimization on the suitable subset of Nehari manifold.

Before stating our main results, we make the following assumptions throughout this paper : Let $f \in C(\Omega\times\mathbb{R},\mathbb{R})$ is positively homogeneous of degree $r-1$, that is, $f(x,tu) = t^{r-1}f(x,u)$ hold for all $(x,u) \in \Omega\times\mathbb{R}$ and we suppose that the function F satisfying suitable growth conditions. Precisely, we assume the following:

(H$_1$) $F : \overline{\Omega}\times\mathbb{R} \longrightarrow \mathbb{R}$ is homogeneous of degree r, that is,

$$F(x,tu) = t^r F(x,u)(t > 0) \text{ for all } x \in \overline{\Omega},\ u \in \mathbb{R}.$$

(H$_2$) $F^{\pm}(x,u) = max(\pm F(x,u),0) \neq 0$ for all $u \neq 0$.

From **(H$_1$)**, f leads to the so-called Euler identity

$$uf(x,u) = rF(x,u)$$

and

$$|F(x,u)| \leqslant C|u|^r \quad \text{for some constant } C > 0. \tag{1.2}$$

Let $K : \mathbb{R}^N \setminus \{0\} \longrightarrow (0,\infty)$ is a function satisfying the following properties:

(K$_1$) $\gamma K \in L^1(\mathbb{R})$, with $\gamma(x) = min(|x|^2,1)$.

(K$_2$) There exists $\mu > 0$ such that $K(x) > \mu|x|^{-(N+2s)}$, $\forall\, x \in \mathbb{R}^n \setminus \{0\}$.

(K$_3$) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

We give below the precise statements of results that we will prove.

THEOREM 1.1. *Suppose that the condition (H_1)–(H_2) and (K_1)–(K_3) are satisfied. Then, there exists $T_{r,\gamma} > 0$ such that, problem (P_λ) has at least two positive solutions for all $\lambda \in (0, T_{r,\gamma})$.*

This paper is organized as follows: The section 2 is devoted to proof some lemmas in preparation for the proof of our main result. While, existence of two solutions (Theorem 1.1) will be presented in section 3.

2. Main results

In this section, we collect some basic results that will be used in the forthcoming sections. Let $T_{r,\gamma}$ be a constant given by

$$T_{r,\gamma} = \frac{1+\gamma}{r-2}\left(\frac{r-2}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{1+\gamma}} \frac{S_{2^*}^{\frac{r+\gamma-1}{1+\gamma}}}{K\|a\|_\infty^{\frac{r-2}{1+\gamma}}|\Omega|^{\frac{(2^*-2)}{2^*(1+\gamma)}}},$$

where S_{2^*} is the best Sobolev constant of the embedding from E into $L^{2^*}(\Omega)$ given by

$$S_{2^*} := \inf_{u \in E\setminus\{0\}} \frac{\left(\int_{\mathcal{Q}} |u(x)-u(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}}}{\left(\int_\Omega |u|^{2^*}dx\right)^{\frac{1}{2^*}}}.$$

Define the constraint set

$$\mathcal{N}_\lambda := \{t(u)u : \ u \in E\setminus\{0\}\}$$

where $t(u)$ are the zeros of the map $\Phi_u : (0,\infty) \to \mathbb{R}$ defined as

$$\Phi_u(t) = \frac{t^2}{2}||u||^2 - \lambda\frac{t^r}{r}\int_\Omega F(x,u(x))dx - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega a(x)u(x)^{1-\gamma}dx.$$

Note that, it is clear that $u \in \mathcal{N}_\lambda$ if and only if

$$||u||^2 - \lambda\int_\Omega F(x,u(x)dx = \int_\Omega a(x)u(x)^{1-\gamma}dx \tag{2.1}$$

and it is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(t) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(1) = 0$.

To investigate the existence of multiple solutions, we decompose $\mathcal{N}_\lambda$ into three measurable sets defined as follows:

$$\mathcal{N}_\lambda^+ \stackrel{\text{def}}{=} \left\{u \in \mathcal{N}_\lambda : (\gamma+1)||u||^2 - \lambda(\gamma+r-1)\int_\Omega F(x,u(x))dx > 0\right\},$$

$$\mathcal{N}_\lambda^- \stackrel{\text{def}}{=} \left\{u \in \mathcal{N}_\lambda : (\gamma+1)||u||^2 - \lambda(\gamma+r-1)\int_\Omega F(x,u(x))dx < 0\right\},$$

$$\mathcal{N}_\lambda^0 \stackrel{\text{def}}{=} \left\{ u \in \mathcal{N}_\lambda : (\gamma+1)||u||^2 - \lambda(\gamma+r-1)\int_\Omega F(x,u(x))dx = 0 \right\}.$$

Our first result is the following

LEMMA 2.1. *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$. Then, using (1.2) and the fact that the embedding $E \hookrightarrow L^{2^*}(\Omega)$ is continuous combined with the Hölder's inequality, we obtain

$$\int_\Omega F(x,u(x))dx \leqslant C\int_\Omega |u|^r dx \leqslant C|\Omega|^{\frac{2^*}{2^*-r}}|u|_{2^*}^r. \tag{2.2}$$

Moreover, as above, one has

$$\int_\Omega a(x)u^{1-\gamma}dx \leqslant \|a\|_\infty \int_\Omega |u|^{1-\gamma}dx \leqslant \|a\|_\infty |\Omega|^{\frac{2^*}{2^*+\gamma-1}}|u|_{2^*}^{1-\gamma}. \tag{2.3}$$

Consequently, from (2.2) and (2.3), we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}||u||^2 - \frac{1}{1-\gamma}\int_\Omega \frac{a(x)}{u(x)^{\gamma-1}}dx - \frac{\lambda}{r}\int_\Omega F(x,u(x))dx \\ &= \frac{r-2}{2r}||u||^2 - \frac{r+\gamma-1}{r(\gamma-1)}\int_\Omega a(x)u^{1-\gamma}dx \\ &\geqslant \frac{r-2}{2r}S_{2^*}|u|_{2^*}^2 - \frac{r+\gamma-1}{r(\gamma-1)}\|a\|_\infty|\Omega|^{\frac{2^*}{2^*+\gamma-1}}|u|_{2^*}^{1-\gamma}. \end{aligned}$$

Now, since $0<\gamma<1$ and $2<r$, the functional J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. which give the proof of the Lemma 2.1. $\square$

LEMMA 2.2. *Let $\lambda \in (0, T_{r,\gamma})$. Then, there exist t_0^+ and t_0^- such that*

$$\Phi_u(t_0^+) = \lambda\int_\Omega F(x,u)\,\mathrm{d}x = \Phi_u(t_0^-)$$

and

$$\Phi_u'(t_0^+) < 0 < \Phi_u'(t_0^-);$$

that is, $t_0^+u \in \mathcal{N}_\lambda^+$ and $t_0^-u \in \mathcal{N}_\lambda^-$.

