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EXISTENCE OF POSITIVE SOLUTIONS FOR COUPLED SYSTEMS OF HALF–LINEAR BOUNDARY VALUE PROBLEMS INVOLVING CAPUTO FRACTIONAL DERIVATIVES

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Abstract. In this paper, we study coupled systems of the half-linear boundary value problems involving left sided Caputo fractional derivatives. The main goal of this paper is restricted to the existence verification of positive solutions for mentioned fractional boundary value problems. To this aim we use nonlinear alternative of Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems. At the end we present some numerical examples to illustrate the obtained theoretical results.

1. Introduction

The fractional differential equations in recent decades have been recognized to be excellent tools for studying natural phenomena, particularly in description of memory and hereditary processes. Maybe this property can be considered as the main advantage of the fractional differential equations with respect to the integer-order differential equations, see [11], [13], [14], [17]. On the other hand studying each differential based phenomenon leads us to a differential system, that is why we should estimate the solvability possibility of such systems. In this way using nonlinear analysis techniques such as fixed point theory provides us an effective tool for studying solvability of the differential systems. There are numerous fixed point theorems that have been applied for proving existence of solutions for a nonlinear differential equations. Here we suggest some of the most famous papers dealt with solvability of nonlinear fractional boundary value problems and references cited therein.

Z. Bai and H. Lu in [2], considered nonlinear Riemann-Liouville fractional boundary value problem

$$\begin{aligned} &D_{0^+}^{\alpha}u(t)+f(t,u(t))=0, \quad 1<\alpha\leqslant 2,\ \ 0<t<1,\\ &u(0)=u(1)=0, \end{aligned} \tag{1}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, then using Guo-Krasnoselskii and Legget-Williams fixed point theorems obtained some existence and multiplicity results for positive solutions of (1).

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S. Zhang in [20], studied the following nonlinear Caputo fractional boundary value problem

$$\begin{aligned} &{}^cD_{0+}^{\alpha}u(t) = f(t,u(t)), \quad 1<\alpha\leqslant 2,\ 0<t<1,\\ &u(0)+u'(0)=0, \quad u(1)+u'(1)=0, \end{aligned} \tag{2}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. The author using the same fixed point theorems obtained the same solvability results for (2).

The authors in [1], considered the following coupled system of Riemann-Liouville fractional boundary value problems

$$\begin{cases} D^{\alpha}u(t) = f(t,v(t),D^{p}v(t)), & t\in(0,1),\\ D^{\beta}v(t) = g(t,u(t),D^{q}u(t)), & t\in(0,1),\\ u(0)=0, \quad u(1)=\gamma u(\eta), \\ v(0)=0, \quad v(1)=\gamma v(\eta), \end{cases} \tag{3}$$

where $1<\alpha,\beta<2$ and $p,q,\gamma>0$, $0<\eta<1$, $\alpha-q\geqslant 1$, $\beta-p\geqslant 1$, $\gamma\eta^{\alpha-1}<1$, $\gamma\eta^{\beta-1}<1$. They using the Schauder fixed point theorem, proved the existence results for fractional coupled system (3).

Beside these papers, we invite eager followers to consultation in papers [1], [6]–[10], [15], [16], [18], [19], [21] and references cited therein for more interesting results about fractional boundary value problems.

S. Dhar and Q. Kong in [4], studied the half-linear ordinary differential equation

$$\left(\phi_{\alpha_2}\left(\phi_{\alpha_1}\left(x'\right)\right)'\right)' + q(t)\phi_{\alpha_1\alpha_2}(x) = 0, \tag{4}$$

where $q\in C(\mathbb{R},\mathbb{R})$, $\phi_p(x)=|x|^{p-1}x$, $p\in(0,\infty)$. The authors imposing the boundary conditions $x(a)=x(b)=0$, $-\infty<a<b<\infty$ and some additional conditions, obtained the following Lyapunov-type inequality

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q_+(s)ds > \left(\frac{2}{b-a}\right)^{\alpha}, \quad \xi\in[a,b],\ \alpha=(\alpha_1+1)\alpha_2. \tag{5}$$

In above inequality q_- and q_+ denote the negative and positive parts of the function q, respectively.

Motivated by the above works, in this paper we consider the following coupled system of the half-linear (λ,μ)-parametric boundary value problems

$$\begin{cases} \Theta_{\beta_2}\left({}^cD_{a+}^{\alpha}\left(\Theta_{\beta_1}(u)\right)\right) + \lambda\Theta_{\beta_1\beta_2}(f(t,v)) = 0,\\ \\ \Theta_{\gamma_2}\left({}^cD_{a+}^{\beta}\left(\Theta_{\gamma_1}(v)\right)\right) + \mu\Theta_{\gamma_1\gamma_2}(g(t,u)) = 0, \end{cases} \quad \alpha,\beta\in(1,2), \quad t\in(a,b), \tag{6}$$

subject to the Dirichlet boundary conditions

$$\begin{cases} u(a)=u(b)=0,\\ v(a)=v(b)=0, \end{cases} \tag{7}$$

where $\Theta_p(u) = |u|^{p-1}u$ and $p, \beta_i, \gamma_i \in (1, +\infty)$ for $i = 1,2$. ${}^cD^{\alpha}_{a^+}$ denotes the left sided Caputo fractional derivative of order $\alpha > 0$ and λ, μ are positive real parameters. Throughout this paper $f, g \in C([a,b] \times \mathbb{R}, \mathbb{R}^+)$. Our solvability results rely on the nonlinear alternative of the Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems rather than Guo-Krasnoselskii and Legget-Williams fixed point theorems. We notice that in view point of terminology the half-linearity of the fractional coupled system (6) turns to the solution space of the (6) that characterize the homogeneity but not additivity of solutions. See [5] for details.

2. Preliminaries

In the sequel we represent some standard definitions and lemmas from theory of fractional calculus.

