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AN INVERSE FRACTIONAL SOURCE PROBLEM IN A SPACE OF PERIODIC SPATIAL DISTRIBUTIONS

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Abstract. For a time fractional diffusion equation and diffusion-wave equation with Caputo partial derivatives we prove the correctness of an inverse problem. This problem is to find a solution of direct problem, which is classical in time with values in the space of periodic spatial distributions, and a source term of the equation. A time integral over-determination condition is used.

1. Introduction

Sufficient conditions of classical solvability of fractional Cauchy problems and boundary-value problems to a time fractional diffusion equation were obtained, for example, in [1, 5, 9, 10, 18]. Inverse problems to equations of fractional order with respect to time with different unknown quantities, under different over-determination conditions, are actively studied in connection with their applications (see, for instance, [2, 4, 6, 7, 12, 16, 19]).

In this paper, for a time fractional diffusion (or diffusion-wave) equation we study the inverse problem consisting in the restoration of a solution for the direct problem, which is classical in time with values in the space of periodic spatial distributions and a source term of the equation. We use a time integral over-determination condition.

Note that the solvability of some nonclassical direct problems for partial differential equations with integral initial conditions, in particular, in the space of periodic spatial variable functions have been established, for example, in [15]. The inverse problem for restoration of an initial data of the solution, classical in time with values in a space of periodic spatial distributions was studied in [7]. The inverse fractional source problem with a space integral over-determination condition was studied in [4]. The existence of a solution to the fractional Cauchy problem, which is classical in time with values in Bessel potential spaces, was proved in [8], the existence and uniqueness theorems to the boundary-value problems for partial differential equations in Sobolev spaces were obtained by Yu. Berezansky, Ya. Roitberg, J.-L. Lions, E. Magenes, V. A. Mikhailets, A. A. Murach and others (see [11] and references therein).

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2. Main definitions

Assume that $\mathbb{N}$ is a set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathscr{D}(\mathbb{R})$ is the space of indefinitely differentiable functions with compact supports, $\mathscr{S}(\mathbb{R})$ is the space of rapidly decreasing indefinitely differentiable functions [17, p. 90], while $\mathscr{D}'(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$ are the spaces of linear continuous functionals (distributions) respectively over $\mathscr{D}(\mathbb{R})$ and $\mathscr{S}(\mathbb{R})$, and the symbol (f,φ) stands for the value of the distribution f on the test function φ. Note that $\mathscr{S}'(\mathbb{R})$ is the space of slowly increasing distributions. Recall that the regularized time fractional derivative (the Caputo derivative, or the Caputo-Djrbashian derivative) is defined in [14] by

$$
\begin{aligned}
{}^cD_t^\alpha v(x,t) &= \frac{1}{\Gamma(1-\alpha)}\Big[\frac{\partial}{\partial t}\int_0^t \frac{v(x,\tau)}{(t-\tau)^\alpha}d\tau - \frac{v(x,0)}{t^\alpha}\Big] \quad \text{for} \quad \alpha \in (0,1), \\
{}^cD_t^\alpha v(x,t) &= \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{v_{\tau\tau}(x,\tau)}{(t-\tau)^{\alpha-1}}d\tau \\
&= \frac{1}{\Gamma(2-\alpha)}\Big[\frac{\partial}{\partial t}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^{\alpha-1}}d\tau - \frac{u_t(x,0)}{t^{\alpha-1}}\Big] \quad \text{for} \quad \alpha \in (1,2), \\
{}^cD_t^1 v(x,t) &= \frac{\partial v(x,t)}{\partial t}.
\end{aligned}
$$

We use the Mittag–Leffler function $E_{\alpha,\mu}(z) = \sum_{p=0}^\infty z^p/\Gamma(p\alpha+\mu)$.

