Fractional Differential Calculus
Volume 6, Number 2 (2016), 233–248
doi:10.7153/fdc-06-15

TWO–WEIGHTED INEQUALITY FOR (p,q)–ADMISSIBLE $B_{k,n}$–POTENTIAL OPERATORS IN WEIGHTED LEBESGUE SPACES

S. K. ABDULLAYEV, E. A. GADJIEVA AND F. A. ISAYEV

(*Communicated by V. S. Guliyev*)

Abstract. In this paper, we study the boundedness of (p,q)-admissible potential operators, associated with the Laplace-Bessel differential operator $B_{k,n} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{k} \frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$ ((p,q)-admissible $B_{k,n}$-potential operators) on a weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ including their weak versions. These conditions are satisfied by most of the operators in harmonic analysis, such as the $B_{k,n}$-fractional maximal operator, $B_{k,n}$-potential integral operators and so on. Sufficient conditions on weighted functions ω and ω_1 are given so that (p,q)-admissible $B_{k,n}$-potential operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1<p<q<\infty$ and weak (p,q)-admissible $B_{k,n}$-potential operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1\leqslant p<q<\infty$.

1. Introduction and preliminaries

The singular integral operators that have been considered S. Mihlin [23] and A. Calderon and A. Zygmund [7] are playing an important role in the theory of Harmonic Analysis and in the theory of partial differential equations. M. Klyuchantsev [22] and I. Kipriyanov and M. Klyuchantsev [21] have firstly introduced and investigated the boundedness in L_p-spaces of multidimensional singular integrals, generated by the $B_{1,n}$-Laplace-Bessel differential operator ($B_{1,n}$-singular integrals), where

$$B_{1,n} = B_1 + \sum_{j=2}^{n} \frac{\partial^2}{\partial x_j^2}, \quad B_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\gamma_1}{x_1}\frac{\partial}{\partial x_1}, \ \ \gamma_1 > 0,$$

I. A. Aliev and A. D. Gadjiev [5], A. D. Gadjiev and E. V. Guliyev [10] and E. V. Guliyev [11] have studied the boundedness of $B_{1,n}$ singular integrals in weighted L_p-spaces with radial and general weights consequently. The maximal functions, singular integrals, potentials and related topics associated with the Laplace-Bessel differential operator $B_{k,n}$-which is known as an important differential operator in analysis and its applications, have been the research areas many mathematicans such as I. Kipriyanov and M. Klyuchantsev [21, 22], L. Lyakhov [26, 27], A. D. Gadjiev and I. A. Aliev [4, 5], I. A. Aliev and S. Bayrakci [2, 3], V. S. Guliyev [12, 13, 14] and others.

Mathematics subject classification (2010): 42B25.
Keywords and phrases: Weighted Lebesgue space, $B_{k,n}$-Laplace-Bessel differential operator, (p,q)-admissible $B_{k,n}$-potential operators, two-weighted inequality.

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x=(x_1,\dots,x_n)$, $\xi=(\xi_1,\dots,\xi_n)$ are vectors in $\mathbb{R}^n$, $(x,\xi)=x_1\xi_1+\dots+x_n\xi_n$, $|x|=\sqrt{(x,x)}$, $x=(x',x'')$, $x'=(x_1\dots,x_k)$, $x''=(x_{k+1},\dots,x_n)$. Let $\mathbb{R}^k_{++}=\{x\in\mathbb{R}^k:\ x_1>0\dots,x_k>0\}$, $\mathbb{R}^n_{k,+}=\{x=(x_1,\dots,x_n):\ x_1,x_2,\dots,x_k>0\}$, $1\leqslant k\leqslant n$, $S_{k,+}=\{x\in\mathbb{R}^n_{k,+}:\ |x|=1\}$.

For $x\in\mathbb{R}^n_{k,+}$ and $r>0$, we denote by $E(x,r)=\{y\in\mathbb{R}^n_{k,+}:\ |x-y|<r\}$ the open ball centered at x of radius r, and by ${}^{\complement}E(x,r)=\mathbb{R}^n_{k,+}\setminus E(x,r)$ denote its complement, $E'(x',r)=\{y'\in\mathbb{R}^k_{++}:\ |x'-y'|<r\}$, ${}^{\complement}E'(x',r)=\mathbb{R}^k_{++}\setminus E'(x',r)$. For measurable set $E\subset\mathbb{R}^n_{k,+}$ let $|E|_\gamma=\int_E(x')^\gamma dx$, then $|E(0,r)|_\gamma=\omega(n,\gamma)r^{n+|\gamma|}$, where $\gamma=(\gamma_1\dots,\gamma_k)$, $|\gamma|=\gamma_1+\dots+\gamma_k$, $(x')^\gamma=x_1^{\gamma_1}\dots x_k^{\gamma_k}$ and $\omega(n,\gamma)=|E(0,1)|_\gamma$.

An almost everywhere positive and locally integrable function $\omega:\ \mathbb{R}^n_{k,+}\to\mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ the set of all measurable functions f on $\mathbb{R}^n_{k,+}$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})}\equiv\|f\|_{p,\omega,\gamma;\mathbb{R}^n_{k,+}}=\left(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x)(x')^\gamma dx\right)^{1/p},\qquad 1\leqslant p<\infty$$

is finite. For $\omega=1$ the space $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ is denoted by $L_{p,\gamma}(\mathbb{R}^n_{k,+})$, and the norm $\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})}$ by $\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})}$.

The operator of generalized shift ($B_{k,n}$-shift operator) is defined by the following way (see [15], [27]):

$$T^yf(x)=C_{\gamma,k}\int_0^\pi\dots\int_0^\pi f\left((x',y')_\beta,x''-y''\right)d\nu(\beta),$$

where
$C_{\gamma,k}=\pi^{-\frac{k}{2}}\Gamma^{-1}\left(\frac{|\gamma|}{2}\right)\prod_{i=1}^k\Gamma(\frac{\nu_i+1}{2})$, $(x',y')_\beta=((x_1,y_1)_{\beta_1}\dots(x_k,y_k)_{\beta_k}),(x_i,y_i)_{\beta_i}=(x_i^2-2x_iy_i\cos\beta_i+y_i^2)^{1/2}$, $1\leqslant i\leqslant k$, $d\nu(\beta)=\prod_{i=1}^k\sin^{\gamma_i-1}\beta_i\,d\beta_1\dots d\beta_k$.

Note that this shift operator is closely connected with $B_{k,n}$-Laplace-Bessel singular differential operators (see [15], [27], in the case $n=1$ see also [24]).

The translation operator T^y generated the corresponding $B_{k,n}$-convolution

$$(f\otimes g)(x)=\int_{\mathbb{R}^n_{k,+}}f(y)[T^yg(x)](y')^\gamma dy,$$

for which the Young inequality holds:

$$\|f\otimes g\|_{L_{q,\gamma}}\leqslant\|f\|_{L_{p,\gamma}}\|g\|_{L_{r,\gamma}},\quad 1\leqslant p,q,r\leqslant\infty,\quad\frac{1}{p}+\frac{1}{r}=\frac{1}{q}+1.$$

The following generalized Hardy inequalities have an important role in proofs of our main results see [9], Chapter 1 (see also [1, 8, 20]).