DEFINITION 1. [3] Let $\alpha \in (0, +\infty)$. The operator $I^{\alpha}_{a^+}$ defined on $L_1[a,b]$ by

$$I^{\alpha}_{a^+} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \tag{8}$$

for $a \leqslant t \leqslant b$, is called the left-sided *Riemann-Liouville* fractional integral operator of order α.

Under same hypotheses, the right-sided *Riemann-Liouville* fractional integral operator is given by

$$_{b\_}I^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds. \tag{9}$$

DEFINITION 2. [3] Suppose $\alpha > 0$ with $n = [\alpha] + 1$. Then the left and right sided *Caputo* fractional derivatives defined on absolutely continuous functions space $AC^n[a,b]$ are given by

$$\left({}^cD^{\alpha}_{a^+} u\right)(t) = \left(I^{n-\alpha}_{a^+} D^n u\right)(t), \tag{10a}$$

$$\left({}^c_{b\_}D^{\alpha} u\right)(t) = (-1)^n \left({}_{b\_}I^{n-\alpha} D^n u\right)(t), \tag{10b}$$

where $D^n \equiv d^n/dt^n$.

LEMMA 1. [11] *Assume that* $\alpha > 0$. *Then*

(i) for $u(t) \in L_1(a,b)$, *we have*

$$\left({}^cD^{\alpha}_{a^+} I^{\alpha}_{a^+} u\right)(t) = u(t), \quad \left({}^c_{b\_}D^{\alpha}\ {}_{b\_}I^{\alpha} u\right)(t) = u(t). \tag{11}$$

(ii) for $u(t) \in AC^n[a,b]$, *we have*

$$\left(I^{\alpha}_{a^+}\, {}^cD^{\alpha}_{a^+} u\right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t-a)^k, \tag{12a}$$

$$\left({}_{b-}I^{\alpha}\, {}^{c}_{b-}D^{\alpha} u \right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{(-1)^k u^{(k)}(b)}{k!} (b-t)^k. \tag{12b}$$

In this position, the nonlinear alternative of the Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems that will be applied to verify existence at least one positive solution for fractional coupled system (6) can be stated as follows, respectively.

THEOREM 1. [21], [10] *Let C be a convex subset of a Banach space, U be an open subset of C with $0 \in U$. Then every completely continuous map $T : \overline{U} \to C$ has at least one of the two following properties:*

(E_1) *There exist an $u \in \overline{U}$ such that $Tu = u$.*

(E_2) *There exist an $v \in \partial U$ and $\lambda \in (0,1)$ such that $v = \lambda T v$.*

THEOREM 2. [12], [9] *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous mapping. If $L : X \to X$ be a linear bounded mapping such that 1 is not an eigenvalue of L and*

$$\lim_{\|u\| \to \infty} \frac{\|Tu - Lu\|}{\|u\|} = 0, \tag{13}$$

then T has a fixed point in X.

At the end of this section we introduce the Banach spaces needed in what follows.

$$Y = X \times X, \quad X = (C[a,b], \|.\|_X), \tag{14}$$

endowed with the norm

$$\|(u,v)\|_Y = \|u\|_X + \|v\|_X, \quad \|u\|_X = \sup_{t \in [a,b]} |u(t)|. \tag{15}$$

3. Existence results

Taking the importance of the Green function of the coupled system (6) into account, we begin with characterization the Green function corresponding to the (6) as follows.

LEMMA 2. *Let $h \in C(\mathbb{R})$. Then for the fractional half-linear boundary value problem*

$$\begin{cases} \Theta_{\beta_2}\left({}^{c}D^{\alpha}_{a^+}\left(\Theta_{\beta_1}(u)\right)\right) + h(t) = 0, \ 1 < \alpha < 2, \ a < t < b, \\ u(a) = u(b) = 0, \end{cases} \tag{16}$$

the equivalent unique integral equation reads as follows

$$u(t) = \Theta_{\beta_1^{-1}}\left(\int_a^b G_\alpha(t,s) \Theta_{\beta_2^{-1}} h(s) ds \right), \tag{17}$$

where

$$G_\alpha(t,s) = \frac{1}{(b-a)\Gamma(\alpha)} \begin{cases} (t-a)(b-s)^{\alpha-1} - (b-a)(t-s)^{\alpha-1}; & a < s \leqslant t < b, \\ (t-a)(b-s)^{\alpha-1}; & a < t \leqslant s < b. \end{cases} \tag{18}$$

Proof. Using the identity (12a), the fractional half-linear differential equation

$$\Theta_{\beta_2}\left({}^cD^\alpha_{a^+}\left(\Theta_{\beta_1}(u)\right)\right) + h(t) = 0,$$

reduces to the fractional integral equation

$$\Theta_{\beta_1}(u)(t) = c_0 + c_1(t-a) - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Theta^{-1}_{\beta_2}h(s)ds. \tag{19}$$

Imposing the boundary condition $u(a) = 0$, immediately it follows that $c_0 = 0$. Next, second boundary condition $u(b) = 0$, explicitly gives us the coefficient c_1 as follows

$$c_1 = \frac{1}{(b-a)\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\Theta^{-1}_{\beta_2}h(s)ds. \tag{20}$$

Substituting c_0, c_1 obtained above in (19), implies that

$$\begin{aligned}
\Theta_{\beta_1}(u)(t) &= \frac{t-a}{b-a}\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\Theta^{-1}_{\beta_2}h(s)ds - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Theta^{-1}_{\beta_2}h(s)ds \\
&= \frac{1}{b-a}\left\{(t-a)\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\Theta^{-1}_{\beta_2}h(s)ds - (b-a)\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Theta^{-1}_{\beta_2}h(s)ds\right\} \\
&= \frac{1}{(b-a)\Gamma(\alpha)}\int_a^t [(t-a)(b-s)^{\alpha-1} - (b-a)(t-s)^{\alpha-1}]\Theta^{-1}_{\beta_2}h(s)ds \\
&\quad + \frac{1}{(b-a)\Gamma(\alpha)}\int_t^b (t-a)(b-s)^{\alpha-1}\Theta^{-1}_{\beta_2}h(s)ds \\
&= \int_a^b G_\alpha(t,s)\Theta_{\beta_2^{-1}}h(s)ds.
\end{aligned}$$

So we have

$$\Theta_{\beta_1}(u)(t) = \int_a^b G_\alpha(t,s)\Theta_{\beta_2^{-1}}h(s)ds. \tag{21}$$

Finally, taking inverse operator $\Theta_{\beta_1^{-1}}$ on both sides of (21), we achieve the following

$$u(t) = \Theta_{\beta_1^{-1}}\left(\int_a^b G_\alpha(t,s)\Theta_{\beta_2^{-1}}h(s)ds\right),$$

that completes the proof. □

LEMMA 3. *The Green function $G_\alpha(t,s)$ given by (18) satisfies in the following properties:*

(*i*) $G_\alpha(t,s) > 0$ *for* $(t,s) \in (a,b)\times(a,b)$.

