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NEW BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS AND THEIR SOLVABILITY

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Abstract. Firstly, the surveys for studies on boundary value problems for higher order ordinary differential equations and for higher order fractional differential equations are given. Secondly a simple review for studies on solvability of boundary value problems for impulsive fractional differential equations is presented. Thirdly we propose four classes of higher order linear fractional differential equations and give their exact piecewise continuous solutions. Fourthly we propose some new classes of boundary value problems for higher order fractional differential equations with impulse effects. Fifthly we establish new general methods for converting boundary value problems of impulsive fractional differential equations with the Riemann-Liouville fractional derivatives or Caputo fractional derivatives to equivalent integral equations. Sixthly by employing fixed point theorems in Banach space, we establish new existence results of solutions for these boundary value problems. Seventh, some remarks are given to show that many methods in known papers are un-suitable. Eighth, some examples are presented to illustrate the efficiency of the results obtained. Finally, possible trends of researches are given at the end of the paper.

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1. Introduction

Fractional differential equation is a generalization of ordinary differential equation to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [62]. Fractional differential equations therefore find numerous applications in different branches of physics, chemistry and biological sciences such as visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles and neuron modelling [64]. Readers may refer to the books and monographs [34, 63] for fractional calculus and developments on fractional differential and fractional integro-differential equations with applications.

On the other hand, theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such characteristics arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the readers to [49].

Benchohra and Slimani [19] initiated the study of fractional differential equations with impulses. Agarwal, Benchohra and Slimani [4] considered a class of initial value problems for differential equations involving the Caputo fractional derivative of order $\alpha \in (0,1]$ and the impulsive effects. Since 2009, there have been many papers concerned with the solvability of initial or boundary value problems for impulsive fractional differential equations. Studies on impulsive fractional differential equations have been of great interest because it is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc.

Now, we firstly present surveys for studies on BVPs for higher order ordinary differential equations, BVPs for fractional differential equations and BVPs for impulsive fractional differential equations. Then we illustrate our purposes in this paper. Finally, we list the organization of this paper.

1.1. BVPs for higher order ordinary differential equations

Solvability of boundary value problems for higher order ordinary differential equations were investigated by many authors. These boundary value problems mainly contain $2m$-th order Lidstone BVPs, $(n,n-p)$ type BVPs, anti-periodic BVPs, periodic BVPs and Neumann BVPs.

For examples, in [20, 22, 23, 66, 82], solvability of the following problems were investigated:

$$\begin{cases} y^{(2m)}(t) = f(t,y(t),\cdots,y^{(2j)}(t),\cdots,y^{(2(m-1))}(t)), & t\in(0,1),\\ y^{(2i)}(0)=0=y^{(2i)}(1), & i\in \mathbb{N}_0^{m-1} \end{cases} \tag{1.1.1}$$

$$\begin{cases} y^{(2m)}(t) = f(t,y(t),\cdots,y^{(2j)}(t),\cdots,y^{(2(m-1))}(t)), & t\in(0,1),\\ y^{(2i)}(0)=0=y^{(2i+1)}(1), & i\in \mathbb{N}_0^{m-1}, \end{cases} \tag{1.1.2}$$

and

$$\begin{cases} y^{(2m)}(t) = f(t,y(t),\cdots,y^{(2j)}(t),\cdots,y^{(2(m-1))}(t)), & t\in(0,1),\\ y^{(2i+1)}(0)=y^{(2i+1)}(1)=0, & i\in \mathbb{N}_0^{m-1} \end{cases} \tag{1.1.3}$$

In [5, 6, 24, 26, 38, 39, 40, 53, 61, 69, 71], the following $(n,n-k)$ type problems were studied:

$$\begin{cases} (-1)^{n-k}y^{(n)} = f(t,y), & t\in(0,1),\\ y^{(i)}(0)=0, \quad i\in \mathbb{N}_0^{k-1}, & y^{(j)}(1)=0, \quad j\in \mathbb{N}_0^{n-k-1}. \end{cases} \tag{1.1.4}$$

In [37, 41], the following more general boundary value problems were studied:

$$\begin{cases} (-1)^{n-k}y^{(n)} = f(t,y), & t\in(0,1),\\ y^{(i)}(0)=0, \quad i\in \mathbb{N}_0^{k-1}, & y^{(j)}(1)=0, \quad j\in \mathbb{N}_q^{n+q-k-1}, \end{cases} \tag{1.1.5}$$

where $k\in \mathbb{N}_1^{n-1}$, $q\in \mathbb{N}_0^{k}$.