Proof. From the definition of the function $\Phi_u(t)$ the function $\Phi_u'(t)$ is defined by

$$\Phi_u'(t) = t||u||^2 - \lambda t^{r-1}\int_\Omega F(x,u(x))dx - t^{-\gamma}\int_\Omega a(x)u(x)^{1-\gamma}dx$$

Then, for $\Phi_u'(t)=0$ it is simple to verify that Φ_u attains it's maximum at

$$t_{\max} = \left(\frac{(r+\gamma-1)\int_\Omega a(x)u^{1-\gamma}dx}{(r-2)||u||^2}\right)^{\frac{1}{\gamma+1}}.$$

Moreover, $\Phi_u'(t) > 0$ for all $0 < t < t_{\max}$ and $\Phi_u'(t) < 0$ for all $t > t_{\max}$. On the other hand,

$$\Phi(t_{\max}) = \frac{\gamma+1}{r-2}\left(\frac{r-2}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{\gamma+1}} \frac{||u||^{\frac{2(r+\gamma-1)}{\gamma+1}}}{(\int_\Omega a(x)u^{1-\gamma}dx)^{\frac{r-2}{\gamma+1}}}.$$

Using (2.2) and (2.3), one has

$$\begin{aligned}
&\Phi_u(t_{\max}) - \lambda \int_\Omega F(x,u(x))dx \\
&\geqslant \frac{\gamma+1}{r-2}\left(\frac{r-2}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{\gamma+1}} \frac{(S_{2^*}|u|_{2^*}^2)^{\frac{r+\gamma-1}{\gamma+1}}}{\left(\|a\|_\infty|\Omega|^{\frac{2^*+\gamma-1}{2^*}}|u|_{2^*}^{1-\gamma}\right)^{\frac{r-2}{\gamma+1}}} - \lambda C|\Omega|^{\frac{2^*-r}{2^*}}|u|_{2^*}^r, \\
&= \frac{\gamma+1}{r-2}\left(\frac{r-2}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{\gamma+1}} \frac{S_{2^*}^{\frac{r+\gamma-1}{\gamma+1}}|u|_{2^*}^{\frac{2(r+\gamma-1)}{\gamma+1}}|u|_{2^*}^{-\frac{(r-2)(1-\gamma)}{\gamma+1}}}{\left(\|a\|_\infty|\Omega|^{\frac{2^*+\gamma-1}{2^*}}\right)^{\frac{r-2}{\gamma+1}}} - \lambda C|\Omega|^{\frac{2^*-r}{2^*}}|u|_{2^*}^r \\
&= C|\Omega|^{\frac{2^*-r}{2^*}}\left(T_{r,\gamma}-\lambda\right)|u|_{2^*}^r > 0, \qquad (2.4)
\end{aligned}$$

for all $\lambda \in (0, T_{r,\gamma})$. Hence, there exist tow numbers denoted t_0^+, t_0^- such that $0 < t_0^+ < t_{\max} < t_0^-$ and verifies

$$\Phi_u(t_0^+) = \lambda \int_\Omega F(x,u)\,\mathrm{d}x = \Phi_u(t_0^-)$$

and

$$\Phi_u'(t_0^+) < 0 < \Phi_u'(t_0^-).$$

This implies that, $t_0^+ u \in \mathcal{N}_\lambda^+$ and $t_0^- u \in \mathcal{N}_\lambda^-$. The proof of Lemma 2.2 is now completed. $\square$

As a consequence of Lemma 2.2, we have the following result:

LEMMA 2.3. *For all $\lambda \in (0, T_{r,\gamma})$ the set $\mathcal{N}_\lambda^\pm \neq \emptyset$ and the set $\mathcal{N}_\lambda^0 = \emptyset$. Moreover, $\mathcal{N}_\lambda^-$ is a closed set in $E-$topology.*

Proof. From Lemma 2.2, we can assume that $\mathcal{N}^\pm$ are non-empty sets for $\lambda \in (0, T_{r,\gamma})$. Now, to prove the result, we proceed by contradiction. For, this purpose there exists $u_0 \in \mathcal{N}_\lambda^0$. It follows that

$$(\gamma+1)\|u_0\|^2 - \lambda(r+\gamma-1)\int_\Omega F(x,u_0(x))dx = 0,$$

that is,

$$\begin{aligned}
0 &= \|u_0\|^2 - \int_\Omega a(x)u_0^{1-\gamma}dx - \lambda\int_\Omega F(x,u_0)dx \\
&= \frac{r-2}{r+\gamma-1}\|u_0\|^2 - \int_\Omega a(x)u_0^{1-\gamma}dx.
\end{aligned}$$