(*ii*) $\sup_{t\in(a,b)} G_\alpha(t,s) = G_\alpha(s,s), \quad \sup_{s\in(a,b)} G_\alpha(s,s) = \frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)(\alpha-1)}{\alpha}\right)^{\alpha-1}.$

(*iii*) *There exists positive constant* $\gamma_\alpha \in (0,1)$ *such that*

$$\min_{t\in\left[\frac{b(3+\alpha)+3a(\alpha-1)}{4\alpha},\frac{b(3\alpha+1)+a(\alpha-1)}{4\alpha}\right]} G_\alpha(t,s) \geqslant \gamma_\alpha \sup_{t\in(a,b)} G_\alpha(t,s).$$

Proof. Let us rewrite the Green function $G_\alpha(t,s)$ defined by (18) as below:

$$G_\alpha(t,s) = \frac{1}{(b-a)\Gamma(\alpha)}\begin{cases} G_{1,\alpha}(t,s); & a<s\leqslant t<b,\\ G_{2,\alpha}(t,s); & a<t\leqslant s<b,\end{cases}$$

where

$$G_{1,\alpha}(t,s) = (t-a)(b-s)^{\alpha-1} - (b-a)(t-s)^{\alpha-1}, \tag{22a}$$

and

$$G_{2,\alpha}(t,s) = (t-a)(b-s)^{\alpha-1}. \tag{22b}$$

Obviously $G_{2,\alpha}(t,s) > 0$ and $G_{2,\alpha}(t,s) > G_{1,\alpha}(t,s)$. So for positivity of the Green function $G_\alpha(t,s)$ it suffices that we prove the positivity of $G_{1,\alpha}(t,s)$. To this aim if we take $t=s$, so $G_\alpha(t,s)$ is obviously positive, otherwise we proceed as follows:

$$G_{1,\alpha}(t,s)>0 \Leftrightarrow (t-a)(b-s)^{\alpha-1}-(b-a)(t-s)^{\alpha-1}>0 \Leftrightarrow \frac{(b-s)^{\alpha-1}}{b-a} > \frac{(t-s)^{\alpha-1}}{t-a}.$$

Now let us define

$$\phi(w) = \frac{(w-s)^{\alpha-1}}{w-a}. \tag{23}$$

Therefore

$$\phi'(w) = \frac{(w-s)^{\alpha-2}[(\alpha-1)-\frac{w-s}{w-a}]}{w-a}, \quad w\neq a,s.$$

Thus for each $w<\frac{b+a(\alpha-1)}{2-\alpha}$, ϕ is increasing and for each $w>\frac{b-a(\alpha-1)}{2-\alpha}$ is decreasing. Taking $w=b$ in the first inequality, we conclude the obvious result $b<\frac{b+a(\alpha-1)}{2-\alpha}$. since $b>t$, then $\phi(b)>\phi(t)$ and consequently

$$\frac{(b-s)^{\alpha-1}}{b-a} > \frac{(t-s)^{\alpha-1}}{t-a}.$$

Thereby $G_{1,\alpha}(t,s)>0$. This completes the proof of the item (i).

Proving the item (ii), we note that, since $\frac{\partial}{\partial t}G_{2,\alpha}(t,s)>0$

$$\sup_{\substack{t\in(a,b),\\ s\in(a,b)}} G_\alpha(t,s)=\sup_{\substack{t,s\in(a,b),\\ t\leqslant s}} G_{2,\alpha}(t,s)=\sup_{s\in(a,b)} G_{2,\alpha}(s,s).$$

On the other hand

$$G'_{2,\alpha}(s,s)=(b-s)^{\alpha-2}[-\alpha s+b+a(\alpha-1)].$$

Thus for each $s<\frac{b+a(\alpha-1)}{\alpha}$, $G_{2,\alpha}(s,s)$ is increasing and for each $s>\frac{b+a(\alpha-1)}{\alpha}$ is decreasing. Therefore we deduce that

$$\sup_{s\in(a,b)} G_{2,\alpha}(s,s)=G_{2,\alpha}\left(\frac{b+a(\alpha-1)}{\alpha},\frac{b+a(\alpha-1)}{\alpha}\right)=\left(\frac{b-a}{\alpha}\right)^{\alpha}(\alpha-1)^{\alpha-1}.$$

Hence, it follows that

$$\sup_{s\in(a,b)} G_\alpha(s,s)=\frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)(\alpha-1)}{\alpha}\right)^{\alpha-1}.$$

To prove the last item (iii), we set

$$\gamma_\alpha=\frac{\min_{t\in\left[\frac{b(3+\alpha)+3a(\alpha-1)}{4\alpha},\frac{b(3\alpha+1)+a(\alpha-1)}{4\alpha}\right]} G_\alpha(t,s)}{\sup_{s\in(a,b)} G_\alpha(s,s)}. \tag{24}$$