The function $E_{\alpha,\mu}(-x)$ $(x>0)$ is indefinitely differentiable for $\alpha \in (0,2)$, $\mu \in \mathbb{R}$. It has the bounds [13]

$$E_{\alpha,\mu}(-x) \leqslant \frac{r_{\alpha,\mu}}{1+z}, \quad x>0,$$

where $r_{\alpha,\mu}$ is a positive constant, and the asymptotic behavior [3]

$$E_{\alpha,\mu}(-x) = O\Big(\frac{1}{x}\Big), \quad x \to +\infty.$$

Let $X_k(x) = \sin kx$, $k \in \mathbb{N}$. Similarly to [17, p. 120], we denote by $\mathscr{D}'_{2\pi}(\mathbb{R})$ the space of periodic distributions, i.e., the space of $v \in \mathscr{D}'(\mathbb{R})$ such that

$$v(x+2\pi) = v(x) = -v(-x) \qquad \forall x \in \mathbb{R}.$$

The formal series

$$\sum_{k=1}^\infty v_k X_k(x), \quad x \in \mathbb{R} \tag{1}$$

is the Fourier series of the distribution $v \in \mathscr{D}'_{2\pi}(\mathbb{R})$, and numbers

$$v_k = \frac{2}{\pi}(v,X_k)_{2\pi} = \frac{2}{\pi}(v,hX_k)$$

are its Fourier coefficients. Here $h(x)$ is an even function from $\mathscr{D}(\mathbb{R})$ possessing the properties:

$$h(x) = \begin{cases} 1, & x \in (-\pi+\varepsilon, \pi-\varepsilon) \\ 0, & x \in \mathbb{R} \setminus (-\pi, \pi) \end{cases}, \quad 0 \leqslant h(x) \leqslant 1.$$

Note that

$$v_k = \frac{2}{\pi} \int_0^\pi v(x) X_k(x) dx \quad \text{for} \quad v \in \mathscr{D}'_{2\pi}(\mathbb{R}) \cap L^1_{loc}(\mathbb{R}),$$

and then the series (1) is the classical Fourier series of v by the system X_k, $k \in \mathbb{N}$.

As is known (see [17, p. 123]) $\mathscr{D}'_{2\pi}(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R})$, the series (1) of $v \in \mathscr{D}'_{2\pi}(\mathbb{R})$ converges in $\mathscr{S}'(\mathbb{R})$ to v, and the Fourier coefficients (clearly defined) have the estimates

$$|v_k| \leqslant C_0(m) C(v,m) (1+k)^m \quad \forall k \in \mathbb{N}$$

with some $m \in \mathbb{Z}_+$ where $C_0(m)$, $C(v,m)$ are positive constants, the same for all $k \in \mathbb{N}$, in particular, $C(v,m) = \left(\int_{\mathbb{R}} (1+x^2)^{-m/2} |v(x)| dx \right)^{1/2}$. The number m is called the order of the distribution v. Note that the order of a regular periodic distribution is a nonpositive number.

We assume next that for $\gamma \in \mathbb{R}$

$$H^\gamma(\mathbb{R}) = \left\{ v \in \mathscr{D}'_{2\pi}(\mathbb{R}) : \|v\|_{H^\gamma(\mathbb{R})} = \sup_{k \in \mathbb{N}} |v_k| (1+k)^\gamma < +\infty \right\}$$

(functions from $H^\gamma(\mathbb{R})$ have the order $-\gamma$ in the sense of the above definition), $C\big([0,T]; H^\gamma(\mathbb{R})\big)$ is the space of continuous in $t \in [0,T]$ functions $v(x,t)$ with values $v(\cdot,t) \in H^\gamma(\mathbb{R})$ endowed with the norm

$$\|v\|_{C([0,T];H^\gamma(\mathbb{R}))} = \max_{t \in [0,T]} \|v(\cdot,t)\|_{H^\gamma(\mathbb{R})},$$

$$C_{2,\alpha}\big([0,T]; H^\gamma(\mathbb{R})\big) = \left\{ v \in C\big([0,T]; H^{2+\gamma}(\mathbb{R})\big) : {}^c D^\alpha v \in C\big([0,T]; H^\gamma(\mathbb{R})\big) \right\}$$

is its subspace endowed with the norm

$$\|v\|_{C_{2,\alpha}([0,T];H^\gamma(\mathbb{R}))} = \max \left\{ \|v\|_{C([0,T];H^{2+\gamma}(\mathbb{R}))}, \|{}^c D^\alpha v\|_{C([0,T];H^\gamma(\mathbb{R}))} \right\}.$$

Note that $H^{\gamma+\varepsilon}(\mathbb{R}) \subset H^\gamma(\mathbb{R})$ for all $\varepsilon > 0$, $\gamma \in \mathbb{R}$.