Using, (2.4) we get

$$\begin{aligned}0 &< \phi_{u_0}(t_{\max}) - \lambda \int_\Omega F(x,u_0)dx \\ &= \frac{\gamma+1}{r-2}\left(\frac{r-2}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{\gamma+1}} \frac{||u_0||^{\frac{2(r+\gamma-1)}{\gamma+1}}}{\left(\int_\Omega a(x)u_0^{1-\gamma}dx\right)^{\frac{r-2}{\gamma+1}}} - \lambda \int_\Omega F(x,u_0)dx \\ &\leqslant \frac{\gamma+1}{r-2}\left(\frac{r-2}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{\gamma+1}} \frac{||u_0||^{\frac{2(r+\gamma-1)}{\gamma+1}}}{\left(\frac{(r-2)\|u_0\|^2}{r+\gamma-1}\right)^{\frac{r-2}{\gamma+1}}} - \frac{\gamma+1}{r+\gamma-1}\|u_0\|^2 = 0 \end{aligned} \tag{2.5}$$

which is impossible. Thus, $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, T_{r,\gamma})$. Now, to prove that $\mathcal{N}_\lambda^-$ is a closed set for all $\lambda \in (0, T_{r,\gamma})$. We can introduce the sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that $u_n \to u$ in E. Since $\{u_n\} \subset \mathcal{N}_\lambda^-$, then we have

$$||u_n||^2 - \int_\Omega a(x)u_n^{1-\gamma}dx - \lambda \int_\Omega F(x,u_n)dx = 0,$$

and

$$(\gamma+1)||u_n||^2 - \lambda(\gamma+r-1)\int_\Omega F(x,u_n)dx < 0. \tag{2.6}$$

Therefore, we get

$$||u||^2 - \int_\Omega a(x)u^{1-\gamma}dx - \lambda \int_\Omega F(x,u)dx = 0,$$

and

$$(\gamma+1)||u||^2 - \lambda(\gamma+r-1)\int_\Omega F(x,u)dx \leqslant 0,$$

which implies that $u \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^- = \mathcal{N}_\lambda^-$. Thus $u \in \mathcal{N}_\lambda^-$ for all $\lambda \in (0, T_{r,\gamma})$. This completes the proof of the Lemma 2.3. □

LEMMA 2.4. *Given $u \in \mathcal{N}_\lambda^-$ (respectively $\mathcal{N}_\lambda^+$) with $u \geqslant 0$, for all $v \in E$ with $v \geqslant 0$, there exist $\varepsilon > 0$ and a continuous function h such that for all $s \in \mathbb{R}$ with $|s| < \varepsilon$ we have*

$$h(0) = 1 \quad \textit{and} \quad h(s)(u+sv) \in \mathcal{N}_\lambda^- \ (\textit{respectively } \mathcal{N}_\lambda^+).$$

Proof. We introduce the function $\psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ define by:

$$\psi(t,s) = t^{\gamma+1}||u+sv||^2 - \int_\Omega a(x)(u+sv)^{1-\gamma}dx - \lambda t^{r+\gamma-1}\int_\Omega F(x,u+sv)dx.$$

That is, the first derivative of the function ψ is given by

$$\psi_t(t,s) = (\gamma+1)t^\gamma||u+sv||^2 - \lambda(r+\gamma-1)t^{r+\gamma-2}\int_\Omega F(x,u+sv)dx,$$

which is continuous on $\mathbb{R}\times\mathbb{R}$. Now, since $u\in\mathcal{N}_\lambda^-\subset\mathcal{N}_\lambda$, we have that that $\psi(1,0)=0$, and

$$\psi_t(1,0)=(\gamma+1)||u||^2-\lambda(r+\gamma-1)\int_\Omega F(x,u)dx<0.$$

Thus, using the implicit function theorem on the function ψ at the point $(1,0)$ we get the existence of a constant $\delta>0$ and a function h such that

$$h(0)=1,\ h(s)(u+sv)\in\mathcal{N}_\lambda,\ \forall\, s\in\mathbb{R},\ |s|<\delta.$$

Hence, taking $\varepsilon>0$ possibly smaller enough, we get

$$h(s)(u+sv)\in\mathcal{N}_\lambda^-,\ \forall\, s\in\mathbb{R},\ |s|<\varepsilon.$$

The case $u\in\mathcal{N}_\lambda^+$ may be obtained in the same way. Therefore the proof of Lemma 2.4 is now completed. □

3. Multiplicity of solutions to the problem (P_λ) for all $\lambda\in(0,T_{r,\gamma})$

Since $J_\lambda(u)=J_\lambda(|u|)$, we can assume that all the price elements in $\mathcal{N}_\lambda$ are nonnegative. On the other hand, according to Lemma 2.1 and Lemma 2.3, for all $\lambda\in(0,T_{r,\gamma})$

$$m^+:=\inf_{u\in\mathcal{N}_\lambda^+}J_\lambda(u)\ \text{and}\ m^-:=\inf_{u\in\mathcal{N}_\lambda^-}J_\lambda(u)$$

are well defined. Moreover, for all $u\in\mathcal{N}_\lambda^+$, it follows that

$$(\gamma+1)||u||^2-\lambda(\gamma+r-1)\int_\Omega F(x,u(x))dx>0,$$

and consequently, since $0<\gamma<1$, $2<r$ and $u\neq0$, we have

$$\begin{aligned}J_\lambda(u)&=\frac{1}{2}||u||^2-\frac{1}{1-\gamma}\int_\Omega a(x)u(x)^{1-\gamma}dx-\frac{\lambda}{r}\int_\Omega F(x,u(x))dx\\&=\Big(\frac{1}{2}-\frac{1}{1-\gamma}\Big)||u||^2+\lambda\Big(\frac{1}{1-\gamma}-\frac{1}{r}\Big)\int_\Omega F(x,u(x))dx\\&<\frac{-1-\gamma}{2(1-\gamma)}||u||^2+\frac{\gamma+1}{r(1-\gamma)}||u||^2\\&=-\frac{(r-2)(\gamma+1)}{2r(1-\gamma)}||u||^2<0.\end{aligned}$$

Thus,

$$m^+=\inf_{u\in\mathcal{N}_\lambda^+}J_\lambda(u)<0 \tag{3.1}$$

for all $\lambda\in(0,T_{r,\gamma})$.

Proof of Theorem 1.1. The proof is done in two steps:

Step 1: (P_λ) have a positive solution in $\mathcal{N}_\lambda^+$.