In [6, 25], authors studied existence of solutions of the following problems:

$$\begin{cases} (-1)^{n-p}y^{(n)} = f(t,y,y',\cdots,y^{(p-1)}), & t\in(0,1),\\ y^{(i)}(0)=0, \quad i\in \mathbb{N}_0^{p-1}, & y^{(j)}(1)=0, \quad j\in \mathbb{N}_p^{n-1}. \end{cases} \tag{1.1.6}$$

Solvability of $(n,1)$ type boundary value problems for higher order ordinary differential equations were investigated by many authors. For examples, the following boundary value problems were studied in [27, 28, 74]:

$$\begin{cases} y^{(n)}+f(t,y,y',\cdots,y^{(p-1)})=0, \quad t\in(0,1),\\ y^{(i)}(0)=0, \quad i\in\mathbb{N}_0^{n-2}, \quad y^{(p)}(1)=0 \end{cases} \tag{1.1.7}$$

or it special cases, where $p\in\mathbb{N}_0^{n-1}$.

As we know that the general anti-periodic, periodic and Neumann boundary value problems are as follows:

$$y^{(n)}=f(t,y), \quad t\in(0,1), \quad y^{(i)}(0)=-y^{(i)}(1), \quad i\in\mathbb{N}_0^{n-1}, \tag{1.1.8}$$

$$y^{(n)}=f(t,y), \quad t\in(0,1), \quad y^{(i)}(0)=y^{(i)}(1), \quad i\in\mathbb{N}_0^{n-1}, \tag{1.1.9}$$

and

$$y^{(n)}=f(t,y), \quad t\in(0,1), \quad y^{(i)}(0)=y^{(j)}(1)=0, \quad i\in\mathbb{N}_1^{k}, \quad j\in\mathbb{N}_1^{n-k}. \tag{1.1.10}$$

Readers may see [2] in which BVP(1.1.8) was studied with $n=5$. Some special cases of BVP(1.1.9) and BVP(1.1.10) were studied in [85] and [57].

We note that BVP(1.1.1)–BVP(1.1.10) have many applications and been studied by many authors see the monograph [1]. We also know that higher order impulsive differential equations can be changed to lower order impulsive differential systems. So existence results for solutions of impulsive differential systems can be applied to solve higher order differential equations see the book [18].

Since $D_{0+}^{\alpha}D_{0+}^{\beta}x\neq D_{0+}^{\alpha+\beta}x\neq D_{0+}^{\beta}D_{0+}^{\alpha}x$ see [63, 65] and ${}^cD_{0+}^{\alpha}\,{}^cD_{0+}^{\beta}x\neq {}^cD_{0+}^{\alpha+\beta}x\neq {}^cD_{0+}^{\beta}\,{}^cD_{0+}^{\alpha}x$ see (6.4) on page 105 in [65], we know that higher order fractional differential equations can not be converted to lower fractional differential systems. Hence it is interesting to generalize BVP(1.1.1)–BVP(1.1.10) to higher order fractional differential equations.

1.2. BVPs for higher order fractional differential equations

There has been not many papers discussed the solvability of BVPs for higher order fractional differential equations. In [81], authors studied existence of solutions of the following boundary value problem for higher order fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, \quad 0<t<b, \quad \lambda>0, \quad \alpha\in[n,n+1),\\ u^{(j)}(0)=0, \quad j\in\mathbb{N}_0^{n-1}, \quad u^{(n-1)}(b)=0. \end{cases} \tag{1.2.1}$$

In [86], Zhang used a fixed-point theorem for the mixed monotone operator to show the existence of positive solutions to the following singular fractional differential equation

$$\begin{cases} -D_{0+}^{\alpha}u(t)=f(t,u(t),u'(t),\cdots,u^{(n-2)}(t)), \quad 0<t<1, \quad \alpha\in[n-1,n),\\ u^{(i)}(0)=0, \quad i=1,2,\cdots,n-2, \quad u^{(n-1)}(1)=0. \end{cases} \tag{1.2.2}$$

methods used in these papers are based upon transforming boundary value problems into integral equations and using fixed point theorems to establish existence and uniqueness results on solutions. Obviously, BVP(1.2.1) and BVP(1.2.2) are generalized forms of BVP(1.1.7) with $p=n-1$.

In [83], investigated the existence of solutions of the following problem BVP for higher order fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)+p(t)u(t)=0, \quad 0<t<1, \quad \lambda>0, \quad \alpha\in[n-1,n), \\ u^{(j)}(0)=0, \quad j\in\mathbb{N}_0^{n-2}, \quad u(1)=0. \end{cases} \tag{1.2.3}$$

In [7], authors established some existence results in a Banach space for a nonlocal boundary value problem involving a nonlinear differential equation of fractional order

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t)=f(t,u(t)), \quad 0<t<1, \quad \alpha\in[n-1,n), \\ u^{(j)}(0)=0, \quad j\in\mathbb{N}_0^{n-2}, \quad u(1)=\alpha u(\eta). \end{cases} \tag{1.2.4}$$

Feng [31] generalized above mentioned BVP and studied the following higher-order singular boundary value problem of fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,u(t)), \quad 0<t<1, \quad \alpha\in[n-1,n), \\ u^{(j)}(0)=0, \quad j\in\mathbb{N}_0^{n-2}, \quad u(1)=\int_0^1 h(s)u(s)ds. \end{cases} \tag{1.2.5}$$

BVP(1.2.3), BVP(1.2.4) and BVP(1.2.5) are generalized forms of BVP(1.1.7) with $p=0$.