3. The inverse problem and its correctness

We study the problem

$${}^c D_t^\alpha u - u_{xx} = F_0(x), \quad (x,t) \in Q_T := \mathbb{R} \times (0,T], \tag{2}$$

$$u(x,0) = F_1(x), \quad u_t(x,0) = F_2(x), \quad x \in \mathbb{R}, \tag{3}$$

$$\int_0^{t_0} u(x,t)dt = \Phi(x), \quad x \in \mathbb{R},\ t_0 \in (0,T] \tag{4}$$

where $\alpha \in (0,2)$, F_1, F_2, Φ are the given functions, T is a given positive number, u, F_0 are unknown functions. The second condition in (3) is absent in the case $\alpha \in (0,1]$.

Let the following assumption holds: (A) $\gamma \in \mathbb{R}$, $\theta \in (0,1)$, $F_j \in H^{\gamma+2+2\theta}(\mathbb{R})$, $j = 1,2$ ($F_2 = 0$ if $\alpha \in (0,1]$); $\Phi \in H^{\gamma+4}(\mathbb{R})$ if $\alpha \in (0,1]$, $\Phi \in H^{\gamma+4+2\theta}(\mathbb{R})$ and, in addition, $t_0 \in (0,T]$ is such that $E_{\alpha,2}(-k^2 t_0^{\alpha}) \neq 1$ for all $k \in \mathbb{N}$ if $\alpha \in (1,2)$.

REMARK 1. We have $0 < E_{\alpha,\mu}(-k^2 t^{\alpha}) < 1$ for all $t > 0$, $\mu \geqslant \alpha$ if $\alpha \in (0,1]$ (see [13]). In the case $\alpha \in (1,2)$, the function $1 - E_{\alpha,2}(-z)$ has a finite number of real positive zeroes [13], therefore, there exists a certain $t_0 \in (0,T]$ such that

$$E_{\alpha,2}(-k^2 t_0^{\alpha}) \neq 1 \qquad \forall k \in \mathbb{N}.$$

Decompose the functions $F_j(x)$, $j \in \{0,1,2\}$, $\Phi(x)$ in the formal Fourier series by the system $X_k(x)$, $k \in \mathbb{N}$:

$$F_j(x) = \sum_{k=1}^{\infty} F_{jk} X_k(x), \quad x \in \mathbb{R}, \ \ j = 0,1,2, \tag{5}$$

$$\Phi(x) = \sum_{k=1}^{\infty} \Phi_k X_k(x), \quad x \in \mathbb{R}.$$

DEFINITION 1. A pair of functions

$$(u, F_0) \in \mathscr{M}_{\alpha,\gamma,\theta} := C_{2,\alpha}\big([0,T]; H^{\gamma}(\mathbb{R})\big) \times H^{\gamma+2\theta}(\mathbb{R})$$
$$\big((u,F_0) \in \mathscr{M}_{\alpha,\gamma} = \mathscr{M}_{\alpha,\gamma,0} \ \text{ if } \ \alpha \in (0,1)\big)$$

given by the series

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x), \quad (x,t) \in Q_T \tag{6}$$

and (5) with $j = 0$, satisfying the equation (2) in $\mathscr{S}'(\mathbb{R})$ and the conditions (3), (4), is called a solution of the problem (2)–(4) under the assumption (A).

Substituting the function (6) in the equation (2) and the conditions (3), (4), we obtain the problems

$$\begin{gathered} {}^{c}D^{\alpha} u_k + k^2 u_k = F_{0k}, \ \ t \in (0,T], \\ u_k(0) = F_{1k}, \ \ u_k'(0) = F_{2k}, \end{gathered} \tag{7}$$

$$\int_0^{t_0} u_k(t)dt = \Phi_k, \quad k \in \mathbb{N} \tag{8}$$

for the unknown $u_k(t)$, $t \in [0,T]$ and F_{0k}, $k \in \mathbb{N}$.