Let us consider the sequence $\{u_n\}\subset\mathcal{N}_\lambda^+$ and applying Ekeland's variational principle (see [2]), we obtain

(i) $J_\lambda(u_n) < m^+ + \frac{1}{N}$,

(ii) $J_\lambda(u) \geqslant J_\lambda(u_n) - \frac{1}{N}\|u - u_n\|$, for all $u \in \mathcal{N}^+$.

Since $J_\lambda(u) = J_\lambda(|u|)$, we can assume that $u_n(x) \geqslant 0$. Consequently, as J_λ is coercive on $\mathcal{N}_\lambda$, $\{u_n\}$ is a bounded sequence in E, going to a sub-sequence denoted by $\{u_n\}$, and $u_0 \geqslant 0$ such that $u_n \rightharpoonup u_0$, weakly in E, $u_n \to u_0$, strongly in $L^{1-\gamma}(\Omega)$, and $L^s(\Omega)$, for $1 \leqslant s < 2^*$, and $u_n(x) \to u_0(x)$, a.e. in Ω, as $n \to \infty$. Now, from (3.1) and using the weak lower semi-continuity of norm $J_\lambda(u_0) \leqslant \liminf J_\lambda(u_n) = \inf\limits_{\mathcal{N}^+} J_\lambda$, we see that $u_0 \not\equiv 0$ in Ω.

Claim 1. $u_0(x) > 0$ a.e. in Ω.

Firstly, we start by observing that, since $u_n \in \mathcal{N}_\lambda^+$, one has

$$(\gamma+1)||u_n||^2 - \lambda(\gamma+r-1)\int_\Omega F(x,u_n)dx > 0 \tag{3.2}$$

equivalent to

$$(\gamma+1)\int_\Omega a(x)u_n^{1-\gamma}dx - \lambda(r-2)\int_\Omega F(x,u_n)dx > C_1. \tag{3.3}$$

Now, using Hölder's inequality, we get that, as $n \to \infty$,

$$\begin{aligned}\int_\Omega u_n^{1-\gamma}\mathrm{d}x &\leqslant \int_\Omega u_0^{1-\gamma}\mathrm{d}x + \int_\Omega |\, u_n - u_0 \,|^{1-\gamma}\,\mathrm{d}x\\ &\leqslant \int_\Omega u_0^{1-\gamma}\mathrm{d}x + C \parallel u_n - u_0 \parallel_{L^2(\Omega)}^{1-\gamma}\\ &= \int_\Omega u_0^{1-\gamma}\mathrm{d}x + o(1).\end{aligned}$$

Similarly

$$\begin{aligned}\int_\Omega u_0^{1-\gamma}\mathrm{d}x &\leqslant \int_\Omega u_n^{1-\gamma}\mathrm{d}x + \int_\Omega |\, u_n - u_0 \,|^{1-\gamma}\,\mathrm{d}x\\ &\leqslant \int_\Omega u_0^{1-\gamma}\mathrm{d}x + C \parallel u_n - u_0 \parallel_{L^2(\Omega)}^{1-\gamma}\\ &= \int_\Omega u_n^{1-\gamma}\mathrm{d}x + o(1).\end{aligned}$$

Thus,

$$\int_\Omega u_n^{1-\gamma}\mathrm{d}x = \int_\Omega u_0^{1-\gamma}\mathrm{d}x + o(1). \tag{3.4}$$

On the other hand, using Vitali's convergence Theorem, we have

$$\lim_{n\to\infty}\int_\Omega F(x,u_n)dx = \int_\Omega F(x,u_0)dx. \tag{3.5}$$

Therefore, from (3.4) and (3.5), it follows that

$$\begin{aligned}&\lim_{n\to\infty}\left((\gamma+1)\int_\Omega a(x)u_n^{1-\gamma}dx - \lambda(r-2)\int_\Omega F(x,u_n)dx\right)\\ &= (\gamma+1)\int_\Omega a(x)u_0^{1-\gamma}dx - \lambda(r-2)\int_\Omega F(x,u_0)dx \geqslant 0.\end{aligned}$$

Now, we assume that

$$(\gamma+1)\int_\Omega a(x)u_0^{1-\gamma}dx-\lambda(r-2)\int_\Omega F(x,u_0)dx=0. \tag{3.6}$$

Consequently, combining (3.4)–(3.5) and the weakly lower semi-continuity of the norm, we obtain

$$\begin{aligned}0 &\geqslant ||u_0||^2-\int_\Omega a(x)u_0^{1-\gamma}dx-\lambda\int_\Omega F(x,u_0)dx\\ &=||u_0||^2-\frac{r+\gamma-1}{r-2}\int_\Omega a(x)u_0^{1-\gamma}dx\\ &=||u_0||^2-\lambda\frac{r+\gamma-1}{\gamma+1}\int_\Omega F(x,u_0)dx\end{aligned} \tag{3.7}$$

and consequently, from (2.5) one has a contradiction. That is

$$(\gamma+1)\int_\Omega a(x)u_0^{1-\gamma}dx-\lambda(r-2)\int_\Omega F(x,u_0)dx>0. \tag{3.8}$$

Now, let us consider the function $\varphi\in E$, with $\varphi\geqslant 0$. From Lemma 2.3 with $u=u_n$, there exits a sequence of continuous functions $g_n=g_n(t)$ such that $g_n(t)(u_n+t\varphi)\in\mathscr{N}_\lambda^+$ and $g_n^2(0)=1$. So,

$$\begin{gathered}[g_n(t)]^2||u_n+t\varphi||^2-[g_n(t)]^{1-\gamma}\int_\Omega a(x)(u_n+t\varphi)^{1-\gamma}dx\\ -\lambda[g_n(t)]^r\int_\Omega F(x,u_n+t\varphi)dx=0.\end{gathered}$$