In [92], authors studied the following singular eigenvalue problem for a higher order fractional differential equation

$$\begin{cases} -D_{0+}^{\alpha}u(t)=\lambda f(t,u(t),D_{0+}^{\mu_1}u(t),\cdots,D_{0+}^{\mu_{n-1}}u(t)), \quad 0<t<1, \quad \alpha\in[n-1,n), \\ u(0)=0, \quad D_{0+}^{\mu_i}u(0)=0, \quad i=1,2,\cdots,n-1, \quad D_{0+}^{\mu}u(1)=\sum_{i=1}^{m}a_iD_{0+}^{\mu}u(\xi_i), \end{cases} \tag{1.2.6}$$

where $n-i-1<\mu_i<\alpha-i$ $(i=1,2,\cdots,n-2)$, $\mu-\mu_{n-1}>0$, $\alpha-\mu_{n-1}\leqslant 2$, $\alpha-\mu>1$, $a_i\geqslant 0$ $(i=1,2,\cdots,m)$, $0<\xi_1<\cdots<\xi_m<1$, D_{0+}^{*} is the standard Riemann-Liouville derivative of order $*$.

BVP(1.2.6) generalizes BVP(1.1.7). We do not find published papers concerned with BVPs which generalize BVP(1.1.1)–BVP(1.1.6) and BVP(1.1.8)–BVP(1.1.10) though there have been some papers concerned with the solvability of boundary value problems for lower order impulsive fractional differential equations.

1.3. BVPs for impulsive fractional differential equations

Impulsive differential equations (IFDEs for short), which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. Hence impulsive boundary value problems for nonlinear fractional differential equations have been addressed by several authors see [3, 4, 8, 9, 29, 35, 44, 59, 68, 84]. The first kind of such problems is concerned with impulsive fractional differential equations with multiple starting points $t = t_i \ (i \in \mathbb{N}_0^m)$. The second kind of such problems is concerned with impulsive fractional differential equations involved a single starting point $t = 0$.

(A) Studies on boundary value problems of impulsive fractional differential equations with multiple starting points $t = t_i \ (i \in \mathbb{N}_0^m)$.

Recently, Wang [73] studied the existence and uniqueness of solutions of the following initial value problem of the impulsive fractional differential equation

$$\begin{cases} {}^cD_{t_i^+}^{\alpha} u(t) = f(t,u(t)), \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^p, \\ u^{(j)}(0) = u_j, \quad j \in \mathbb{N}_0^{n-1}, \\ \Delta u^{(j)}(t_i) = I_{ji}(u(t_i)), \quad i \in \mathbb{N}_1^p, \quad j \in \mathbb{N}_0^{n-1}, \end{cases} \tag{1.3.1}$$

where $\alpha \in (n-1,n)$ with n being a positive integer, ${}^cD_{t_i^+}^{\alpha}$ represents the standard Caputo fractional derivatives of order α, $\mathbb{N}_a^b = \{a, a+1, \cdots, b\}$ with a,b being integers, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_{ji} \in C(\mathbb{R},\mathbb{R}) \ (i \in \mathbb{N}_1^p, \ j \in \mathbb{N}_0^{n-1})$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Henderson and Ouahab [36] studied the existence of solutions of the following initial value problem and periodic boundary value problem of impulsive fractional differential equations:

$$\begin{cases} {}^cD_{t_i^+}^{\alpha} u(t) = f(t,u(t)), \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^p, \\ u^{(j)}(0) = u_j, \quad j \in \mathbb{N}_0^{1}, \\ u^{(j)}(t_i) = I_{ji}(u(t_i)), \quad i \in \mathbb{N}_1^p, \quad j \in \mathbb{N}_0^{1}, \end{cases} \tag{1.3.2}$$

and

$$\begin{cases} {}^cD_{t_i^+}^{\alpha} u(t) = f(t,u(t)), \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^p, \\ u^{(j)}(0) = u^{(j)}(b), \quad j \in \mathbb{N}_0^{1}, \\ u^{(j)}(t_i) = I_{ji}(u(t_i)), \quad i \in \mathbb{N}_1^p, \quad j \in \mathbb{N}_0^{1}, \end{cases} \tag{1.3.3}$$

where $\alpha \in (1,2]$, $b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = b$, $f : [0,b] \times \mathbb{R} \to \mathbb{R}$, $I_{ji} : \mathbb{R} \to \mathbb{R}$ are continuous functions. Readers should also refer [75]. Obviously, the boundary conditions in IVP(1.3.1) and IVP(1.3.2) are different and BVP(1.3.3) is a generalized form of BVP(1.1.9) $(n = 2)$.