We know that $\min_{t,s\in(a,b)} G_\alpha(t,s)=G_{1,\alpha}(t,s)$. Also we know that $\frac{\partial}{\partial s}G_{1,\alpha}(t,s)>0$. So it follows that

$$\sup_{s\leqslant t} G_{1,\alpha}(t,s)=G_{2,\alpha}(t,t),\quad t\in(a,b).$$

Let us point out this fact that $\dfrac{b(3+\alpha)+3a(\alpha-1)}{4\alpha}>\dfrac{b+a(\alpha-1)}{\alpha}$. Since for each $t>\dfrac{b+a(\alpha-1)}{\alpha}$, $G_{2,\alpha}(t,t)$ is decreasing, so we have

$$\begin{aligned}
&\min_{t\in\left[\frac{b(3+\alpha)+3a(\alpha-1)}{4\alpha},\frac{b(3\alpha+1)+a(\alpha-1)}{4\alpha}\right]} G_\alpha(t,s)\\
&=G_{2,\alpha}\left(\frac{b(3\alpha+1)+a(\alpha-1)}{4\alpha},\frac{b(3\alpha+1)+a(\alpha-1)}{4\alpha}\right)\\
&=\frac{1}{4^\alpha\Gamma(\alpha+1)}\left(\frac{(b-a)(\alpha-1)}{\alpha}\right)^{\alpha-1}(3\alpha+1).
\end{aligned} \tag{25}$$

Thereby, using item (ii) we conclude that

$$\gamma_\alpha=\frac{\frac{1}{4^\alpha\Gamma(\alpha+1)}\left(\frac{(b-a)(\alpha-1)}{\alpha}\right)^{\alpha-1}(3\alpha+1)}{\frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)(\alpha-1)}{\alpha}\right)^{\alpha-1}}=\frac{3\alpha+1}{4^\alpha}\in(0,1).$$

This completes the proof of the item (iii). □

We are preparing ourselves to link the fractional half-linear coupled system (6) and introduced fixed point theorems. To this aim we define the integral operator $A : C \subset Y \to Y$ as follows

$$A(u,v) = (A_1 v, A_2(u)), \tag{26}$$

where

$$(A_1 v)(t) = \lambda^{\frac{1}{\beta_1 \beta_2}} \Theta_{\beta_1^{-1}} \left(\int_a^b G_\alpha(t,s) \Theta_{\beta_1}(f(s,v))\, ds \right), \tag{27a}$$

and

$$(A_2 u)(t) = \mu^{\frac{1}{\gamma_1 \gamma_2}} \Theta_{\gamma_1^{-1}} \left(\int_a^b G_\beta(t,s) \Theta_{\gamma_1}(g(s,u))\, ds \right), \tag{27b}$$

and

$$\begin{aligned}
&C = C_1 \oplus C_2, \\
&C_1 = \left\{ (0,v) \in Y \,\middle|\, v(t) \geqslant 0,\ t \in (a,b), \quad \min_{t \in \Delta_\alpha} v(t) \geqslant \gamma_\alpha \|v\|_X \right\}, \\
&C_2 = \left\{ (u,0) \in Y \,\middle|\, u(t) \geqslant 0,\ t \in (a,b), \quad \min_{t \in \Delta_\beta} u(t) \geqslant \gamma_\beta \|u\|_X \right\}, \\
&\Delta_\alpha = \left[\frac{b(3+\alpha) + 3a(\alpha - 1)}{4\alpha}, \frac{b(3\alpha + 1) + a(\alpha - 1)}{4\alpha} \right], \\
&\Delta_\beta = \left[\frac{b(3+\beta) + 3a(\beta - 1)}{4\beta}, \frac{b(3\beta + 1) + a(\beta - 1)}{4\beta} \right].
\end{aligned} \tag{27c}$$

LEMMA 4. *Assume the operator A and the cone C are given by (26) and (27c), respectively. Then A leaves the cone C invariant, i.e. $A(C) \subset C$.*

Proof. Suppose $(u,v) \in C$. Trivially we observe that $(A_1 v)(t) \geqslant 0$ and $(A_2 u)(t) \geqslant 0$.

Consider the integral operator

$$(A_1 v)(t) = \lambda^{\frac{1}{\beta_1 \beta_2}} \Theta_{\beta_1^{-1}} \left(\int_a^b G_\alpha(t,s) \Theta_{\beta_1}(f(s,v))\, ds \right).$$

So we have

$$\begin{aligned}\min_{t\in\Delta_\alpha}(A_1v)(t) &= \lambda^{\frac{1}{\beta_1\beta_2}}\min_{t\in\Delta_\alpha}\left[\Theta_{\beta_1^{-1}}\left(\int_a^b G_\alpha(t,s)\Theta_{\beta_1}(f(s,v))\,ds\right)\right]\\ &\geqslant \lambda^{\frac{1}{\beta_1\beta_2}}\Theta_{\beta_1^{-1}}\left(\int_a^b \min_{t\in\Delta_\alpha} G_\alpha(t,s)\Theta_{\beta_1}(f(s,v))\,ds\right)\\ &\geqslant \lambda^{\frac{1}{\beta_1\beta_2}}\Theta_{\beta_1^{-1}}\left(\int_a^b \gamma\sup_{t\in(a,b)} G_\alpha(t,s)\Theta_{\beta_1}(f(s,v))\,ds\right)\\ &\geqslant \gamma_\alpha^{\frac{1}{\beta_1}}\sup_{t\in(a,b)}\left[\lambda^{\frac{1}{\beta_1\beta_2}}\Theta_{\beta_1^{-1}}\left(\int_a^b G_\alpha(t,s)\Theta_{\beta_1}(f(s,v))\,ds\right)\right]\\ &\geqslant \gamma_\alpha\|A_1v\|_X.\end{aligned} \tag{28}$$

Similarly one can show that

$$\min_{t\in\Delta_\beta}(A_2u)(t)\geqslant \gamma_\beta\|A_2u\|_X. \tag{29}$$

Therefore by means of (28) and (29), we conclude that for each $(u,v)\in C$,

$$\left\{(A_1v)(t),(A_2u)(t)\geqslant 0,\ t\in(a,b),\quad \begin{array}{l}\min_{t\in\Delta_\alpha}(A_1v)(t)\geqslant\gamma_\alpha\|A_1v\|_X,\\ \min_{t\in\Delta_\beta}(A_2u)(t)\geqslant\gamma_\beta\|A_2u\|_X\end{array}\right\}. \tag{30}$$