Since

$$||u_n||^2-\int_\Omega a(x)u_n^{1-\gamma}dx-\lambda\int_\Omega F(x,u_n)dx=0, \tag{3.9}$$

it follows that, for t small enough

$$\begin{aligned}0&=(g_n(t)^2-1)||u_n+t\varphi||^2+\left(||u_n+t\varphi||^2-||u_n||^2\right)\\ &\quad-(g_n(t)^{1-\gamma}-1)\int_\Omega a(x)(u_n+t\varphi)^{1-\gamma}dx-\int_\Omega a(x)\left((u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}\right)dx\\ &\quad-\lambda(g_n(t)^r-1)\int_\Omega F(x,u_n+t\varphi)dx-\lambda\int_\Omega F(x,u_n+t\varphi)-F(x,u_n)dx\\ &\leqslant(g_n(t)^2-1)||u_n+t\varphi||^2+\left(||u_n+t\varphi||^2-||u_n||^2\right)\\ &\quad-(g_n(t)^{1-\gamma}-1)\int_\Omega a(x)(u_n+t\varphi)^{1-\gamma}dx\\ &\quad-\lambda(g_n(t)^r-1)\int_\Omega F(x,u_n+t\varphi)dx-\lambda\int_\Omega F(x,u_n+t\varphi)-F(x,u_n)dx,\end{aligned}$$

dividing by $t>0$ and passing to the limit as $t\to 0$, we get

$$\begin{aligned}0 &\leqslant 2g_n'(0)\|u_n\|^2 - g_n'(0)(1-\gamma)\int_\Omega u_n^{1-\gamma}\mathrm{d}x - \lambda r g_n'(0)\int_\Omega F(x,u_n)\mathrm{d}x\\ &\quad + 2\int_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)}{|x-y|^{N+2s}}dxdy\\ &= g_n'(0)\left((2-r)\|u_n\|^2 + (r+\gamma-1)\int_\Omega u_n^{1-\gamma}\mathrm{d}x\right)\\ &\quad + 2\int_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)}{|x-y|^{N+2s}}dxdy,\end{aligned} \tag{3.10}$$

where $g_n'(0)\in[-\infty,\infty]$ denotes the right derivate of $g_n(t)$ at zero. Since $u_n\in\mathcal{N}^+$, $g_n'(0)\neq -\infty$. For simplicity, we assume that the right derivate of g_n at $t=0$ exists. Moreover, from (3.7) $g_n'(0)$ is uniformly bounded from below. Now, using the condition (ii) ,

$$\begin{aligned}&|g_n(t)-1|\frac{\|u_n\|}{n} + tg_n(t)\frac{\|\varphi\|}{n}\\ &\geqslant J_\lambda(u_n) - J_\lambda(g_n(t)(u_n+t\varphi))\\ &= -\frac{\gamma+1}{2(1-\gamma)}\|u_n\|^2 + \lambda\frac{r+\gamma-1}{r(1-\gamma)}\int_\Omega F(x,u_n)dx + \frac{\gamma+1}{2(1-\gamma)}g_n(t)^2\|u_n+t\varphi|^2\\ &\quad -\lambda\frac{r+\gamma-1}{r(1-\gamma)}g_n(t)^r\int_\Omega F(x,u_n+t\varphi)dx\\ &= \frac{\gamma+1}{2(1-\gamma)}\left[\|u_n+t\varphi\|^2 - \|u_n\|^2 + (f_n(t)^2-1)\|u_n+t\varphi\|^2\right]\\ &\quad -\lambda\frac{r+\gamma-1}{r(1-\gamma)}\left[\int_\Omega F(x,u_n+t\varphi)-F(x,u_n)dx + (g_n(t)^r-1)\int_\Omega F(x,u_n+t\varphi)dx\right].\end{aligned}$$

Then, dividing the above inequality by $t>0$, and passing to the limit $t\to 0$, we obtain

$$\begin{aligned}&\frac{1}{N}\left(|g_n'(0)|\|u_n\| + tg_n(t)\|\varphi\|\right)\\ &\geqslant \frac{g_n'(0)}{1-\gamma}\left[\frac{r+\gamma-1}{(1-\gamma)}\int_\Omega a(x)u_n^{1-\gamma}dx - \frac{\gamma+1}{(1-\gamma)}\|u_n\|^2\right]\\ &\quad + \frac{2(\gamma+1)}{(1-\gamma)}\int_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)}{|x-y|^{N+2s}}dxdy\\ &\quad -\lambda\left(\frac{r+\gamma-1}{(1-\gamma)}\right)\int_\Omega f(x,u_n)\varphi dx.\end{aligned} \tag{3.11}$$

Then from (3.8), there exists a positive constant C such that

$$-\frac{1}{1-\gamma}\left((r-2)\|u_n\|^2 - (r+\gamma-1)\int_\Omega u_n^{1-\gamma}\mathrm{d}x\right) - \frac{||u_n||}{n} \geqslant C > 0. \tag{3.12}$$

Thus, according to (3.12) and (3.11), $g_n'(0)$ is uniformly bounded from above. Consequently,

$$g_n'(0) \quad \text{is uniformly bounded for } n \text{ large enough.} \tag{3.13}$$

Thus from condition (ii) it follows that for $t>0$ small enough,

$$J_\lambda(u_n) \leqslant J_\lambda(g_n(t)(u_n+t\varphi)) + \frac{1}{N}||g_n(t)(u_n+t\varphi)-u_n||. \tag{3.14}$$

That is,

$$\begin{aligned}
&\frac{1}{N}\left(|g_n(t)-1|\|u_n\|+tg_n(t)\|\varphi\|\right)\\
&\geqslant \frac{1}{N}\|g_n(t)(u_n+t\varphi)-u_n\|\\
&\geqslant J_\lambda(u_n)-J_\lambda\left(g_n(t)(u_n+t\varphi)\right)\\
&= -\frac{g_n^2(t)-1}{2}\|u_n\|^2+\frac{g_n^{1-\gamma}(t)-1}{1-\gamma}\int_\Omega a(x)u_n^{1-\gamma}dx\\
&\quad +\frac{g_n^{1-\gamma}(t)}{1-\gamma}\int_\Omega a(x)\left((u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}\right)dx+\frac{g_n^2(t)}{2}\left(\|u_n\|^2-\|u_n+t\varphi\|^2\right)\\
&\quad +\frac{\lambda}{r}g_n^r(t)\int_\Omega F(x,u_n+t\varphi)-F(x,u_n)dx+\frac{\lambda}{r}(g_n^r(t)-1)\int_\Omega F(x,u_n)dx,
\end{aligned}$$