LEMMA 1.1. *Suppose that* $1 \leqslant p \leqslant q \leqslant \infty$, $p' = p/(p-1)$ *and* $\omega(x)$ *and* $v(x)$ *are positive functions defined on* $\mathbb{R}^n$.

1. For the n*-dimensional Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|<|x|/2}|f(y)|\,dy\right)^q\omega(x)\,dx\right)^{1/q}\leqslant C_5\left(\int_{\mathbb{R}^n}|f(x)|^p\upsilon(x)\,dx\right)^{1/p}$$

with a constant C_5, *independent on* f, *to hold, it is necessary and sufficient that the following condition be satisfied:*

$$\sup_{r>0}\left(\int_{|x|>2r}\omega(x)\,dx\right)^{1/q}\left(\int_{|x|<r}\upsilon^{1-p'}(x)\,dx\right)^{1/p'}<\infty.$$

2. For the n*-dimensional (dual) Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|>2|x|}|f(y)|\,dy\right)^q\omega(x)\,dx\right)^{1/q}\leqslant C_6\left(\int_{\mathbb{R}^n}|f(x)|^p v(x)\,dx\right)^{1/p}$$

with a constant C_6, *independent on* f, *to hold, it is necessary and sufficient that the following condition be satisfied:*

$$\sup_{r>0}\left(\int_{|x|<r}\omega(x)\,dx\right)^{1/q}\left(\int_{|x|>2r}v^{1-p'}(x)\,dx\right)^{1/p'}<\infty.$$

This lemma could be directly deduced from results proved by P. Drabek, H. Heinig and A. Kufner (see Theorem 2.1, p. 4 and Theorem 2.2, p. 7 in [18]).

In this paper we study the boundedness of (p,q)-admissible potential operators, associated with the Laplace-Bessel differential operator $B_{k,n}=\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}+\sum\limits_{j=1}^{k}\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$ ((p,q)-admissible $B_{k,n}$-potential operators) on a weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ including their weak versions. These conditions are satisfied by most of the operators in harmonic analysis, such as the $B_{k,n}$-fractional maximal operator, $B_{k,n}$-potential integral operators and so on. Sufficient conditions on weighted functions ω and ω_1 are given so that (p,q)-admissible $B_{k,n}$-potential operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1<p<q<\infty$ and weak (p,q)-admissible $B_{k,n}$-potential operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1\leqslant p<q<\infty$.

2. Main results

The operator T is called sublinear, if for all $\lambda,\mu>0$ and for all f and g in the domain of T

$$|T(\lambda f+\mu g)(x)|\leqslant\lambda|Tf(x)|+\mu|Tg(x)|.$$

DEFINITION 2.1. (p-admissible $B_{k,n}$-singular operator). Let $1<p<\infty$. A sublinear operator T_γ will be called p-admissible $B_{k,n}$-singular operator, if:

1) T_γ satisfies the size condition of the form

$$\chi_{E(x,r)}(z)\left|T_\gamma\left(f\chi_{\mathbb{R}^n_{k,+}\setminus E(x,2r)}\right)(z)\right| \leqslant C\chi_{E(x,r)}(z)\int_{\mathbb{R}^n_{k,+}\setminus E(x,2r)} T^y|x|^{-n-|\gamma|}\,|f(y)|\,(y')^\gamma dy \tag{2.1}$$

for $x\in\mathbb{R}^n_{k,+}$ and $r>0$, where χ_E is the characteristic function of the set E;

2) T_γ is bounded in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$.

DEFINITION 2.2. (weak p-admissible $B_{k,n}$-singular operator). Let $1\leqslant p<\infty$. A sublinear operator T_γ will be called the weak p-admissible $B_{k,n}$-singular operator, if:

1) T_γ satisfies the size condition (2.1).

2) T_γ is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to the weak $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$.

DEFINITION 2.3. ((p,q)-admissible $B_{k,n}$-potential operator). Let $1<p<\infty$. A sublinear operator $T_{\alpha,\gamma}$, $0<\alpha<n+|\gamma|$ will be called (p,q)-admissible $B_{k,n}$-potential operator, if:

1) $T_{\alpha,\gamma}$ satisfies the size condition of the form

$$\chi_{E(x,r)}(z)\left|T_{\alpha,\gamma}\left(f\chi_{\mathbb{R}^n_{k,+}\setminus E(x,2r)}\right)(z)\right| \leqslant C\chi_{E(x,r)}(z)\int_{\mathbb{R}^n_{k,+}\setminus E(x,2r)} T^y|x|^{\alpha-n-|\gamma|}\,|f(y)|\,(y')^\gamma dy \tag{2.2}$$

for $x\in\mathbb{R}^n_{k,+}$ and $r>0$;

2) $T_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.

DEFINITION 2.4. (weak (p,q)-admissible $B_{k,n}$-potential operator). Let $1\leqslant p<q<\infty$. A sublinear operator $T_{\alpha,\gamma}$, $0<\alpha<n+|\gamma|$ will be called the weak (p,q)-admissible $B_{k,n}$-potential operator, if:

1) $T_{\alpha,\gamma}$ satisfies the size condition (2.2).

2) $T_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to the weak $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

REMARK 2.1. Note that p-admissible singular operators were introduced and studied their boundedness on vanishing generalized Morrey spaces in [28]. Also Φ-admissible singular operators and weak Φ-admissible singular operators was introduced and studied the boundedness of them on generalized Orlicz-Morrey spaces in [16, 19]. Also, p-admissible singular operators, associated with the Laplace-Bessel differential operator $B_{k,n}$ were introduced and studied their boundedness on weighted Lebesgue spaces in [17], see also [29].

DEFINITION 2.5. A function k defined on $\mathbb{R}^n_{k,+}$, is said to be $B_{k,n}$-singular kernel in the space $\mathbb{R}^n_{k,+}$ if

i) $k \in C^\infty(\mathbb{R}^n_{k,+})$;

ii) $k(rx) = r^{-n-|\gamma|}k(x)$ for each $r > 0$, $x \in \mathbb{R}^n_{k,+}$;

iii) $\int_{S_{k,+}} k(x)x^\gamma d\sigma(x) = 0$, where $d\sigma$ is the element of area of the $S_{k,+}$.

The $B_{k,n}$-fractional maximal function (see [12, 13, 15]) is defined by

$$M_{\alpha,\gamma}f(x) = \sup_{r>0} \frac{1}{\omega(n,\gamma)r^{n+|\gamma|}} \int_{E(0,r)} T^y|f(x)|\,(y')^\gamma dy,\quad 0 \leqslant \alpha < n+|\gamma|$$

and the $B_{k,n}$-Riesz potential (see [4, 12, 13, 15]) is defined by

$$I_{\alpha,\gamma}f(x) = \int_{\mathbb{R}^n_{k,+}} T^y|x|^{\alpha-n-|\gamma|}\, f(y)\,(y')^\gamma dy,\quad 0 < \alpha < n+|\gamma|.$$

Note that, $M_\gamma \equiv M_{0,\gamma}$ is the $B_{k,n}$-maximal function.