Wang, Ahmad and Zhang [77] studied the existence and uniqueness of solutions of the following periodic boundary value problems for nonlinear impulsive fractional differential equation

$$\begin{cases} {}^cD_{t_i^+}^{\alpha} u(t) = f(t,u(t)), \quad t \in (0,T] \setminus \{t_1, \cdots, t_p\}, \\ \Delta u(t_i) = I_i(u(t_i)), \quad i \in \mathbb{N}_1^p, \\ \Delta u'(t_i) = I_i^*(u(t_i)), \quad i \in \mathbb{N}_1^p, \\ u'(0) + (-1)^{\theta} u(T) = bu(T), \quad u(0) + (-1)^{\theta} u(T) = 0, \end{cases} \tag{1.3.4}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $\theta = 1,2$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, I_i, $I_i^* \in C(\mathbb{R},\mathbb{R})$ $(i \in \mathbb{N}_1^p)$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [10, 11, 91, 67], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}x(t) = w(t)f(t,x(t),x'(t)), & t \in (0,1] \setminus \{t_1,\cdots,t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), & i \in \mathbb{N}_1^p, \\ \Delta x'(t_i) = J_i(x(t_i)), & i \in \mathbb{N}_1^p, \\ ax(0) \pm bx'(0) = g_1(x), & cx(1) + dx'(1) = g_2(x), \end{cases} \tag{1.3.5}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $a,b,c,d \geqslant 0$ with $ac + ad + bc \neq 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})$ $(i \in \mathbb{N}_1^p)$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $w : [0,1] \to [0,+\infty)$ is a continuous function, $g_1, g_2 : PC(0,1] \to \mathbb{R}$ are two continuous functions.

In 2015, Zhou, Liu and Zhang [93] studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}x(t) = \lambda x(t) + f(t,x(t),(Kx)(t),(Hx)(t)), & t \in (0,1] \setminus \{t_1,\cdots,t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), & i \in \mathbb{N}_1^p, \\ \Delta x'(t_i) = J_i(x(t_i)), & i \in \mathbb{N}_1^p, \\ ax(0) - bx'(0) = x_0, & cx(1) + dx'(1) = x_1, \end{cases} \tag{1.3.6}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $a \geqslant 0$, $b > 0$, $c \geqslant 0$, $d > 0$ with $\delta = ac + ad + bc \neq 0$, $\lambda > 0$, x_0, $x_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, I_i, $J_i \in C(\mathbb{R},\mathbb{R})$ $(i \in \mathbb{N}_1^p)$, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $(Hx)(t) = \int_0^1 h(t,s)x(s)ds$ and $(Kx)(t) = \int_0^t k(t,s)x(s)ds$. BVP(1.3.4)–BVP(1.3.6) are generalized forms of the Sturm-LIouville boundary value problems for second order differential equations see [47].

In [90], Zhao and Gong studied existence of positive solutions of the following nonlinear impulsive fractional differential equation with generalized periodic boundary value conditions

$$\begin{cases} {}^cD^{q}_{t_i^+}u(t) = f(t,u(t)), & t \in (0,T] \setminus \{t_1,\cdots,t_p\}, \\ \Delta u(t_i) = I_i(u(t_i)), & i \in \mathbb{N}_1^p, \\ \Delta u'(t_i) = J_iu(t_i)), & i \in \mathbb{N}_1^p, \\ \alpha u(0) - \beta u(1) = 0, & \alpha u'(0) - \beta u'(1) = 0, \end{cases} \tag{1.3.7}$$

where $q \in (1,2)$, ${}^CD^{q}_{t_i^+}$ represents the standard Caputo fractional derivatives of order q, $\alpha > \beta > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, I_i, $J_i \in C([0,+\infty),[0,+\infty))$ $(i \in \mathbb{N}_1^p)$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

In [50, 55], authors studied the existence of solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1,\cdots,t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ ax(0)+bx(1) = g_1(x), \quad ax'(0)+bx'(1) = g_2(x), \end{cases} \tag{1.3.8}$$

where $\alpha \in (1,2)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $a,b \in \mathbb{R}$ with $a \geqslant b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, I_i, $J_i \in C(\mathbb{R},\mathbb{R})$ $(i \in \mathbb{N}_1^p)$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $g_1, g_2 : PC(0,1] \to R$ are two continuous functions. BVP(1.3.7) and BVP(1.3.8) are generalized forms of BVP(1.1.8) and BVP(1.1.9) $(n = 2)$.