Then dividing by $t>0$, and passing to the limit $t\to 0$, we obtain

$$\begin{aligned}
&\frac{1}{N}\left(|g_n'(0)|\|u_n\|+\|\varphi\|\right)\\
&\geqslant -g_n'(0)\left[\|u_n\|^2+\int_\Omega a(x)u_n^{1-\gamma}dx+\lambda\int_\Omega F(x,u_n)dx\right]+\lambda\int_\Omega f(x,u_n)\varphi dx\\
&\quad -\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)}{|x-y|^{N+2s}}dxdy\\
&\quad +\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega\left(\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}x\right)\\
&= -\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)}{|x-y|^{N+2s}}dxdy\\
&\quad +\lambda\int_\Omega f(x,u_n)\varphi dx+\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega a(x)\left(\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\right)\mathrm{d}x.
\end{aligned} \tag{3.15}$$

From (3.15) we deduce that

$$\begin{aligned}
&\liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\mathrm{d}x\\
&\leqslant \int_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)}{|x-y|^{N+2s}}dxdy\\
&\quad -\lambda\int_\Omega f(x,u_n)\varphi dx+\frac{1}{N}\left(|f_n'(0)|\|u_n\|+\|\varphi\|\right)
\end{aligned} \tag{3.16}$$

Since

$$a(x)[(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}]\geqslant 0,\quad \forall x\in\Omega,\ \forall t>0,$$

using Fatou's Lemma we get

$$\int_\Omega a(x)u_n^{-\gamma}\varphi\,\mathrm{d}x \leqslant \liminf_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega a(x)\left(\frac{(u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma}}{t}\,\mathrm{d}x\right).$$

Hence, using (3.16), it follows that

$$\begin{aligned}\int_\Omega u_n^{-\gamma}\varphi\,\mathrm{d}x \leqslant \int_{\mathbb{R}^{2N}} &\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)}{|x-y|^{N+2s}}dxdy\\ &-\lambda\int_\Omega f(x,u_n)\varphi\,\mathrm{d}x + f_n'(0)\frac{\|u_n\|+\|\varphi\|}{n}\end{aligned}$$

for n large enough. Using (3.13) and applying Fatou's Lemma again, to conclude that $u_0(x)>0$ a.e. in Ω and

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))(u_0(x)-u_0(y))K(x-y)}{|x-y|^{N+2s}}dxdy\\ &-\int_\Omega a(x)u_0^{-\gamma}\varphi\,\mathrm{d}x-\lambda\int_\Omega f(x,u_0)\varphi\,\mathrm{d}x\geqslant 0,\end{aligned} \tag{3.17}$$

for all $\varphi\in E$, with $\varphi\geqslant 0$. Now, we prove that $u_0\in\mathscr{N}_\lambda^+$ for all $\lambda\in(0,T_{r,\gamma})$. Then, choosing $\varphi=u_0$ in (3.17), we get

$$\|u_0\|^2\geqslant\lambda\int_\Omega F(x,u_0)dx+\int_\Omega a(x)(u_0)^{1-\gamma}dx.$$

On the other hand, from (3.7) it follows that,

$$\|u_0\|^2\leqslant\lambda\int_\Omega F(x,u_0)dx+\int_\Omega a(x)(u_0)^{1-\gamma}dx.$$

Thus

$$\|u_0\|^2=\lambda\int_\Omega F(x,u_0)dx+\int_\Omega a(x)(u_0)^{1-\gamma}dx, \tag{3.18}$$

this implies that $u^+\in\mathscr{N}_\lambda$. Moreover from (3.9), ones gets

$$\lim_{n\to\infty}\|u_n\|^2=\lambda\int_\Omega F(x,u^+)dx+\int_\Omega a(x)(u^+)^{1-\gamma}dx.$$

Hence according to (3.18), we have $u_n\to u_0$ in E as $n\to\infty$. In particular, combining (3.8) with (3.18), we obtain

$$(\gamma+1)||u_0||^2-\lambda(\gamma+r-1)\int_\Omega F(x,u_0)dx>0,$$

and therefore $u_0\in\mathscr{N}_\lambda^+$.

Claim 2. u_0 is a solution of problem (P_λ).

Our proof is inspired by Ghanmi-Saoudi [13]. Let $\phi \in E$ and $\varepsilon > 0$. We define $\Psi \in E$ by $\Psi := (u_0+\varepsilon\phi)^+$ where $(u_0+\varepsilon\phi)^+ = \max\{u_0+\varepsilon\phi, 0\}$. Let $\Omega_\varepsilon = \{u_0+\varepsilon\phi \leqslant 0\}$ and $\Omega^\varepsilon = \{u_0+\varepsilon\phi < 0\}$. Replace φ with Ψ in (3.17) and combining with (3.18) we obtain