Let k is a $B_{k,n}$-singular kernel and K_γ be the $B_{k,n}$-singular integral operator (see [5, 10, 11, 17, 21, 22, 26, 27])

$$K_\gamma f(x) = p.v. \int_{\mathbb{R}^n_{k,+}} T^y k(x)\, f(y)\,(y')^\gamma dy.$$

REMARK 2.2. Note that, the conditions p-admissible $B_{k,n}$-singular operators and (p,q)-admissible $B_{k,n}$-potential operators are satisfied by many interesting operators in harmonic analysis, such as the $B_{k,n}$-maximal operator, the $B_{k,n}$-fractional maximal operator, $B_{k,n}$-potential operators, $B_{k,n}$-singular integral operators and so on.

First, we establish the boundedness from weighted $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to weighted $L_{q,\gamma}(\mathbb{R}^n_{k,+})$ for a wide class of (p,q)-admissible $B_{k,n}$-potential operator.

THEOREM 2.1. *Let* $1 < p < q < \infty$ *and* $T_{\alpha,\gamma}$, $0 < \alpha < n+|\gamma|$ *be a* (p,q)-*admissible* $B_{k,n}$-*potential operator.*

Moreover, let $\omega(x)$, $\omega_1(x)$ *be weight functions on* $\mathbb{R}^n_{k,+}$ *and the following three conditions are satisfied:*

(a) there exist $b > 0$ *such that*

$$\sup_{|x|/8<|y|\leqslant 8|x|} \omega_1(y)^{1/q} \leqslant b\,\omega(x)^{1/p} \quad \text{for a.e. } x \in \mathbb{R}^n_{k,+},$$

$$(b)\quad \mathscr{A} \equiv \sup_{r>0}\left(\int_{\complement_{E(0,2r)}} \omega_1(x)|x|^{-(n+|\gamma|-\alpha)q}(x')^\gamma dx\right)^{1/q}\left(\int_{E(0,r)} \omega^{1-p'}(x)(x')^\gamma dx\right)^{1/p'} < \infty,$$

$$(c)\quad \mathscr{B} \equiv \sup_{r>0}\left(\int_{E(0,r)} \omega_1(x)(x')^\gamma dx\right)^{1/q}\left(\int_{\complement_{E(0,2r)}} \omega^{1-p'}(x)|x|^{-(n+|\gamma|-\alpha)(1-p')}(x')^\gamma dx\right)^{1/p'} < \infty.$$

Then there exists a constant c*, independent of* f*, such that for all* $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_\alpha f(x)|^q \omega_1(x)(x')^\gamma dx\right)^{1/q} \leqslant c\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx\right)^{1/p}. \tag{2.3}$$

Moreover, condition (a) *can be replaced by the condition*
(a') *there exist* $b > 0$ *such that*

$$\omega_1(x)^{1/q}\left(\sup_{|x|/8<|y|\leqslant 8|x|} \frac{1}{\omega(y)^{1/p}}\right) \leqslant b \quad \text{for a.e. } x \in \mathbb{R}^n_{k,+}.$$

Similarly we can prove the following weak variant of the Theorem 2.1.

THEOREM 2.2. *Let* $1 \leqslant p < q < \infty$ *and let* $T_{\alpha,\gamma}$, $0 < \alpha < n+|\gamma|$ *be a weak* (p,q)*-admissible* $B_{k,n}$*-potential operators. Moreover, let* $\omega(x)$, $\omega_1(x)$ *be weight functions on* $\mathbb{R}^n_{k,+}$ *and conditions* (a), (b), (c) *be satisfied.*

Then there exists a constant c*, independent of* f*, such that for all* $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$

$$\left(\int_{\left\{x\in\mathbb{R}^n_{k,+}:|T_\alpha f(x)|>\lambda\right\}} \omega_1(x)(x')^\gamma dx\right)^{1/q} \leqslant \frac{c}{\lambda^q}\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx\right)^{1/p}. \tag{2.4}$$

Note that, the operators $M_{\alpha,\gamma}$ and $I_{\alpha,\gamma}$ are (p,q)-admissible $B_{k,n}$-potential operator for $1 < p < q < \infty$, $0 < \alpha < n+|\gamma|$ and $1/p - 1/q = \alpha/(n+|\gamma|)$ and weak (p,q)-admissible $B_{k,n}$-potential operators for $1 \leqslant p < q < \infty$, $0 < \alpha < n+|\gamma|$ and $1/p-1/q = \alpha/(n+|\gamma|)$. Thus, we have

COROLLARY 2.1. *Let* $1 < p < q < \infty$, $0 < \alpha < n+|\gamma|$ *and* $1/p-1/q = \alpha/(n+|\gamma|)$*. Moreover, let* $\omega(x)$, $\omega_1(x)$ *be weight functions on* $\mathbb{R}^n_{k,+}$ *and conditions* (a), (b), (c) *be satisfied. Then the operators* $M_{\alpha,\gamma}$ *and* $I_{\alpha,\gamma}$ *are bounded from* $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

COROLLARY 2.2. *Let* $1 \leqslant p < q < \infty$, $0 < \alpha < n+|\gamma|$ *and* $1/p-1/q = \alpha/(n+|\gamma|)$*. Moreover, let* $\omega(x)$, $\omega_1(x)$ *be weight functions on* $\mathbb{R}^n_{k,+}$ *and conditions* (a), (b), (c) *be satisfied. Then the operators* $M_{\alpha,\gamma}$ *and* $I_{\alpha,\gamma}$ *are bounded from* $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

THEOREM 2.3. *Let* $1 < p < q < \infty$ *and* $T_{\alpha,\gamma}$, $0 < \alpha < n+|\gamma|$ *be a* (p,q)*-admissible* $B_{k,n}$*-potential operator.*

Moreover, let $\omega(x')$, $\omega_1(x')$ *be a weight functions on* $\mathbb{R}^k_{++}$ *and the following three conditions be satisfied*
(a_1) *there exists a constant* $b > 0$ *such that*

$$\sup_{|x'|/8<|y'|<8|x'|} (\omega_1(y'))^{1/q} \leqslant b\,\omega(x')^{1/p} \quad \text{for a.e. } x' \in \mathbb{R}^k_{++},$$

$$(b_1) \quad \mathscr{A}_1 \equiv \sup_{r>0} \left(\int_{\complement_{E'(0,2r)}} \omega_1(x') |x'|^{(n-k)(1+q/p')-(n+|\gamma|-\alpha)q} (x')^\gamma dx' \right)^{1/q}$$

$$\times \left(\int_{E'(0,r)} \omega^{1-p'}(x')(x')^\gamma dx' \right)^{1/p'} < \infty,$$

$$(c_1) \quad \mathscr{B}_1 \equiv \sup_{r>0} \left(\int_{E'(0,r)} \omega_1(x')(x')^\gamma dx' \right)^{1/q}$$

$$\times \left(\int_{\complement_{E'(0,2r)}} \omega^{1-p'}(x') |x'|^{-((n-k)(1/q+1/p')-n-|\gamma|+\alpha)(1-p')} (x')^\gamma dx' \right)^{1/p'} < \infty.$$

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n_{k,+})$

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_\alpha f(x)|^q \omega_1(x')(x')^\gamma dx \right)^{1/q} \leqslant c \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx \right)^{1/p}. \quad (2.5)$$

Moreover, condition (a) can be replaced by the condition
$(a_1{}')$ *there exists a constant $b > 0$ such that*

$$\omega_1(x')^{1/q} \left(\sup_{|x'|/8<|y'|<8|x'|} \frac{1}{\omega(y')^{1/p}} \right) \leqslant b \quad \text{for a.e. } x' \in \mathbb{R}^k_{++}.$$

Similarly we can prove the following weak variant of the Theorem 2.3.