Equivalently has shown that $A(C)\subset C$. The proof is completed now. □

LEMMA 5. *The integral operator $A:Y\to Y$ defined by (26)–(27b) is completely continuous.*

Proof. First we notice that continuity of the Green function $G_\alpha(t,s)$ and half-linear operator $\phi_p u$ for $p\in(1,+\infty)$, ensure the continuity of the integral operator A_1v defined by

$$(A_1v)(t)=\lambda^{\frac{1}{\beta_1\beta_2}}\Theta_{\beta_1^{-1}}\left(\int_a^b G_\alpha(t,s)\Theta_{\beta_1}(f(s,v))\,ds\right).$$

Let us consider the bounded set $\Omega_1\subset X$. So there exists positive real constant L_1 such that

$$\|v\|_X\leqslant L_1,\quad v\in X.$$

Also suppose that

$$M_1=\sup_{\substack{t\in(a,b),\\ \|v\|_X\leqslant L_1}}|f(t,v)|.$$

Therefore we have

$$\begin{aligned} |A_1 v(t)| &= \left| \lambda^{\frac{1}{\beta_1 \beta_2}} \Theta_{\beta_1^{-1}} \left(\int_a^b G_\alpha(t,s) \Theta_{\beta_1}(f(s,v))\, ds \right) \right| \\ &\leqslant \lambda^{\frac{1}{\beta_1 \beta_2}} \left(\int_a^b G_\alpha(t,s) |\Theta_{\beta_1}(f(s,v))|\, ds \right)^{\frac{1}{\beta_1}} \\ &\leqslant \lambda^{\frac{1}{\beta_1 \beta_2}} \left(\int_a^b G_\alpha(s,s) |(f(s,v))|^{\beta_1} ds \right)^{\frac{1}{\beta_1}} \\ &\leqslant \lambda^{\frac{1}{\beta_1 \beta_2}} \left(\frac{1}{\Gamma(\alpha)} \left(\frac{b-a}{\alpha} \right)^{\alpha} (\alpha-1)^{\alpha-1} M_1^{\beta_1} \right)^{\frac{1}{\beta_1}} . \end{aligned} \tag{31}$$

Thereby we conclude that

$$\|A_1 v\|_X \leqslant M_1 . \lambda^{\frac{1}{\beta_1 \beta_2}} \left(\frac{1}{\Gamma(\alpha)} \left(\frac{b-a}{\alpha} \right)^{\alpha} (\alpha-1)^{\alpha-1} \right)^{\frac{1}{\beta_1}} . \tag{32}$$

Similarly we deduce that

$$\|A_2 u\|_X \leqslant M_2 . \mu^{\frac{1}{\gamma_1 \gamma_2}} \left(\frac{1}{\Gamma(\beta)} \left(\frac{b-a}{\beta} \right)^{\beta} (\beta-1)^{\beta-1} \right)^{\frac{1}{\gamma_1}} , \tag{33}$$

in which there exists a bounded subset $\Omega_2 \subset X$ and there exists a positive constant L_2 such that for each $u \in \Omega_2$, we have $\|u\|_X \leqslant L_2$. In addition

$$M_2 = \sup_{\substack{t \in (a,b), \\ \|u\|_X \leqslant L_2}} |g(t,u)|.$$

Thus one can conclude that

$$\begin{aligned} \|A(u,v)\|_Y &= \|A_1 v\|_X + \|A_2 u\|_X \\ &\leqslant 2 \max \left\{ M_1 . \lambda^{\frac{1}{\beta_1 \beta_2}} \left(\frac{1}{\Gamma(\alpha)} \left(\frac{b-a}{\alpha} \right)^{\alpha} (\alpha-1)^{\alpha-1} \right)^{\frac{1}{\beta_1}} , \right. \\ &\left. M_2 . \mu^{\frac{1}{\gamma_1 \gamma_2}} \left(\frac{1}{\Gamma(\beta)} \left(\frac{b-a}{\beta} \right)^{\beta} (\beta-1)^{\beta-1} \right)^{\frac{1}{\gamma_1}} \right\}. \end{aligned} \tag{34}$$

Therefore setting $\Omega = \Omega_1 \times \Omega_1$, it has shown that $A(\Omega)$ is bounded.

In the last step we are going to prove the equicontinuity of the integral operator $A(u,v)$. Let $v \in \Omega_1$ and $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. So we have

$$
\begin{aligned}
|A_1v(t_2)-A_1v(t_1)| &\leqslant M_1.\lambda^{\frac{1}{\beta_1\beta_2}}\left|\int_a^b[G_\alpha(t_2,s)-G_\alpha(t_1,s)]ds\right|^{\frac{1}{\beta_1}} \\
&\leqslant \frac{M_1.\lambda^{\frac{1}{\beta_1\beta_2}}}{\Gamma^{\frac{1}{\beta_1}}(\alpha)}\left|\int_a^{t_2}\left[(t_2-a)(b-s)^{\alpha-1}-(b-a)(t_2-s)^{\alpha-1}\right]ds+\int_{t_2}^b(t_2-a)(b-s)^{\alpha-1}ds\right. \\
&\quad\left.-\int_a^{t_1}\left[(t_1-a)(b-s)^{\alpha-1}-(b-a)(t_1-s)^{\alpha-1}\right]ds-\int_{t_1}^b(t_1-a)(b-s)^{\alpha-1}ds\right|^{\frac{1}{\beta_1}} \\
&\leqslant \frac{M_1.\lambda^{\frac{1}{\beta_1\beta_2}}}{\Gamma^{\frac{1}{\beta_1}}(\alpha+1)}\left\{\left|(t_2-a)\left[(b-a)^\alpha-(b-t_2)^\alpha\right]-(t_1-a)\left[(b-a)^\alpha-(b-t_1)^\alpha\right]\right|\right. \\
&\quad\left.+(b-a)\left|\left[(t_2-a)^\alpha-(t_1-a)^\alpha\right]\right|+\left|(t_2-a)(b-t_2)^\alpha-(t_1-a)(b-t_1)^\alpha\right|\right\}^{\frac{1}{\beta_1}}.
\end{aligned}
\tag{35}
$$