$$
\begin{aligned}
0 \leqslant & \int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\psi(x)-\psi(y))K(x-y)}{|x-y|^{N+2s}} dxdy \\
& - \int_\Omega a(x) u_0^{-\gamma} \Psi \, \mathrm{d}x - \lambda \int_\Omega f(x,u_0)\Psi \, \mathrm{d}x \\
= & \int_{\{(x,y)\in\, \Omega^\varepsilon\times\Omega^\varepsilon\}} \frac{(u_0(x)-u_0(y))((u_0+\varepsilon\phi)(x)-(u_0+\varepsilon\phi)(y))K(x-y)}{|x-y|^{N+2s}} dxdy \\
& - \int_{\{(x,y)\in\, \Omega^\varepsilon\times\Omega^\varepsilon\}} \Big(a(x) u_0^{-\gamma}(u_0+\varepsilon\phi) + \lambda f(x,u_0)(u_0+\varepsilon\phi) \Big) \, \mathrm{d}x \\
= & \int_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon\phi)(x)-(u_0+\varepsilon\phi)(y))}{|x-y|^{N+2s}} dxdy \\
& - \int_\Omega \Big(a(x) u_0^{-\gamma}(u_0+\varepsilon\phi) + \lambda f(x,u_0)(u_0+\varepsilon\phi) \Big) \, \mathrm{d}x \\
= & \int_{\{(x,y)\in\, \Omega_\varepsilon\times\Omega_\varepsilon\}} \frac{(u_0(x)-u_0(y))((u_0+\varepsilon\phi)(x)-(u_0+\varepsilon\phi)(y))K(x-y)}{|x-y|^{N+2s}} dxdy \\
& - \int_{\{(x,y)\in\, \Omega_\varepsilon\times\Omega_\varepsilon\}} \Big(a(x) u_0^{-\gamma}(u_0+\varepsilon\phi) + \lambda f(x,u_0)(u_0+\varepsilon\phi) \Big) \, \mathrm{d}x \\
= & \|u_0\|^2 - \int_\Omega u_0^{1-\gamma} \, \mathrm{d}x - \lambda \int_\Omega F(x,u_0) \, \mathrm{d}x - \int_\Omega \Big(u_0^{-\gamma}\phi + \lambda f(x,u_0)\phi \Big) \, \mathrm{d}x \\
& + \varepsilon \int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\phi(x)-\phi(y))K(x-y)}{|x-y|^{N+2s}} dxdy \\
& - \int_{\{(x,y)\in\Omega_\varepsilon\times\Omega_\varepsilon\}} \frac{(u_0(x)-u_0(y))((u_0+\varepsilon\phi)(x)-(u_0+\varepsilon\phi)(y))K(x-y)}{|x-y|^{N+2s}} dxdy \\
& - \int_{\{(x,y)\in\, \Omega_\varepsilon\times\Omega_\varepsilon\}} \Big(a(x) u_0^{-\gamma}(u_0+\varepsilon\phi) + \lambda f(x,u_0)(u_0+\varepsilon\phi) \Big) \, \mathrm{d}x \\
= & \, \varepsilon \int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\phi(x)-\phi(y))K(x-y)}{|x-y|^{N+2s}} dxdy - \varepsilon \int_\Omega \Big(u_0^{-\gamma}\phi + \lambda f(x,u_0)\phi \Big) \, \mathrm{d}x \\
& - \int_{\{(x,y)\in\, \Omega_\varepsilon\times\Omega_\varepsilon\}} \frac{(u_0(x)-u_0(y))((u_0+\varepsilon\phi)(x)-(u_0+\varepsilon\phi)(y))K(x-y)}{|x-y|^{N+2s}} dxdy \\
& - \varepsilon \int_{\{(x,y)\in\, \Omega_\varepsilon\times\Omega_\varepsilon\}} \Big(a(x) u_0^{-\gamma}(u_0+\varepsilon\phi) + \lambda f(x,u_0)(u_0+\varepsilon\phi) \Big) \, \mathrm{d}x \\
\leqslant & \, \varepsilon \int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\phi(x)-\phi(y))K(x-y)}{|x-y|^{N+2s}} dxdy \\
& - \varepsilon \int_\Omega \Big(a(x) u_0^{-\gamma}\phi + \lambda f(x,u_0)\phi \Big) \, \mathrm{d}x \\
& - \int_{\{(x,y)\in\, \Omega_\varepsilon\times\Omega_\varepsilon\}} \frac{(u_0(x)-u_0(y))((u_0+\varepsilon\phi)(x)-(u_0+\varepsilon\phi)(y))K(x-y)}{|x-y|^{N+2s}} dxdy.
\end{aligned}
$$

By Claim 1 we derive that the measure of the domain of integration Ω_ε tends to zero as $\varepsilon \to 0^+$. It follows as $\varepsilon \to 0^+$ that,

$$\int_{\{(x,y)\in \Omega_\varepsilon \times \Omega_\varepsilon\}} \frac{(u_0(x)-u_0(y))((u_0+\varepsilon\phi)(x)-(u_0+\varepsilon\phi)(y))K(x-y)}{|x-y|^{N+2s}} dxdy \to 0.$$

Dividing by ε and letting $\varepsilon \to 0^+$, we get

$$\int_{\mathbb{R}^{2N}} \frac{(u_0(x)-u_0(y))(\phi(x)-\phi(y))K(x-y)}{|x-y|^{N+2s}} dxdy - \int_\Omega \left(u_0^{-\gamma}\phi + \lambda f(x,u_0)\phi\right) \mathrm{d}x \geqslant 0.$$

Since the equality holds if we replace φ by $-\varphi$ which implies that u_0 is a positive solution of problem (P_λ).

Step 2: (P_λ) have a positive solution in $\mathscr{N}_\lambda^-$.

Similarly to the first Step, applying Ekeland's variational principle to the minimization problem $m^- = \inf_{v\in\mathscr{N}_\lambda^-} J_\lambda(v)$ there exists a sequence $\{v_n\} \subset \mathscr{N}_\lambda^-$ such that

(i) $J_\lambda(v_n) < m^+ + \frac{1}{N}$,

(ii) $J_\lambda(v) \geqslant J_\lambda(v_n) - \frac{1}{N}\|v-v_n\|$, for all $v \in \mathscr{N}^-$.