THEOREM 2.4. *Let $1 \leqslant p < q < \infty$ and let $T_{\alpha,\gamma}$, $0 < \alpha < n+|\gamma|$ be a weak (p,q)-admissible $B_{k,n}$-potential operators. Moreover, let $\omega(x)$, $\omega_1(x')$ be weight functions on $\mathbb{R}^k_{++}$ and conditions (a_1), (b_1), (c_1) be satisfied.*

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$

$$\left(\int_{\left\{x \in \mathbb{R}^n_{k,+} : |T_\alpha f(x)| > \lambda\right\}} \omega_1(x')(x')^\gamma dx \right)^{1/q} \leqslant \frac{c}{\lambda^q} \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx \right)^{1/p}. \quad (2.6)$$

COROLLARY 2.3. *Let $1 < p < q < \infty$, $0 < \alpha < n+|\gamma|$ and $1/p - 1/q = \alpha/(n+|\gamma|)$. Moreover, let $\omega(x')$, $\omega_1(x')$ be weight functions on $\mathbb{R}^k_{++}$ and the following three conditions be satisfied (a_1),*

$$(b_1)' \quad \sup_{r>0} \left(\int_{\complement_{E'(0,2r)}} \omega_1(x') |x'|^{-(k+|\gamma|)(1-\alpha/(n+|\gamma|)q} (x')^\gamma dx' \right)^{1/q}$$

$$\times \left(\int_{E'(0,r)} \omega^{1-p'}(x')(x')^\gamma dx' \right)^{1/p'} < \infty,$$

$$(c_1)' \quad \sup_{r>0}\left(\int_{E'(0,r)} \omega_1(x')(x')^{\gamma}dx'\right)^{1/q}$$
$$\times\left(\int_{\complement_{E'(0,2r)}} \omega^{1-p'}(x')|x'|^{(k+|\gamma|)(1-\alpha/(n+|\gamma|))(1-p')}(x')^{\gamma}dx'\right)^{1/p'} < \infty.$$

Then the operators $M_{\alpha,\gamma}$ *and* $I_{\alpha,\gamma}$ *are bounded from* $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

COROLLARY 2.4. *Let* $1 \leqslant p < q < \infty$, $0 < \alpha < n+|\gamma|$ *and* $1/p - 1/q = \alpha/(n+|\gamma|)$. *Moreover, let* $\omega(x')$, $\omega_1(x')$ *be weight functions on* $\mathbb{R}^n_{k,+}$ *and conditions* (a_1), $(b_1)'$, $(c_1)'$ *be satisfied. Then the operators* $M_{\alpha,\gamma}$ *and* $I_{\alpha,\gamma}$ *are bounded from* $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

REMARK 2.3. Note that, if instead of $\omega(x)$, $\omega_1(x)$ respectively put $\omega(x')$, $\omega_1(x')$, then from the conditions (a), (b), (c) will not follows the conditions (a_1), (b_1), (c_1) respectively.

3. Proofs of the main results

Proof of the Theorem 2.1. For $l \in Z$ we define $E_l = \{x \in \mathbb{R}^n_{k,+} : \ 2^l < |x| \leqslant 2^{l+1}\}$, $E_{l,1} = \{x \in \mathbb{R}^n_{k,+} : \ |x| \leqslant 2^{l-1}\}$, $E_{l,2} = \{x \in \mathbb{R}^n_{k,+} : \ 2^{l-1} < |x| \leqslant 2^{l+2}\}$, $E_{l,3} = \{x \in \mathbb{R}^n_{k,+} : \ |x| > 2^{l+2}\}$. Then $E_{l,2} = E_{l-1} \cup E_l \cup E_{l+1}$ and the multiplicity of the covering $\{E_{l,2}\}_{l\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$\begin{aligned}
|T_\alpha f(x)| &= \sum_{l\in Z} |T_\alpha f(x)|\chi_{E_l}(x)\\
&\leqslant \sum_{l\in Z}\left|T_\alpha f_{l,1}(x)\right|\chi_{E_l}(x) + \sum_{l\in Z}\left|T_\alpha f_{l,2}(x)\right|\chi_{E_l}(x) + \sum_{l\in Z}\left|T_\alpha f_{l,3}(x)\right|\chi_{E_l}(x)\\
&\equiv T_{\alpha,1}f(x) + T_{\alpha,2}f(x) + T_{\alpha,3}f(x),
\end{aligned}$$

where χ_{E_l} is the characteristic function of the set E_l, $f_{l,i} = f\chi_{E_{l,i}}$, $i = 1,2,3$.

First we shall estimate $\|T_{\alpha,1}f\|_{L_{q,\omega_1,\gamma}}$. Note that for $x \in E_l$, $y \in E_{k,1}$ we have $|y| \leqslant 2^{l-1} \leqslant |x|/2$. Moreover, $E_l \cap supp f_{l,1} = \varnothing$ and $|x-y| \geqslant |x|/2$. Hence, by (2.1)

$$\begin{aligned}
T_{\alpha,1}f(x) &\leqslant c_0 \sum_{l\in Z}\left(\int_{\mathbb{R}^n_{k,+}} T^y|x|^{\alpha-n-|\gamma|}\ |f_{l,1}(y)|(y')^{\gamma}dy\right)\chi_{E_l}\\
&\leqslant c_0 \int_{E(0,|x|/2)} |x-y|^{\alpha-n-|\gamma|}|f(y)|\ (y')^{\gamma}dy\\
&\leqslant 2^{n+|\gamma|-\alpha}c_0|x|^{\alpha-n-|\gamma|}\int_{E(0,|x|/2)} |f(y)|\ (y')^{\gamma}dy
\end{aligned}$$

for any $x \in E_l$. Then we have

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1}f(x)|^q \omega_1(x)\ (x')^\gamma dx\right)^{1/q}$$
$$\leqslant 2^{n+|\gamma|-\alpha} c_0 \left(\int_{\mathbb{R}^n_{k,+}} \left(\int_{E(0,|x|/2)} |f(y)|\ (y')^\gamma dy\right)^q |x|^{-(n+|\gamma|-\alpha)q} \omega_1(x)\ (x')^\gamma dx\right)^{1/q}.$$

Since $\mathscr{A} < \infty$, the Hardy inequality

$$\left(\int_{\mathbb{R}^n_{k,+}} \omega_1(x)|x|^{-(n+|\gamma|-\alpha)q} \left(\int_{E(0,|x|/2)} |f(y)|\ (y')^\gamma dy\right)^q (x')^\gamma dx\right)^{1/q}$$
$$\leqslant C\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)\ (x')^\gamma dx\right)^{1/p}$$

holds and $C \leqslant c'\mathscr{A}$, where c' depends only on n and p. In fact the condition $\mathscr{A} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [8]). Hence, we obtain

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1}f(x)|^q \omega_1(x)\ (x')^\gamma dx\right)^{1/q} \leqslant c_1 \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)\ (x')^\gamma dx\right)^{1/p}, \quad (3.1)$$

where c_1 is independent of f.