In [54], Liu and Li investigated the existence and uniqueness of solutions for the following nonlinear impulsive fractional differential equations

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}u(t) = f(t,u(t),u'(t),u''(t)), \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^p, \\ u(0) = \lambda_1 u(T) + \xi_1 \int_0^T q_1(s,u(s),u'(s),u''(s))ds, \\ u'(0) = \lambda_2 u'(T) + \xi_2 \int_0^T q_2(s,u(s),u'(s),u''(s))ds, \\ u''(0) = \lambda_3 u''(T) + \xi_3 \int_0^T q_3(s,u(s),u'(s),u''(s))ds, \\ \Delta u(t_i) = A_i(u(t_i)), \quad i \in \mathbb{N}_1^p, \\ \Delta u'(t_i) = B_i(u(t_i)), \quad i \in \mathbb{N}_1^p, \\ \Delta u''(t_i) = C_i(u(t_i)), \quad i \in \mathbb{N}_1^p, \end{cases} \tag{1.3.9}$$

where $\alpha \in (2,3)$, ${}^cD^{\alpha}_{t_i^+}$ represents the standard Caputo fractional derivatives of order α, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $\lambda_i, \xi_i \in \mathbb{R}$ $(i = 1,2,3)$ are constants, $A_i, B_i, C_i \in C(\mathbb{R},\mathbb{R})$ $(i \in \mathbb{N}_1^p)$, $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. BVP(1.3.9) generalizes BVP(1.1.8) $(n = 3)$.

Recently, in [17], to extend the problem for impulsive differential equation $u''(t) - \lambda u(t) = f(t,u(t))$, $u(0) = u(T) = 0$, $\Delta u'(t_i) = I_i(u(t)i))$, $i \in \mathbb{N}_1^p$ to impulsive fractional differential equation, the authors studied the existence and the multiplicity of solutions for the Dirichlet's boundary value problem for impulsive fractional order differential equation

$$\begin{cases} {}^cD^{\alpha}_{T^-}({}^CD^{\alpha}_{0^+}x(t) + a(t)x(t) = \lambda f(t,x(t)), \quad t \in [0,T], \quad t \neq t_i, \quad i \in \mathbb{N}_1^m, \\ \Delta {}^CD^{\alpha-1}_{T^-}({}^CD^{\alpha}_{0^+}x(t_i) = \mu I_i(x(t_i^-)), \quad i \in \mathbb{N}_1^m, \quad x(0) = x(T) = 0, \end{cases} \tag{1.3.10}$$

where $\alpha \in (1/2,1]$, $\lambda, \mu > 0$ are constants, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_i : \mathbb{R} \to \mathbb{R}$ $(i \in \mathbb{N}_1^m)$ are continuous functions, ${}^cD^{\alpha}_{0^+}$ (or ${}^cD^{\alpha}_{T^-}$) is the standard left (or right) Caputo fractional derivative of order α, $a \in C[0,T]$ and there exist constants $a_1, a_2 > 0$ such that $a_1 \leqslant a(t) \leqslant a_2$ for all $t \in [0,T]$, $\Delta x|_{t=t_i} = \lim_{t\to t_i^+} x(t) - \lim_{t\to t_i^-} x(t) = x(t_i^+) - x(t_i^-)$ and $x(t_i^+)$, $x(t_i^-)$ represent the right and left limits of $x(t)$ at $t = t_i$ respectively.

In [16], authors the existence of solutions of the following periodic boundary value problem for impulsive fractional differential equation

$$\begin{cases} D_{t_k^+}^{\alpha} u(t) - \lambda u(t) = f(t,u(t)), \quad t \in (0,1], \quad t \neq t_k, \quad k \in \mathbb{N}_0^p, \\ \lim\limits_{t\to t_k^+} (t-t_k)^{1-\alpha}[u(t) - u(t_k)] = I_k(u(t_k)), \quad k \in \mathbb{N}_1^p, \\ \lim\limits_{t\to 0} t^{1-\alpha} u(t) = u(1), \end{cases} \tag{1.3.11}$$

where $\alpha \in (0,1)$, $D_{t_k^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α with the base point t_k, $\lambda \in \mathbb{R}$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, p is a positive integer, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$. BVP(1.3.10) and BVP(1.3.11) have applications in applied sciences see [16, 17].

(B) Studies on boundary value problems of impulsive fractional differential equations with a single starting point $t = 0$.