Therefore, one can derive that the right hand side of the inequality (35), tends to zero provided that $t_2 \to t_1$. Hence the Arzela-Ascoli theorem implies that the integral operator A_1v is completely continuous. In similar manner one can prove that the integral operator A_2u is also completely continuous. Thereafter completely continuity of the operators A_1v and A_2u ensure the completely continuity of the integral operator $A(u,v)=(A_1v,A_2u)$. The proof is completed. □

In order to apply the nonlinear alternative of Leray-Schauder fixed point theorem, assume that the following stipulations are valid.

HYPOTHESES 1. There exist positive continuous functions ϕ_i, ψ_i, $i=1,2$ with ψ_i increasing, such that

(H_1) $|f(t,v)| \leqslant \phi_1(t)\psi_1(|v|)$, $(t,v)\in(a,b)\times\mathbb{R}$;

(H_2) $|g(t,u)| \leqslant \phi_2(t)\psi_2(|u|)$, $(t,u)\in(a,b)\times\mathbb{R}$.

THEOREM 3. *Let the hypotheses (H_1) and (H_2) are satisfied. Assume that there exists real constant $\theta>0$ such that*

$$
\frac{1}{\lambda^{\frac{1}{\beta_1\beta_2}}\Lambda_1^{\frac{1}{\beta_1}}}+\frac{1}{\mu^{\frac{1}{\gamma_1\gamma_2}}\Lambda_2^{\frac{1}{\gamma_1}}} > \frac{1}{\theta}\left\{\psi_1(\theta)\int_a^b|\phi_1(s)|ds+\psi_2(\theta)\int_a^b|\phi_2(s)|ds\right\}, \tag{36}
$$

where

$$
\Lambda_1=\frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)(\alpha-1)}{\alpha}\right)^{\alpha-1},
$$

$$
\Lambda_2=\frac{1}{\Gamma(\beta+1)}\left(\frac{(b-a)(\beta-1)}{\beta}\right)^{\beta-1}.
$$

Then the coupled system of fractional half-linear boundary value problems (6) has at least one positive solution in C.

Proof. Consider the coupled system

$$\begin{cases} \Theta_{\beta_2}\left({}^{c}D^{\alpha}_{a^+}\left(\Theta_{\beta_1}(u)\right)\right) + \nu\lambda\Theta_{\beta_1\beta_2}(f(t,v)) = 0, & \\ & \alpha \in (1,2), \quad t \in (a,b), \\ \Theta_{\gamma_2}\left({}^{c}D^{\beta}_{a^+}\left(\Theta_{\gamma_1}(v)\right)\right) + \nu\mu\Theta_{\gamma_1\gamma_2}(g(t,u)) = 0, & \\ u(a) = u(b) = 0, & \\ v(a) = v(b) = 0, & \end{cases} \tag{37}$$

where $\nu \in (0,1)$. We define

$$\mathscr{O} = \{(u,v) \in C |\ \|u\|_X < \theta, \|v\|_X < \theta\}. \tag{38}$$

Now, we must prove that $(v,u) \neq \nu A(u,v)$, with $(u,v) \in \partial\mathscr{O}$ and $\nu \in (0,1)$. Therefore assume on contrary that there exists $(u,v) \in \partial\mathscr{O}$ such that $(v,u) = \nu A(u,v) = \nu(A_1(v), A_2(u))$. Thus it follows that:

$$\begin{aligned} \|v\|_X = \nu\|A_1 v\|_X &= \nu\lambda^{\frac{1}{\beta_1\beta_2}} \sup_{t\in(a,b)} \Theta_{\frac{1}{\beta_1}}\left(\int_a^b G_\alpha(t,s)\Theta_{\beta_1} f(s,v)ds\right) \\ &\leqslant \lambda^{\frac{1}{\beta_1\beta_2}} \left| \int_a^b \sup_{t\in(a,b)} G_\alpha(t,s)\Theta_{\beta_1} f(s,v)ds \right|^{\frac{1}{\beta_1}} \\ &\leqslant \lambda^{\frac{1}{\beta_1\beta_2}} \Lambda_1^{\frac{1}{\beta_1}} \int_a^b |f(s,v)|ds \leqslant \lambda^{\frac{1}{\beta_1\beta_2}} \Lambda_1^{\frac{1}{\beta_1}} \psi_1(\theta) \int_a^b |\phi_1(s)|ds. \end{aligned} \tag{39}$$

So we have

$$\theta \leqslant \lambda^{\frac{1}{\beta_1\beta_2}} \Lambda_1^{\frac{1}{\beta_1}} \psi_1(\theta) \int_a^b |\phi_1(s)|ds.$$

Hence the following is immediate

$$\frac{1}{\lambda^{\frac{1}{\beta_1\beta_2}} \Lambda_1^{\frac{1}{\beta_1}}} \leqslant \frac{1}{\theta}\psi_1(\theta) \int_a^b |\phi_1(s)|ds. \tag{40}$$

Similarly one can deduce

$$\frac{1}{\mu^{\frac{1}{\gamma_1\gamma_2}} \Lambda_2^{\frac{1}{\gamma_1}}} \leqslant \frac{1}{\theta}\psi_2(\theta) \int_a^b |\phi_2(s)|ds. \tag{41}$$