Next we estimate $\|T_{\alpha,3}f\|_{L_{q,\omega_1,\gamma}}$. As is easy to verify, for $x \in E_l$, $y \in E_{l,3}$ we have $|y| > 2|x|$ and $|x-y| \geqslant |y|/2$. Since $E_l \cap supp f_{l,3} = \emptyset$, for $x \in E_l$ by (2.1) we obtain

$$\begin{aligned}
T_{\alpha,3}f(x) &\leqslant c_0 \int_{\complement_{E(0,2|x|)}} T^y|x|^{\alpha-n-|\gamma|}|f(y)|\ (y')^\gamma dy \\
&\leqslant 2^{n+|\gamma|-\alpha} c_0 \int_{\complement_{E(0,2|x|)}} |f(y)||x-y|^{\alpha-n-|\gamma|}\ (y')^\gamma dy \\
&\leqslant 2^{n+|\gamma|-\alpha} c_0 \int_{\complement_{E(0,2|x|)}} |f(y)||y|^{\alpha-n-|\gamma|}\ (y')^\gamma dy.
\end{aligned}$$

Hence we have

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3}f(x)|^q \omega_1(x)\ (x')^\gamma dx\right)^{1/q}$$
$$\leqslant 2^{n+|\gamma|-\alpha} c_0 \left(\int_{\mathbb{R}^n_{k,+}} \left(\int_{\complement_{E(0,2|x|)}} |f(y)||y|^{\alpha-n-|\gamma|}\ (y')^\gamma dy\right)^q \omega_1(x)\ (x')^\gamma dx\right)^{1/q}.$$

Since $\mathscr{B} < \infty$, the Hardy inequality

$$\left(\int_{\mathbb{R}^n_{k,+}} \omega_1(x) \left(\int_{\complement_{E(0,2|x|)}} |f(y)||y|^{\alpha-n-|\gamma|} \, (y')^\gamma dy \right)^q (x')^\gamma dx \right)^{1/q} \leqslant C \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) \, (x')^\gamma dx \right)^{1/p}$$

holds and $C \leqslant c'\mathscr{B}$, where c' depends only on n and p. In fact the condition $\mathscr{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [8]). Hence, we obtain

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3} f(x)|^q \omega_1(x) \, (x')^\gamma dx \right)^{1/q} \leqslant c_2 \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) \, (x')^\gamma dx \right)^{1/p}, \qquad (3.2)$$

where c_2 is independent of f.

Finally, we estimate $\|T_{\alpha,2} f\|_{L_{q,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_{k,+}) \to L_{q,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of $T_{\alpha,\gamma}$ and condition (a) we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,2} f(x)|^q \omega_1(x) \, (x')^\gamma dx \right)^{1/q} \\
&= \left(\int_{\mathbb{R}^n_{k,+}} \left(\sum_{l \in Z} \left| T_\alpha f_{l,2}(x) \right| \chi_{E_l}(x) \right)^q \omega_1(x) \, (x')^\gamma dx \right)^{1/q} \\
&= \left(\int_{\mathbb{R}^n_{k,+}} \left(\sum_{l \in Z} \left| T_\alpha f_{l,2}(x) \right|^q \chi_{E_l}(x) \right) \omega_1(x) \, (x')^\gamma dx \right)^{1/q} \\
&= \left(\sum_{l \in Z} \int_{E_l} \left| T_\alpha f_{l,2}(x) \right|^q \omega_1(x) \, (x')^\gamma dx \right)^{1/q} \\
&\leqslant \left(\sum_{l \in Z} \sup_{y \in E_l} \omega_1(y) \int_{\mathbb{R}^n_{k,+}} \left| T_\alpha f_{l,2}(x) \right|^q \, (x')^\gamma dx \right)^{1/q} \\
&\leqslant \|T_\alpha\| \left(\sum_{l \in Z} \sup_{y \in E_l} \omega_1(y) \left(\int_{\mathbb{R}^n_{k,+}} \left| f_{l,2}(x) \right|^p \, (x')^\gamma dx \right)^{q/p} \right)^{1/q} \\
&= \|T_\alpha\| \left(\sum_{l \in Z} \sup_{y \in E_l} \omega_1(y) \left(\int_{E_{l,2}} |f(x)|^p \, (x')^\gamma dx \right)^{q/p} \right)^{1/q},
\end{aligned}$$

where $\|T_\alpha\| \equiv \|T_\alpha\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}) \to L_{q,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in E_{l,2}$, $2^{l-1} < |x| \leqslant 2^{l+2}$, we have by condition (a)

$$\sup_{y \in E_l} (\omega_1(y))^{p/q} = \sup_{2^{l-1} < |y| \leqslant 2^{l+2}} (\omega_1(y))^{p/q} \leqslant \sup_{|x|/8 < |y| \leqslant 8|x|} (\omega_1(y))^{p/q} \leqslant b\,\omega(x)$$

for almost all $x \in E_{l,2}$. Therefore

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,2} f(x)|^q \omega_1(x)(x')^\gamma dx \right)^{1/q} \leqslant \|T_\alpha\| b \left(\sum_{l \in Z} \int_{E_{l,2}} |f(x)|^p \omega(x)(x')^\gamma dx \right)^{1/p}$$
$$\leqslant c_3 \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx \right)^{1/p}, \qquad (3.3)$$

where $c_3 = 3\|T_\alpha\| b$, since the multiplicity of covering $\{E_{l,2}\}_{l \in Z}$ is equal to 3.

Inequalities (3.1), (3.2), (3.3) imply (2.3) which completes the proof. □

Proof of the Theorem 2.3. For $l \in Z$ we define $\widetilde{E}_l = \{x \in \mathbb{R}^n_{k,+} : 2^l < |x'| \leqslant 2^{l+1}\}$, $\widetilde{E}_{l,1} = \{x \in \mathbb{R}^n_{k,+} : |x'| \leqslant 2^{l-1}\}$, $\widetilde{E}_{l,2} = \{x \in \mathbb{R}^n_{k,+} : 2^{l-1} < |x'| \leqslant 2^{l+2}\}$, $\widetilde{E}_{l,3} = \{x \in \mathbb{R}^n_{k,+} : |x'| > 2^{l+2}\}$. Then $\widetilde{E}_{l,2} = \widetilde{E}_{l-1} \cup \widetilde{E}_l \cup \widetilde{E}_{l+1}$ and the multiplicity of the covering $\left\{\widetilde{E}_{l,2}\right\}_{l \in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$\begin{aligned} |T_\alpha f(x)| &= \sum_{l \in Z} |T_\alpha f(x)| \chi_{\widetilde{E}_l}(x) \\ &\leqslant \sum_{l \in Z} \left|T_\alpha f_{l,1}(x)\right| \chi_{\widetilde{E}_l}(x) + \sum_{l \in Z} \left|T_\alpha f_{l,2}(x)\right| \chi_{\widetilde{E}_l}(x) + \sum_{l \in Z} \left|T_\alpha f_{l,3}(x)\right| \chi_{\widetilde{E}_l}(x) \quad (3.4) \\ &\equiv T_{\alpha,1} f(x) + T_{\alpha,2} f(x) + T_{\alpha,3} f(x), \end{aligned}$$