So, the pairs $(u_k(t), F_{0k})$ ($k \in \mathbb{N}$) of the Fourier coefficients of the problem's solution satisfy (7), (8).

THEOREM 1. *Assume that* $\gamma \in \mathbb{R}$, $\theta \in (0,1)$, $F_0 \in H^{\gamma+2\theta}(\mathbb{R})$, $F_j \in H^{\gamma+2}(\mathbb{R})$, $j = 1,2$, *if* $\alpha \in (1,2)$, $F_0 \in H^{\gamma}(\mathbb{R})$, $F_1 \in H^{\gamma+2}(\mathbb{R})$, $F_2 = 0$, *if* $\alpha \in (0,1]$.

Then there exists a unique solution $u \in C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)$ *to the direct problem (2), (3). It is given by (6) where*

$$u_k(t) = F_{0k}k^{-2}\big[1 - E_{\alpha,1}(-k^2t^{\alpha})\big] + F_{1k}E_{\alpha,1}(-k^2t^{\alpha}) + F_{2k}tE_{\alpha,2}(-k^2t^{\alpha}), \quad t \in [0,T], \quad k \in \mathbb{N}. \tag{9}$$

The solution continuously depends on the data (F_0, F_1, F_2 *), and the following inequality of coercivity holds:*

$$||u||_{C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)} \leqslant a_0||F_0||_{H^{\gamma+2\theta}(\mathbb{R})} + \sum_{j=1}^{2} a_j||F_j||_{H^{\gamma+2}(\mathbb{R})}, \tag{10}$$

where a_j, $j \in \{0,1,2\}$ *are positive constants independent of data,* $F_2 = 0$ *and* $\theta = 0$ *in (10) if* $\alpha \in (0,1]$.

Proof. It follows from the theorem 1 in [7] that there exists a unique solution $u \in C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)$ to the problem (2), (3) under the theorem's conditions, that it is given by (6) where

$$u_k(t) = F_{0k}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-k^2\tau^{\alpha})d\tau + F_{1k}E_{\alpha,1}(-k^2t^{\alpha}) + F_{2k}tE_{\alpha,2}(-k^2t^{\alpha}), \quad t \in [0,T], \quad k \in \mathbb{N}.$$

By the link

$$\lambda \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^{\alpha})d\tau = 1 - E_{\alpha,1}(-\lambda t^{\alpha}), \tag{11}$$

we obtain the formulas (9) and, using [7, th.1], we obtain the bounds (10). These bounds imply that a solution of the problem is unique and continuously depends on the data. □

THEOREM 2. *Assume that (A) holds. Then there exists a unique solution* $(u,F_0) \in \mathscr{M}_{\alpha,\gamma,\theta}$ *of the inverse problem (2)–(4). It is given by the Fourier series (6) and (5) with* $j = 0$ *where* $u_k(t)$ *are defined by (9),*

$$F_{0k} = \Big[\Phi_k - F_{1k}t_0E_{\alpha,2}(-k^2t_0^{\alpha}) - F_{2k}t_0^2E_{\alpha,3}(-k^2t_0^{\alpha})\Big]k^2\, G_k^{-1}, \; k \in \mathbb{N} \tag{12}$$

with $G_k = t_0\big[1 - E_{\alpha,2}(-k^2t_0^{\alpha})\big]$. *The solution continuously depends on the data* F_0, F_2, Φ *and the following inequality of coercivity holds:*

$$\begin{gathered} ||u||_{C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)} + ||F_0||_{H^{\gamma+2\theta}(\mathbb{R})} \leqslant b_0||\Phi||_{H^{\gamma+2\theta+4}(\mathbb{R})} + \sum_{j=1}^{2} b_j||F_j||_{H^{\gamma+2+2\theta}(\mathbb{R})}, \; \alpha \in (1,2), \\ ||u||_{C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)} + ||F_0||_{H^{\gamma}(\mathbb{R})} \leqslant b_0||\Phi||_{H^{\gamma+4}(\mathbb{R})} + b_1||F_1||_{H^{\gamma+2}(\mathbb{R})}, \; \alpha \in (0,1] \end{gathered} \tag{13}$$

where $b_j,\ j\in\{0,1,2\}$ *are positive constants independent of data.*

Proof. Using (9), we write the conditions (8) as follows

$$F_{0k}k^{-2}\int_0^{t_0}\Big[1-E_{\alpha,1}(-k^2t^\alpha)\Big]dt+\int_0^{t_0}\Big[F_{1k}E_{\alpha,1}(-k^2t^\alpha)+F_{2k}tE_{\alpha,2}(-k^2t^\alpha)\Big]dt=\Phi_k,\ k\in\mathbb{N}.$$