The inequalities (40) and (41), give us

$$\frac{1}{\lambda^{\frac{1}{\beta_1\beta_2}} \Lambda_1^{\frac{1}{\beta_1}}} + \frac{1}{\mu^{\frac{1}{\gamma_1\gamma_2}} \Lambda_2^{\frac{1}{\gamma_1}}} \leqslant \frac{1}{\theta}\left\{ \psi_1(\theta) \int_a^b |\phi_1(s)|ds + \psi_2(\theta) \int_a^b |\phi_2(s)|ds \right\}, \tag{42}$$

which contradicts with (36). Thus we conclude that for each $(u,v)\in\partial\mathscr{O}$ and for each $\nu\in(0,1)$, $(v,u)\neq \nu A(u,v)$. Therefore in accordance with Theorem 1 particularly the item (E_1), one has that the fixed point problem $(v,u)=(A_1v,A_2u)=A(u,v)$ has at least one fixed point (u,v) in cone C. Equivalently the coupled system of the fractional half-linear boundary value problems (6) has at least one positive solution in C. This completes the proof. □

In the sequel, in order to proper running of the mechanism of the Krasnoselskii-Zabreiko fixed point theorem stated above, we introduce forthcoming hypotheses.

HYPOTHESES 2. Let the following hypotheses hold:

$$(K_1)\quad \lim_{\|v\|_X\to\infty}\frac{A_1v}{v}\leqslant \lambda^{\frac{1}{\beta_1\beta_2}}\Theta_{\beta_1^{-1}}\left(\int_a^b G_{2,\alpha}(t,s)\Theta_{\beta_1}(\xi_1(s))ds\right);$$

$$(K_2)\quad \lim_{\|u\|_X\to\infty}\frac{A_2u}{u}\leqslant \mu^{\frac{1}{\gamma_1\gamma_2}}\Theta_{\gamma_1^{-1}}\left(\int_a^b G_{2,\beta}(t,s)\Theta_{\gamma_1}(\xi_2(s))ds\right).$$

THEOREM 4. *Let the hypotheses (K_1) and (K_2) are satisfied. If*

$$\int_a^b G_{2,\alpha}(t,s)ds<\|\xi_1\|_X^{-\beta_1},\quad \int_a^b G_{2,\beta}(t,s)ds<\|\xi_2\|_X^{-\gamma_1},\tag{43}$$

then the coupled system of fractional half-linear boundary value problems (6) has at east one positive solution in C.

Proof. We begin the proof with introducing the linear bounded mappings $L_i : C_i\subset X\to X$, $i=1,2$ defined by

$$\begin{aligned}L_1v(t)&=\lambda^{\frac{1}{\beta_1\beta_2}}\Theta_{\beta_1^{-1}}\left(\int_a^b G_{2,\alpha}(t,s)\Theta_{\beta_1}(\xi_1(s))ds\right)v(t),\\ L_2u(t)&=\mu^{\frac{1}{\gamma_1\gamma_2}}\Theta_{\gamma_1^{-1}}\left(\int_a^b G_{2,\beta}(t,s)\Theta_{\beta_1}(\xi_2(s))ds\right)u(t).\end{aligned}\tag{44}$$

Using inequalities (43), it is clear that

$$\|L_1v\|_X<\lambda^{\frac{1}{\beta_1\beta_2}}\|v\|_X,\quad \|L_2u\|_X<\mu^{\frac{1}{\gamma_1\gamma_2}}\|u\|_X.\tag{45}$$

It means that 1 is not an eigenvalue of the operators L_i for $i=1,2$. Thus defining $L(u,v)=(L_1v,L_2u)$, it follows that the linear bounded mapping L dos not admit the pair $(1,1)$ as eigenvalue. In the sequel, considering the hypotheses (K_1) and (K_2), one can deduce that for each arbitrary $\varepsilon>0$ there exists a positive constant $N>0$ such that $\|u\|_X > N$ and $\|v\|_X > N$, imply that

$$\begin{aligned} &\left\| A_1 v - \lambda^{\frac{1}{\beta_1\beta_2}} \Theta_{\beta_1^{-1}} \left(\int_a^b G_{2,\alpha}(t,s)\Theta_{\beta_1}(\xi_1(s))ds \right) v \right\|_X < \varepsilon \|v\|_X, \\ &\left\| A_2 u - \mu^{\frac{1}{\gamma_1\gamma_2}} \Theta_{\gamma_1^{-1}} \left(\int_a^b G_{2,\beta}(t,s)\Theta_{\gamma_1}(\xi_2(s))ds \right) u \right\|_X < \varepsilon \|u\|_X. \end{aligned} \tag{46}$$

Therefore we have

$$\lim_{\|v\|_X \to \infty} \frac{\|A_1 v - L_1 v\|_X}{\|v\|_X} = 0, \tag{47a}$$

$$\lim_{\|u\|_X \to \infty} \frac{\|A_2 u - L_2 u\|_X}{\|u\|_X} = 0. \tag{47b}$$

Using (47a) and (47b), we conclude that

$$\begin{aligned} \lim_{\substack{\|u\|_X \to \infty, \\ \|v\|_X \to \infty}} \frac{\|A(u,v) - L(u,v)\|_Y}{\|(u,v)\|_Y} &= \lim_{\substack{\|u\|_X \to \infty, \\ \|v\|_X \to \infty}} \frac{\|\left((A_1 v - L_1 v),(A_2 u - L_2 u)\right)\|_Y}{\|u\|_X + \|v\|_X} \\ &\leqslant \lim_{\|v\|_X \to \infty} \frac{\|A_1 v - L_1 v\|_X}{\|v\|_X} + \lim_{\|u\|_X \to \infty} \frac{\|A_2 u - L_2 u\|_X}{\|u\|_X} \\ &= 0. \end{aligned}$$

Therefore the Krasnoselskii-Zabreiko fixed point theorem ensures that the coupled system of fractional half-linear boundary value problems (6) has at least one positive solution in C. The proof is completed. □

4. Numerical examples

Implementing the obtained theoretical main results, we present some numerical examples as follows.