where $\chi_{\widetilde{E}_l}$ is the characteristic function of the set $\widetilde{E}_l$, $f_{l,i} = f \chi_{\widetilde{E}_{l,i}}$, $i = 1,2,3$. We shall estimate $\|T_{\alpha,1} f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,1}$ we have $|y'| \leqslant 2^{l-1} \leqslant |x'|/2$. Moreover, $\widetilde{E}_l \cap supp f_{l,1} = \emptyset$ and $|x' - y'| \geqslant |x'|/2$. Hence, by (2.1)

$$\begin{aligned} T_{\alpha,1} f(x) &\leqslant c_4 \sum_{l \in Z} \left(\int_{\mathbb{R}^n_{k,+}} |f_{l,1}(y)| T^y |x|^{\alpha - n - |\gamma|} dy \right) \chi_{\widetilde{E}_l} \\ &\leqslant c_4 \int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} T^y |x|^{\alpha - n - |\gamma|} |f(y)| (y')^\gamma dy \\ &\leqslant c_5 \int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} \left(|x'| + |x'' - y''|\right)^{\alpha - n - |\gamma|} |f(y)| (y')^\gamma dy' dy'' \end{aligned}$$

for any $x \in E_l$. Using this last inequality we have

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1} f(x)|^q \omega_1(x')(x')^\gamma dx \right)^{1/q}$$

$$\leqslant c_5 \left(\int_{\mathbb{R}^n_{k,+}} \left(\int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} \left(|x'| + |x'' - y''|\right)^{\alpha - n - |\gamma|} |f(y)| (y')^\gamma dy' dy'' \right)^q \omega_1(x')(x')^\gamma dx \right)^{1/q}.$$

For $x=(x',x'')\in\mathbb{R}^n_{k,+}$ let

$$I(x')=\int_{\mathbb{R}^{n-k}}\left(\int_{\mathbb{R}^{n-k}}\int_{E'(0,|x'|/2)}\left(|x'|+|x''-y''|\right)^{\alpha-n-|\gamma|}|f(y',y'')|(y')^{\gamma}dy'dy''\right)^{q}dx''$$
$$=\int_{\mathbb{R}^{n-k}}\left(\int_{E'(0,|x'|/2)}\left(\int_{\mathbb{R}^{n-k}}\left(|x'|+|x''-y''|\right)^{\alpha-n-|\gamma|}|f(y',y'')|dy''\right)(y')^{\gamma}dy'\right)^{q}dx''.$$

Using the Minkowski and Young inequalities we obtain

$$I(x')\leqslant\left[\int\limits_{E'(0,|x'|/2)}\left(\int\limits_{\mathbb{R}^{n-k}}|f(y',y'')|^{p}dy''\right)^{1/p}\left(\int\limits_{\mathbb{R}^{n-k}}\frac{dx''}{(|x'|+|x''|)^{r(n+|\gamma|-\alpha)}}\right)^{1/r}(y')^{\gamma}dy'\right]^{q}$$
$$=\left(\int_{E'(0,|x'|/2)}\|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}}(y')^{\gamma}dy'\right)^{q}\left(\int_{\mathbb{R}^{n-k}}\frac{dx''}{(|x'|+|x''|)^{r(n+|\gamma|-\alpha)}}\right)^{q/r}$$
$$=\left(\int\limits_{E'(0,|x'|/2)}\|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}}(y')^{\gamma}dy'\right)^{q}\left(\int\limits_{\mathbb{R}^{n-k}}\frac{|x'|^{n-k-(n+|\gamma|-\alpha)r}dx''}{(|x''|+1)^{r(n+|\gamma|-\alpha)}}\right)^{q/r}$$
$$=c_6|x'|^{(n-k)q/r-(n+|\gamma|-\alpha)q}\left(\int_{E'(0,|x'|/2)}\|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}}(y')^{\gamma}dy'\right)^{q}.$$

Integrating in $\mathbb{R}^k_{++}$ we get

$$\left(\int_{\mathbb{R}^n_{k,+}}|T_{\alpha,1}f(x)|^{q}\omega_1(x')(x')^{\gamma}dx\right)^{1/q}$$
$$\leqslant c_7\left(\int\limits_{\mathbb{R}^k_{++}}\omega_1(x')|x'|^{(n-k)q/r-(n+|\gamma|-\alpha)q}\left(\int\limits_{E'(0,|x'|/2)}\|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}}(y')^{\gamma}dy'\right)^{q}(x')^{\gamma}dx'\right)^{1/q}.$$

Since $\mathscr{A}_1<\infty$, the Hardy inequality

$$\left(\int_{\mathbb{R}^k_{++}}\omega_1(x')|x'|^{(n-k)q/r-(n+|\gamma|-\alpha)}\left(\int_{E'(0,|x'|/2)}\|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}}(y')^{\gamma}dy'\right)^{q}(x')^{\gamma}dx'\right)^{1/q}$$
$$\leqslant C\left(\int_{\mathbb{R}^k_{++}}\|f(\cdot,x')\|^{p}_{p,\mathbb{R}^{n-k}}\omega(x')(x')^{\gamma}dx'\right)^{1/p}$$

holds and $C\leqslant c'\mathscr{A}_1$, where c' depends only on n and p. In fact the condition $\mathscr{A}_1<\infty$ is necessary and sufficient for the validity of this inequality (see [6], [20]). Hence, we obtain

$$\left(\int_{\mathbb{R}^n_{k,+}}|T_{\alpha,1}f(x)|^{q}\omega_1(x')(x')^{\gamma}dx\right)^{1/q}\leqslant c_9\left(\int_{\mathbb{R}^n_{k,+}}|f(x)|^{p}\omega(x')(x')^{\gamma}dx\right)^{1/p}. \quad (3.5)$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As is easy to verify, for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,3}$ we have $|y'| > 2|x'|$ and $|x'-y'| \geqslant |y'|/2$. Since $\widetilde{E}_l \cap supp f_{k,3} = \emptyset$, for $x \in \widetilde{E}_l$ by (2.1) we obtain

$$T_{\alpha,3} f(x) \leqslant c_5 \int_{\mathbb{R}^{n-k}} \int_{\complement_{E'}(0,2|x'|)} |f(y)| \left(|y'| + |x''-y''|\right)^{\alpha-n-|\gamma|} (y')^\gamma dy' dy''.$$