Note that [7]

$$\int_0^{t_0}E_{\alpha,1}(-k^2t^\alpha)dt=t_0E_{\alpha,2}(-k^2{t_0}^\alpha),\quad k\in\mathbb{N},$$

and similarly

$$\int_0^{t_0}tE_{\alpha,2}(-k^2t^\alpha)dt=\frac{1}{\alpha k^{4/\alpha}}\int_0^{k^2{t_0}^\alpha}E_{\alpha,2}(-z)z^{\frac{2}{\alpha}-1}dz=\frac{1}{k^{4/\alpha}}\sum_{p=0}^{\infty}\frac{(-1)^p(k^2{t_0}^\alpha)^{p+\frac{2}{\alpha}}}{\Gamma(p\alpha+3)}$$
$$=t_0^2E_{\alpha,3}(-k^2{t_0}^\alpha),\quad k\in\mathbb{N}.$$

From here, according to the assumption (A), we find the expressions (12) for the unknown Fourier coefficients $F_{0k},\ k\in\mathbb{N}$.

Let us show that the founded solution belongs to $\mathscr{M}_{\alpha,\gamma,\theta}$. Given that the functions $E_{\alpha,\mu}(-k^2t^\alpha)$ ($\mu\in\{\alpha,1,2,3\}$) have the same behavior for large k and given the formulas (12) into account, one obtains

$$(1+k)^{\gamma+2\theta}|F_{0k}|$$
$$\leqslant c_0\Big[|\Phi_k|(1+k)^{\gamma+2\theta}+|F_{1k}|(1+k)^{\gamma+2\theta-2}+|F_{2k}|(1+k)^{\gamma+2\theta-2}\Big](1+k)^4$$
$$=c_0\Big[|\Phi_k|(1+k)^{\gamma+4+2\theta}+|F_{1k}|(1+k)^{\gamma+2\theta+2}+|F_{2k}|(1+k)^{\gamma+2\theta+2}\Big],\quad \alpha\in(1,2),$$

$$(1+k)^\gamma|F_{0k}|\leqslant c_0\Big[|\Phi_k|(1+k)^{\gamma+4}+\sup_{t\in(0,T]}|F_{1k}|(1+k)^{\gamma+2}\Big],\quad \alpha\in(0,1),\quad k\in\mathbb{N}$$

where c_0 is a positive constant, and therefore,

$$||F_0||_{H^{\gamma+2\theta}(\mathbb{R})}\leqslant c_0\Big[||\Phi||_{H^{\gamma+4+2\theta}(\mathbb{R})}+\sum_{j=1}^{2}||F_j||_{H^{\gamma+2+2\theta}(\mathbb{R})}\Big],\ \alpha\in(1,2),$$
$$||F_0||_{H^\gamma(\mathbb{R})}\leqslant c_0\Big[||\Phi||_{H^{\gamma+4}(\mathbb{R})}+||F_1||_{H^{\gamma+2}(\mathbb{R})}\Big],\quad \alpha\in(0,1].$$

So, under the theorem's assumptions, $F_0\in H^{\gamma+2\theta}(\mathbb{R})$ ($F_0\in H^\gamma(\mathbb{R})$ if $\alpha\in(0,1]$). Then using (10) we obtain the inequality (13). This inequality implies that a solution of the problem is unique and continuously depends on the problem's data. □

REMARK 2. Uniqueness of a solution of the inverse problem (2)–(4) is obtained for all $t_0 \in (0,T]$ in the case $\alpha \in (0,1]$ and only under an assumption on t_0 in the case $\alpha \in (1,2)$.

The obtained result can be transferred to the case of the boundary value problem for a time fractional diffusion or diffusion-wave equation when the corresponding Sturm–Liouville problem has positive eigenvalues.

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