EXAMPLE 1. Let us consider the following coupled system of fractional half-linear boundary value problems

$$\begin{cases} \Theta_2\left({}^cD_{0^+}^{1.5}(\Theta_2(u)) \right) + \dfrac{1}{10}\Theta_{2\times 2}(e^t(1+|v|)) = 0, \\ \Theta_2\left({}^cD_{0^+}^{1.5}(\Theta_2(v)) \right) + \dfrac{1}{10}\Theta_{2\times 2}\left(e^{-t}\left(1+\frac{|u|}{2}\right)\right) = 0, \end{cases} \quad t \in (0,1), \tag{48}$$

with boundary conditions

$$\begin{cases} u(0) = u(1) = 0, \\ v(0) = v(1) = 0. \end{cases} \tag{49}$$

Indeed, in the above system the setting

$$\phi_1(t) = e^t, \quad \phi_1(t) = e^{-t};$$
$$\psi_1(|v|) = 1 + |v|, \quad \psi_2(|u|) = 1 + \frac{|u|}{2};$$
$$\beta_i = \gamma_i = 2, \quad i = 1,2, \quad \lambda = \mu = \frac{1}{10}, \ \theta = 1, \ \alpha = \beta = 1.5,$$

have implemented. As a result of positivity and increasing nature of the functions ψ_i, $i = 1,2$, we conclude that the hypotheses (H_1) and (H_2) are fulfilled. On the other hand, a direct calculation shows that

$$\frac{1}{\theta}\left\{\psi_1(\theta)\int_a^b |\phi_1(s)|ds + \psi_2(\theta)\int_a^b |\phi_2(s)|ds\right\} = \frac{3}{2}(e - e^{-1}),$$

$$\frac{1}{\lambda^{\frac{1}{\beta_1\beta_2}}\Lambda_1^{\frac{1}{\beta_1}}} + \frac{1}{\mu^{\frac{1}{\gamma_1\gamma_2}}\Lambda_2^{\frac{1}{\gamma_1}}} = \sqrt{\frac{2}{3\sqrt{3\pi}}} \times 4 \times 10^4.$$

So, it is clear that

$$\frac{1}{\lambda^{\frac{1}{\beta_1\beta_2}}\Lambda_1^{\frac{1}{\beta_1}}} + \frac{1}{\mu^{\frac{1}{\gamma_1\gamma_2}}\Lambda_2^{\frac{1}{\gamma_1}}} > \frac{1}{\theta}\left\{\psi_1(\theta)\int_a^b |\phi_1(s)|ds + \psi_2(\theta)\int_a^b |\phi_2(s)|ds\right\},$$

Consequently, since all of the conditions of Theorem 3 hold, then the fractional coupled system (48)–(49) has at least one positive solution in C.

EXAMPLE 2. Consider the coupled system of fractional half-linear boundary value problems

$$\begin{cases} \Theta_{\beta_2}\left({}^cD^{\alpha}_{a^+}\left(\Theta_{\beta_1}(u)\right)\right) + \lambda\Theta_{\beta_1\beta_2}(f(t,v)) = 0, \\ \Theta_{\gamma_2}\left({}^cD^{\beta}_{a^+}\left(\Theta_{\gamma_1}(v)\right)\right) + \mu\Theta_{\gamma_1\gamma_2}(g(t,u)) = 0, \end{cases} \quad \alpha,\beta \in (1,2), \quad t \in (0,1), \quad (50)$$

subject to the boundary conditions

$$\begin{cases} u(0) = u(1) = 0, \\ v(0) = v(1) = 0, \end{cases} \quad (51)$$

where $f(t,v) = v$ and $g(t,u) = u$. Suppose that u and v are two positive increasing continuous functions such that

$$\int_0^1 v^{\beta_1}(s)ds < v^{\beta_1}(t), \quad \int_0^1 u^{\gamma_1}(s)ds < u^{\gamma_1}(t).$$

Before continuing estimation process let us point out the following key inequality.

REMARK 1. Assume that both f and g are increasing(decreasing) on (a,b). Then

$$\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right) \leqslant (b-a)\int_a^b f(x)g(x)dx. \tag{52}$$

If f is decreasing and g is increasing, then the counter-inequality is satisfied.

As we know, $G_\alpha(t,s) \leqslant G_{2,\alpha}(t,s)$ and $G_{2,\alpha}(t,s)$ is a decreasing function with respect to the variable s. On the other hand, increasing nature of the function $v(t)$ implies that

$$\int_0^1 G_\alpha(t,s)f^{\beta_1}(s,v(s))ds \leqslant \int_0^1 G_{2,\alpha}(t,s)v^{\beta_1}(s)ds \leqslant \int_0^1 G_{2,\alpha}(t,s)ds\int_0^1 v^{\beta_1}(s)ds$$
$$< v^{\beta_1}(t)\int_0^1 G_{2,\alpha}(t,s)ds.$$

Hence, one can derive that

$$\frac{A_1 v}{v} \leqslant \lambda^{\frac{1}{\beta_1\beta_2}}\Theta_{\beta_1^{-1}}\left(\int_0^1 G_{2,\alpha}(t,s)ds\right),$$
$$\frac{A_2 u}{u} \leqslant \mu^{\frac{1}{\gamma_1\gamma_2}}\Theta_{\gamma_1^{-1}}\left(\int_0^1 G_{2,\beta}(t,s)ds\right).$$

Thus the hypotheses (K_1) and (K_2) are satisfied. On the other hand, in accordance with definitions of the functions f and g taking $\xi_1 = \xi_2 = 1$, we have

$$\int_0^1 G_{2,\alpha}(t,s) = t\int_0^1 (1-s)^{\alpha-1}ds = \frac{t}{\alpha} < 1, \quad 1<\alpha<2,\ t\in(0,1). \tag{53}$$

Thereby

$$\int_0^1 G_{2,\alpha}(t,s)ds < \|\xi_1\|_X^{-\beta_1}, \quad \int_0^1 G_{2,\beta}(t,s)ds < \|\xi_2\|_X^{-\gamma_1}. \tag{54}$$

Since all of the conditions of Theorem 4 hold, so the coupled system of fractional half-linear boundary value problems (50)–(51) has at least one positive solution in C.

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