Using this last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3} f(x)|^q \omega_1(x')(x')^\gamma dx$$

$$\leqslant c_5^q \int\limits_{\mathbb{R}^n_{k,+}} \left(\int\limits_{\mathbb{R}^{n-k}} \int\limits_{\complement_{E'}(0,2|x'|)} |f(y)| \left(|y'| + |x''-y''|\right)^{\alpha-n-|\gamma|} (y')^\gamma dy' dy'' \right)^q \omega_1(x')(x')^\gamma dx.$$

For $x = (x', x'') \in \mathbb{R}^n$ let

$$I_1(x') = \int\limits_{\mathbb{R}^{n-k}} \left(\int\limits_{\complement_{E'}(0,2|x'|)} \int\limits_{\mathbb{R}^{n-k}} |f(y)| \left(|y'| + |x''-y''|\right)^{\alpha-n-|\gamma|} (y')^\gamma dy' dy'' \right)^q (x')^\gamma dx''.$$

Using the Minkowski and Young inequalities we obtain

$$I_1(x') \leqslant \left[\int_{\complement_{E'}(0,2|x'|)} \left(\int\limits_{\mathbb{R}^{n-k}} |f(y)|^p dy'' \right)^{1/p} \left(\int\limits_{\mathbb{R}^{n-k}} \frac{dy''}{(|y'| + |y''|)^{(n+|\gamma|-\alpha)r}} \right)^{1/r} (y')^\gamma dy' \right]^q$$

$$= c_6 \left(\int\limits_{\complement_{E'}(0,2|x'|)} |y'|^{(n-k)1/r-n-|\gamma|+\alpha} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^q$$

$$\times \left(\int\limits_{\mathbb{R}^{n-k}} \frac{dy''}{(|y''|+1)^{(n+|\gamma|-\alpha)r}} \right)^{q/r}$$

$$= c_7 \left(\int_{\complement_{E'}(0,2|x'|)} |y'|^{(n-k)1/r-n-|\gamma|+\alpha} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^q.$$

Integrating over $\mathbb{R}^k_{++}$ we get

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3} f(x)|^q \omega_1(x')(x')^\gamma dx \right)^{1/q}$$

$$\leqslant c_8 \left(\int_{\mathbb{R}^k_{++}} \left(\int_{\complement_{E'}(0,2|x'|)} |y'|^{(n-k)1/r-n-|\gamma|+\alpha} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'' \right)^q \omega_1(x')(x')^\gamma dx'' \right)^{1/q}.$$

Since $\mathscr{B}_1 < \infty$, the Hardy inequality

$$\left(\int_{\mathbb{R}^k_{++}} \omega_1(x')\left(\int_{\complement_{E'(0,2|x'|)}} |y'|^{-k-|\gamma|+\alpha}\|f(\cdot,y')\|_{p,\mathbb{R}^{n-1}}(y')^\gamma dy'\right)^q (x')^\gamma dx'\right)^{1/q}$$
$$\leqslant C\left(\int_{\mathbb{R}^k_{++}} \|f(\cdot,x')\|^p_{p,\mathbb{R}^{n-k}}|x'|^{(n-k)1/r-n-|\gamma|+\alpha}\omega(x')|x'|^{-((n-k)1/r-n-|\gamma|+\alpha)}(x')^\gamma dx'\right)^{1/p}$$
$$= C\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p\omega(x')(x')^\gamma dx\right)^{1/p}$$

holds and $C \leqslant c'\mathscr{B}_1$, where c' depends only on n, γ and p. In fact the condition $\mathscr{B}_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [6], [20]). Hence, we obtain

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3}f(x)|^q\omega_1(x')(x')^\gamma dx\right)^{1/q} \leqslant c_{10}\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p\omega(x')(x')^\gamma dx\right)^{1/p}. \quad (3.6)$$

Finally, we estimate $\|T_{\alpha,2}f\|_{L_{q,\omega_1,\gamma}}$. From $L_{p,\gamma}(\mathbb{R}^n_{k,+}) \to L_{q,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of $T_{\alpha,\gamma}$ and condition (a_1) we have

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,2}f(x)|^q\omega_1(x')(x')^\gamma dx\right)^{1/q}$$
$$= \left(\int_{\mathbb{R}^n_{k,+}}\left(\sum_{l\in Z}\left|T_\alpha f_{l,2}(x)\right|\chi_{\widetilde{E}_l}(x)\right)^q\omega_1(x')(x')^\gamma dx\right)^{1/q}$$
$$= \left(\int_{\mathbb{R}^n_{k,+}}\left(\sum_{l\in Z}\left|T_\alpha f_{l,2}(x)\right|^q\chi_{\widetilde{E}_l}(x)\right)\omega_1(x')(x')^\gamma dx\right)^{1/q}$$
$$= \left(\sum_{l\in Z}\int_{\widetilde{E}_l}\left|T_\alpha f_{l,2}(x)\right|^q\omega_1(x')(x')^\gamma dx\right)^{1/q}$$
$$\leqslant \left(\sum_{l\in Z}\sup_{y\in\widetilde{E}_l}\omega_1(y')\int_{\mathbb{R}^n}\left|T_\alpha f_{l,2}(x)\right|^q(x')^\gamma dx\right)^{1/q}$$
$$\leqslant \|T_\alpha\|\left(\sum_{l\in Z}\sup_{y\in\widetilde{E}_l}\omega_1(y')\left(\int_{\mathbb{R}^n}\left|f_{l,2}(x)\right|^p(x')^\gamma dx\right)^{q/p}\right)^{1/q}$$
$$= \|T_\alpha\|\left(\sum_{l\in Z}\sup_{y\in\widetilde{E}_l}\omega_1(y')\left(\int_{\widetilde{E}_{l,2}}|f(x)|^p(x')^\gamma dx\right)^{q/p}\right)^{1/q},$$

where $\|T_\alpha\| \equiv \|T_\alpha\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})\to L_{q,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in \widetilde{E}_{l,2}$, $2^{l-1} < |x'| \leqslant 2^{l+2}$, we have by condition (a_1)

$$\sup_{y\in\widetilde{E}_l}(\omega_1(y'))^{p/q}=\sup_{2^{l-1}<|y'|\leqslant 2^{l+2}}(\omega_1(y'))^{p/q}\leqslant\sup_{|x'|/8<|y'|<8|x'|}(\omega_1(y'))^{p/q}\leqslant b\omega(x')$$

for almost all $x\in\widetilde{E}_{l,2}$. Therefore

$$\begin{aligned}\left(\int_{\mathbb{R}^n_{k,+}}|T_{\alpha,2}f(x)|^q\omega_1(x')(x')^\gamma dx\right)^{1/q}&\leqslant\|T_\alpha\|b\sum_{l\in Z}\left(\int_{\widetilde{E}_{l,2}}|f(x)|^p\omega(x')dx\right)^{1/p}\\&\leqslant c_{11}\left(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x')(x')^\gamma dx\right)^{1/p},\quad(3.7)\end{aligned}$$

where $c_{11}=3\|T_\alpha\|b$, since the multiplicity of covering $\left\{\widetilde{E}_{l,2}\right\}_{l\in Z}$ is equal to 3.