Since $J_\lambda(v) = J_\lambda(|v|)$, we can assume that $v_n(x) \geqslant 0$. Consequently, as J_λ is coercive on $\mathscr{N}_\lambda$, $\{v_n\}$ is a bounded sequence in E, going to a sub-sequence denoted by $\{v_n\}$, and $v_0 \geqslant 0$ such that $u_n \rightharpoonup u_0$, weakly in E, $v_n \to v_0$, strongly in $L^{1-\gamma}(\Omega)$, and $L^s(\Omega)$, for $1 \leqslant s < 2^*$, and $v_n(x) \to v_0(x)$, a.e. in Ω, as $n \to \infty$. Now, from (3.1) and using the weak lower semi-continuity of norm $J_\lambda(v_0) \leqslant \liminf J_\lambda(v_n) = \inf_{\mathscr{N}^-} J_\lambda$, we see that $v_0 \not\equiv 0$ in Ω. Now, we prove that $v_0(x) > 0$ a.e. in Ω. Similarly to the arguments in Claim 1, we start by observing that, since $v_n \in \Lambda^-$, one has

$$(\gamma+1)||v_n||^2 - \lambda(\gamma+r-1)\int_\Omega F(x,v_n)dx < 0 \tag{3.19}$$

and consequently,

$$((\gamma+1)\int_\Omega a(x)v_n^{1-\gamma}dx - \lambda(r-2)\int_\Omega F(x,v_n)dx < 0. \tag{3.20}$$

Therefore, from (3.4) and (3.5) it follows that

$$\begin{aligned}&\lim_{n\to\infty}\left[((\gamma+1)\int_\Omega a(x)v_n^{1-\gamma}dx - \lambda(r-2)\int_\Omega F(x,v_n)dx\right]\\ &= ((\gamma+1)\int_\Omega a(x)v_0^{1-\gamma}dx - \lambda(r-2)\int_\Omega F(x,v_0)dx \leqslant 0.\end{aligned}$$

Now, repeating the same arguments as in Claim 1, it follows that

$$(1+\gamma)\int_\Omega |v_0|^{1-\gamma}dx - \lambda(r-2)\int_\Omega F(x,v_0)dx < 0. \tag{3.21}$$

Now, let $\varphi \in E$, with $\varphi \geqslant 0$. From Lemma 2.3 with $u = v_n$, there exits a sequence of continuous functions $g_n = g_n(t)$ such that $g_n(t)(v_n + t\varphi) \in \mathcal{N}^-$ and $g_n(0) = 1$. Therefore, using the same arguments as in Claim 1 we prove that

$$g_n'(0) \quad \text{is uniformly bounded for } n \text{ large enough.} \tag{3.22}$$

Then, as in Step 1 applying (ii) and (3.22), we conclude that $v_0(x) > 0$ a.e. in Ω and

$$\int_{\mathbb{R}^{2N}} \frac{(v_0(x) - v_0(y))(\phi(x) - \phi(y))K(x-y)}{|x-y|^{N+2s}} dxdy$$
$$- \int_\Omega \left(v_0^{-\gamma}\phi + \lambda f(x, v_0)\phi \right) \mathrm{d}x \geqslant 0, \tag{3.23}$$

for all $\varphi \in E$. Finally, as in the arguments of Claim 2, we obtain that $v_0 \in \Lambda^-$ is a positive solution of problem (P_λ). The proof of the Theorem 1.1 is now completed. □

Acknowledgements. The authors would like to thank the anonymous referees for their carefully reading this paper and their useful comments.

REFERENCES

[1] C. O. ALVES, A. EL HAMIDI, *Nehari manifold and existence of positive solutions to a class of quasilinear problems*, Nonlinear Anal. **60** (2005) 611–624.

[2] J. P. AUBIN, I. EKELAND, *Applied Nonlinear Analysis*, Pure Appl. Math., Wiley-Interscience Publications, 1984.

[3] D. APPLEBAUM, *Lévy Processes and Stochastic Calculus*, seconded., Camb. Stud. Adv. Math., **116**, Cambridge University Press, Cambridge, 2009.

[4] B. BARRIOS, E. COLORADO, A. DE PABLO, U. SANCHEZ, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations **252** (2012), 6133–6162.

[5] K. J. BROWN, Y. ZHANG, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differential Equations **193** (2003) 481–499.

[6] X. CABRÉ, J. TAN, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224** (2010), 2052–2093.

[7] L. CAFFARELLI, L. SILVESTRE, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

[8] M. M. COCLITE AND G. PALMIERI, *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations **14** (1989), no. 10, 1315–1327.

[9] H. CHEN, H. HAJAIEJ, Y. WANG, *On a class of semilinear fractional elliptic equations involving outside Dirac data*, Nonlinear Anal. **125** (2015), 639–658.

[10] M. G. CRANDALL, P. H. RABINOWITZ AND L. TARTAR, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), no. 2, 193–222.

[11] R. CONT AND P. TANKOV, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC Financ. Math. Ser. 2004.

[12] P. DRABEK AND S. I. POHOZAEV, *Positive solutions for the p-Laplacian: application of the fibering method*, Proc. Royal Soc. Edinburgh Sect A, **127** (1997) 703–726.

[13] A. GHANMI, K. SAOUDI, *A multiplicity results for a singular problem involving the fractional p−Laplacian operator*, Complex variables and elliptic equations **61**, 9 (2016) 1199–1216.

[14] J. GIACOMONI AND K. SAOUDI, *Multiplicity of positive solutions for a singular and critical problem*, Nonlinear Anal. **71** (2009), no. 9, 4060–4077.

[15] J. GIACOMONI, I. SCHINDLER AND P. TAKÁČ, *Sobolev versus Hölder local minimizers and global multiplicity for a singular and quasilinear equation*, annali della scuola normale superiore di pisa, classe di scienze série V, **6** (2007) 117–158.

[16] E. DI. NEZZA, G. PALATUCCI AND E. VALDINOCI, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012) 521–573.
[17] K. SAOUDI AND M. KRATOU, *Existence of multiple solutions for a singular and quasilinear equation*, Complex Var. Elliptic Equ. **60** (2015), no. 7, 893–925.
[18] R. SERVADEI, E. VALDINOCI, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887–898.
[19] R. SERVADEI, E. VALDINOCI, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), 2105–2137.
[20] R. SERVADEI AND E. VALDINOCI, *On the spectrum of two different fractional operators*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), no. 4, 831–855.
[21] J. TAN, *The Brézis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations **36** (2011), 21–41.

(Received January 26, 2016)

Abdeljabbar Ghanmi
Faculty of Science and Arts, Khulais
University of jeddah
Saoudi Arabia
e-mail: Abdeljabbar.ghanmi@lamsin.rnu.tn

Kamel Saoudi
College of sciences at Dammam
University of Dammam
31441 Dammam, Kingdom of Saudi Arabia
e-mail: kasaoudi@gmail.com