For impulsive fractional differential equations whose derivatives have a single start point $t = 0$, there have been not many papers published. In [70], authors presented a new method to converting the impulsive fractional differential equation (with the Caputo fractional derivative) to an equivalent integral equation and established existence and uniqueness results for some boundary value problems of impulsive fractional differential equations involving the Caputo fractional derivatives with single start point. The existence and uniqueness of solutions of the following initial or boundary value problems were discussed in [70]:

$$\begin{cases} {}^cD_{0^+}^{\alpha} x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ x(0) = x_0, \quad x'(0) = x_1, \end{cases} \tag{1.3.12}$$

$$\begin{cases} {}^cD_{0^+}^{\alpha} x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ x(0) + \phi(x) = x_0, \quad x'(0) = x_1, \end{cases} \tag{1.3.13}$$

$$\begin{cases} {}^cD_{0^+}^{\beta} x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ ax(0) + bx(1) = 0, \end{cases} \tag{1.3.14}$$

$$\begin{cases} {}^cD_{0^+}^{\alpha} x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1, \cdots, t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ ax(0) - bx'(0) = x_0, \quad cx(1) + dx'(1) = x_1, \end{cases} \tag{1.3.15}$$

and

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1,\cdots,t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ x(0) - ax(\xi) = x(1) - bx(\eta) = 0, \end{cases} \tag{1.3.16}$$

where $\alpha \in (1,2]$, $\beta \in (0,1]$, $D_{0^+}^{*}$ is the Caputo fractional derivative with order $*$ and single starting point $t = 0$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $I_i, J_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $a,b,c,d,x_0,x_1 \in \mathbb{R}$ are constants, $\phi : PC(0,1] \to \mathbb{R}$ is a functional. Our Theorem 3.2.1 generalizes the results (Lemma 2.3) obtained in [70] see Corollary 3.2.1 and Remark 3.2.1 in Section 3.2. We find that some lemmas in [72, 89] concerned with the solvability of BVP(1.3.15) are unsuitable see Remark 6.7.

In [90], Zhao and Gong studied the existence of positive solutions of the following nonlinear boundary value problem of fractional impulsive differential equations

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) = f(t,x(t)), \quad t \in (0,1] \setminus \{t_1,\cdots,t_p\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ \Delta x'(t_i) = J_i(x(t_i)), \quad i \in \mathbb{N}_1^p, \\ ax(0) - bx(1) = 0, \quad ax'(0) - bx'(1) = 0, \end{cases} \tag{1.3.17}$$

where $\alpha \in (1,2)$, ${}^CD_{0^+}^{\alpha}$ represents the standard Caputo fractional derivatives of order α, $a > b > 0$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $I_i, J_i \in C(\mathbb{R},\mathbb{R})$ $(i \in \mathbb{N}_1^p)$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. BVP(1.3.17) generalizes BVP(1.1.8) $(n = 2)$. So one can apply our results (Theorem 5.3.1) to solve BVPs more general than BVP(1.3.12)–BVP(1.3.17) see Section 5.3.

In recent paper [45], Liu studied existence of positive solutions for the following boundary value problems (BVP) of fractional impulsive differential equations

$$\begin{cases} D_{0^+}^{\alpha}u(t) = -f(t,u(t)), \quad t \in (0,1), \quad t \neq t_k, \quad k \in \mathbb{N}_1^m, \\ u(t_k^+) = (1-c_k)u(t_k^-), \quad k \in \mathbb{N}_1^m, \quad u(0) = u(1) = 0, \end{cases} \tag{1.3.18}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (1,2)$ with the base point 0, m is a positive integer, $c_k \in \left(0,\frac{1}{2}\right)$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a given continuous function, $u(t_k^+)$ and $u(t_k^-)$ denote the right limit and left limit of u at t_k and $u(t_k^+) = u(t_k)$, i.e., u is right continuous at t_k. By constructing a novel transformation, BVP(1.3.18) is convert into a continuous system. using a specially constructed cone, the Krein-Rutman theorem, topological degree theory, and bifurcation techniques, some sufficient conditions are obtained for the existence of positive solutions of BVP(1.3.18). However, we find that Lemma 3.1 [45] is un-correct, see Remark 6.1 and Remark 6.2 in Section 6.

In [12, 78], Bai, Wang and Bai studied the existence and uniqueness of solutions of the following periodic boundary value problem for impulsive fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}u(t)-\lambda u(t)=f(t,u(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m, \\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}[u(t)-u(t_i)]=I(u(t_i)), \\ \lim\limits_{t\to 0}t^{1-\alpha}u(t)-u(1)=k, \end{cases} \tag{1.3.19}$$

where $\alpha\in(0,1)$, $k\in\mathbb{R}$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α with the starting point 0, $\lambda\in\mathbb{R}$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $I:\mathbb{R}\to\mathbb{R}$ are given continuous functions, $0=t_0<t_1<\cdots<t_{m+1}=1$. By Lemma 2.1 in [12], BVP(1.3.19) is converted to an integral equation. We given a new method for converting BVP(1.3.19) to a integral equation see Remark 6.5 in Section 6.