Inequalities (3.4), (3.5), (3.6), (3.7) imply (2.5) which completes the proof. □

Acknowledgements. The authors would like to express their gratitude to the referees for his (her) very valuable comments and suggestions. The research of F. A. Isayev was partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan, Grant EIF-2013-9(15)-46/10/1. The research of S. K. Abdullayev, E. A. Gadjieva and F. A. Isayev was partially supported by the grant of Presidium Azerbaijan National Academy of Science 2015.

REFERENCES

[1] E. ADAMS, *On weighted norm inequalities for the Riesz transforms of functions with vanishing moments*, Studia Math. **78** (1984), 107–153.

[2] I. A. ALIEV, S. BAYRAKCI, *On inversion of B-elliptic potentials by the method of Balakrishnan-Rubin*, Fract. Calc. Appl. Anal. **1** (4) (1998), 365–384.

[3] I. A. ALIEV, S. BAYRAKCI, *On inversion of Bessel potentials associated with the Laplace-Bessel differential operator*, Acta Math. Hungar **95** (1–2) (2002), 125–145.

[4] I. A. ALIEV, A. D. GADJIEV, *On classes of operators of potential types, generated by a generalized shift*, Reports of enlarged Session of the Seminars of I. N. Vekua Inst. of Appl. Math. Tbilisi, **3** (2) (1998), 21–24.

[5] I. A. ALIEV, A. D. GADJIEV, *Weighted estimates of multidimensional singilar integrals generated by the generalized shift operator*, Mat. Sb. **183** (9) (1992), 45–66. English, translated into Russian, Acad. Sci. Sb. Math. **77** (1) (1994), 37–55.

[6] J. S. BRADLEY, *The Hardy's inequalities with mixed norms*, Canad. Math. Bull. **21** (1978), 405–408.

[7] A. P. CALDERON AND A. ZYGMUND, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[8] D. EDMUNDS, P. GURKA, L. PICK, *Compactness of Hardy-type integral operators in weighted Banach function spaces*, Studia Math. **109** (1994), 73–90.

[9] D. EDMUNDS, V. KOKILASHVILI, A. MESKHI, *Bounded and compact integral operators*, Kluwer, Dordrecht, Boston, London, 2002.

[10] A. D. GADJIEV AND E. V. GULIYEV, *Two-weighted inequality for singular integrals in Lebesgue spaces, associated with the Laplace-Bessel differential operator*, Proc. A. Razmadze Math. Inst. **138** (2005), 1–15.

[11] E. V. GULIYEV, *Two-weighted inequality for some sublinear operators in Lebesgue spaces, associated with the Laplace-Bessel differential operators*, Proc. A. Razmadze Math. Inst. **139** (2005), 5–31.

[12] V. S. GULIYEV, *Sobolev theorems for B-Riesz potentials*, Dokl. RAN **358** (4) (1998), 450-451.

[13] V. S. GULIYEV, *Some properties of the anisotropic Riesz-Bessel potential*, Anal. Math. **26** (2) (2000), 99-118.

[14] V. S. GULIYEV, *On maximal function and fractional integral, associated with the Bessel differential operator*, Math. Ineq. Appl. **6** (2) (2003), 317–330.

[15] V. S. GULIYEV, N. N. GARAKHANOVA, Y. ZEREN, *Pointwise and integral estimates for B-Riesz potentials in terms of B-maximal and B-fractional maximal functions*, Siberian Math. J. **49** (6) (2008), 1008–1022.

[16] V. S. GULIYEV, F. DERINGOZ, J. J. HASANOV, *Φ-admissible singular operators and their commutators on vanishing generalized Orlicz-Morrey spaces*, J. Inequal. Appl. 2014, 2014:143.

[17] V. S. GULIYEV, F. A. ISAYEV, Z. V. SAFAROV, *Two-weighted inequality for p admissible $B_{k,n}$ singular operators in weighted Lebesgue spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (1) (2014), 122–146.

[18] P. DRABEK, H. HEINIG, A. KUFNER, *Higher dimensional Hardy inequality. In: General Inequalities VII*, International Series of Numerical Mathematics, vol. 123, pp. 3–16, Birkhäuser, Basel (1997).

[19] J. J. HASANOV, *Φ-admissible sublinear singular operators and generalized Orlicz-Morrey spaces*, J. Funct. Spaces Volume 2014 (2014), Article ID 505237, 7 pages, http://dx.doi.org/10.1155/2014/505237.

[20] V. M. KOKILASHVILII, *On Hardy's inequalities in weighted spaces*, Bull. Acad. Sci. Georgian SSR **96** (1979), 37–40, (Russian).

[21] I. A. KIPRIYANOV, M. I. KLYUCHANTSEV, *On singular integrals generated by the generalized shift operator, II*, Sibirsk. Mat. Zh. **11** (1970), 1060–1083, (Russian) translation in Siberian Math. J. **11** (1970), 787–804.

[22] M. I. KLYUCHANTSEV, *On singular integrals generated by the generalized shift operator, I*, Sibirsk. Math. Zh. **11** (1970), 810–821; (Russian) translation in Siberian Math. J. **11** (1970), 612–620.

[23] S. G. MIHLIN, *Multidimensional singular integrals and integral equations*, Fizmatgiz, Moscow, 1962; english transl. Pergamon Press, NY, 1965.

[24] B. M. LEVITAN, *Bessel function expansions in series and Fourier integrals*, Uspekhi Mat. Nauk **6** (42) (2) (1951), 102–143, (Russian).

[25] J. LÖFSTROM, J. PEETRE, *Approximation theorems connected with generalized translations*, Math. Ann. **181** (1969), 255–268.

[26] L. N. LYAKHOV, *On a class of spherical functions and singular pseudodifferential operators*, Dokl. Akad. Nauk. **272** (4) (1983), 781–784, (Russian) translation in Soviet Math.Dokl. **28** (2) (1983), 431–434.

[27] L. N. LJAKHOV, *Multipliers of the mixed Fourier-Bessel transformation*, Proc. V. A. Steklov Inst. Math. **214** (1997), 234–249.

[28] N. SAMKO, *Maximal, Potential and Singular Operators in Vanishing Generalized Morrey Spaces*, J. Global Optim. **57** (4) (2013), 1385–1399.

[29] Y. ZEREN, V. S. GULIYEV, *Two-weight norm inequalities for some anisotropic sublinear operators*, Turk J. Math. **30** (2006), 329–350.

(Received July 27, 2015)

S. K. Abdullayev
Baku State University, Baku, Azerbaijan
and
Institute of Mathematics and Mechanics of NAS of Azerbaijan
"B. Vahabzade" Str., bl. 9, Baku, Azerbaijan
e-mail: sadig.abdullayev@mail.ru

E. A. Gadjiyeva
Institute of Mathematics and Mechanics of NAS of Azerbaijan
"B. Vahabzade" Str., bl. 9, Baku, Azerbaijan
e-mail: egadjieva@gmail.com

F. A. Isayev
Institute of Mathematics and Mechanics of NAS of Azerbaijan
"B. Vahabzade" Str., bl. 9, Baku, Azerbaijan
e-mail: isayevfatai@yahoo.com