In [13], Bai studied the existence of solutions of the following boundary value problems of nonlinear impulsive fractional differential equation at resonance

$$\begin{cases} D_{0^+}^{\alpha}u(t)=f(t,u(t),D_{0^+}^{\alpha-1}u(t)), \quad t\in(0,1), \quad t\neq t_i, \quad i\in\mathbb{N}_1^k, \\ \lim\limits_{t\to 0^+}t^{2-\alpha}u(t)=\sum\limits_{i=1}^{n}a_iu(\xi_i), \quad u(1)=\sum\limits_{i=1}^{n}b_iu(\eta_i), \\ \Delta u(t_i)=I_i(u(t_i),D_{0^+}^{\alpha-1}u(t_i)), \quad \Delta D_{0^+}^{\alpha-1}u(t_i)=J_i(u(t_i),D_{0^+}^{\alpha-1}u(t_i)), \quad i\in\mathbb{N}_1^k, \end{cases} \tag{1.3.20}$$

where $\alpha\in(1,2)$, $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $*$, $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, and $I_i,J_i:\mathbb{R}^2\to\mathbb{R}$ are continuous, k is a fixed positive integer, t_i $(i\in\mathbb{N}_1^k)$ are fixed points with $0<t_1<t_2<\cdots<t_k<1$, $\Delta u(t_i)=u(t_i+0)-u(t_i-0)$, $\Delta D_{0^+}^{\alpha-1}u(t_i)=D_{0^+}^{\alpha-1}u(t_i+0)-D_{0^+}^{\alpha-1}u(t_i-0)$, $i\in\mathbb{N}_1^k$, $0<\xi_1<\cdots<\xi_n<1$, $0<\eta_1<\cdots<\eta_n<1$, $\xi_i,\eta_j\neq t_k$, and

$$\sum_{i=1}^{n}a_i\xi_i^{\alpha-2}=\sum_{i=1}^{n}b_i\eta_i^{\alpha-2}=1, \quad \sum_{i=1}^{n}a_i\xi_i^{\alpha-1}=0, \quad \sum_{i=1}^{n}b_i\eta_i^{\alpha-1}=1.$$

In [14], Bai studied the existence of solutions of the following boundary value problems of nonlinear impulsive fractional differential equation at resonance

$$\begin{cases} D_{0^+}^{\alpha}u(t)=f(t,u(t),D_{0^+}^{\alpha-1}u(t)), \quad t\in(0,1), \quad t\neq t_i, \quad i\in\mathbb{N}_1^k, \\ D_{0^+}^{\alpha-1}u(0)=\sum\limits_{i=1}^{m}a_iD_{0^+}^{\alpha-1}u(\xi_i), \quad u(1)=\sum\limits_{i=1}^{n}b_i\eta_i^{2-\alpha}u(\eta_i), \\ \Delta u(t_i)=I_i(u(t_i),D_{0^+}^{\alpha-1}u(t_i)), \quad \Delta D_{0^+}^{\alpha-1}u(t_i)=J_i(u(t_i),D_{0^+}^{\alpha-1}u(t_i)), \quad i\in\mathbb{N}_1^k, \end{cases} \tag{1.3.21}$$

where $\alpha\in(1,2)$, $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $*$, $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, and $I_i,J_i:\mathbb{R}^2\to\mathbb{R}$ are continuous, k is a fixed positive integer, t_i $(i\in\mathbb{N}_1^k)$ are fixed points with $0<t_1<t_2<\cdots<t_k<1$, $\Delta u(t_i)=u(t_i+0)-u(t_i-0)$, $\Delta D_{0^+}^{\alpha-1}u(t_i)=D_{0^+}^{\alpha-1}u(t_i+0)-D_{0^+}^{\alpha-1}u(t_i-0)$, $i\in\mathbb{N}_1^k$, $0<\xi_1<\cdots<\xi_m<1$, $0<\eta_1<\cdots<\eta_n<1$, $\xi_i,\eta_j\neq t_k$, and

$$\sum_{i=1}^{m}a_i=\sum_{i=1}^{n}b_i=1.$$

However, we find that the main lemmas in [13, 14] are unsuitable see Remark 6.3. Lemma 2.1 in [95] is also unsuitable see Remark 6.4 in Section 6.

In recent paper [52], Liu and Jia studied the existence of solutions of the following boundary value problem for multi-term impulsive fractional differential equation (BVP for short) with impulse effects

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) = f\left(t, u(t), {}^cD_{0^+}^{\beta}u(t)\right), \quad t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k), D_{0^+}^{\beta}u(t_k)), \quad k \in \mathbb{N}_1^m, \\ \Delta {}^cD_{0^+}^{\beta}u|_{t=t_k} = \overline{I}_k(u(t_k), {}^cD_{0^+}^{\beta}u(t_k)), \quad k \in \mathbb{N}_1^m, \\ u(0) = 0, \quad u(1) = \int_0^1 u(t)g(t)dt, \end{cases} \tag{1.3.22}$$

where ${}^cD_{a^+}^b$ is the Caputo fractional derivative of order $b > 0$ with the starting point a, $\alpha \in (1,2)$, $0 < \beta < \alpha - 1$ and $0 < \beta < 1$, $J' = [0,1] \setminus \{t_1, \cdots, t_m\}$, m is a positive integer, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+), u(t_k^-)$ denote the right limit and the left limit of $u(t)$ at t_k, respectively, $\Delta {}^cD_{0^+}u|_{t=t_k} = {}^cD_{0^+}u(t_k^+) - {}^cD_{0^+}u(t_k^-)$, ${}^cD_{0^+}u(t_k^+), {}^cD_{0^+}u(t_k^-)$ denote the right limit and the left limit of ${}^cD_{0^+}^{\beta}u(t)$ at t_k, respectively. Because $u(t_k), {}^cD_{0^+}^{\beta}u(t_k^-)$ exist, we define $u(t_k) = u(t_k^-)$ and ${}^cD_{0^+}^{\beta}u(t_k) = {}^cD_{0^+}^{\beta}u(t_k^-)$, $k \in \mathbb{N}_1^m$. $f, I_k, \overline{I}_k$ are continuous functions. We find that the impulsive function $\Delta {}^cD_{0^+}^{\beta}u|_{t=t_k} = \overline{I}_k(u(t_k), {}^cD_{0^+}^{\beta}u(t_k))$ is unsuitable. So Lemma 2.3 in [52], see Remark 6.10 in Section 6.

Sequential fractional derivative operators can appear in the formulation of various applied problems in physics and applied science. Indeed, differential equations modelling processes or objects arise usually as a result of a substitution of one relationship involving derivatives into another one. If the derivatives in both relationships are fractional derivatives, then the resulting expression (equation) will contain, in general case, sequential fractional derivative operators ([65], p. 88). Therefore the consideration of sequential fractional derivative operators is of interest [65].

Liu [46] studied the existence of global solutions of the following initial value problem of nonlinear fractional differential system on half line with sequential fractional derivative operators

$$\begin{cases} D^{\sigma_n}x(t) + \phi(t)f(t, y(t), D_{0^+}^{p}y(t)) = 0, \quad t \in (0, +\infty), \\ D^{\tau_m}y(t) + \psi(t)g(t, x(t), D_{0^+}^{q}x(t)) = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t \to 0} t^{1-\alpha_i}D^{\sigma_{i-1}}x(t) = x_{i-1}, \quad i \in \mathbb{N}_1^n, \\ \lim\limits_{t \to 0} t^{1-\beta_i}D^{\tau_{i-1}}y(t) = y_{i-1}, \quad i \in \mathbb{N}_1^n, \end{cases} \tag{1.3.23}$$

where $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $* > 0$, $\alpha_i \in (0,1)$, $\sigma_j = \alpha_1 + \cdots + \alpha_j (j \in \mathbb{N}_1^n)$, $q \in (0,1)$ with $q < \sigma_n$, $D^{\sigma_j}x = D_{0^+}^{\alpha_j} \cdots D_{0^+}^{\alpha_2} D_{0^+}^{\alpha_1}x$ $(j \in \mathbb{N}_1^n)$ is a sequential fractional derivative operator, $D^{\sigma_0}x = x$, $\beta_i \in (0,1)$, $\tau_j = \beta_1 + \cdots + \beta_j$ $(j \in \mathbb{N}_1^m)$, $p \in (0,1)$ with $p < \tau_m$, $D^{\tau_j}y = D_{0^+}^{\beta_j} \cdots D_{0^+}^{\beta_2} D_{0^+}^{\beta_1}y$ $(j \in N_1^m)$ is a sequential fractional derivative operator, $D^{\beta_0}y = y$, $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$ are initial data, $\phi,\psi:(0,+\infty)\to\mathbb{R}$ satisfy that there exist constants $k_i>-1$ $(i=1,2)$ such that $|\phi(t)|\leqslant t^{k_1}$, $|\psi(t)|\leqslant t^{k_2}$, $t\in(0,\infty)$, $f,g:(0,+\infty)\times\mathbb{R}^2\to\mathbb{R}$ and f is a τ-Carathéodory function and g a σ-Carathéodory function.

In [12], Bai, by using monotone iterative method, studied the existence of solutions for nonlinear impulsive fractional differential equation with periodic boundary conditions

$$\begin{cases} D_{0^+}^{2\alpha}u(t)=f(t,u(t),D_{0^+}^{\alpha}u(t)), \quad t\in(0,1), \quad t\neq t_i, \quad i\in\mathbb{N}_1^m,\\ \lim\limits_{t\to0}t^{1-\alpha}u(t)=u(1), \quad \lim\limits_{t\to0}t^{1-\alpha}D_{0^+}^{\alpha}u(t)=D_{0^+}^{\alpha}u(1),\\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}[u(t)-u(t_i)]=I_i(u(t_i)), \quad i\in\mathbb{N}_1^m,\\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}[D_{0^+}^{\alpha}u(t)-D_{0^+}^{\alpha}u(t_i)]=\overline{I}_i(u(t_i)), \quad i\in\mathbb{N}_1^m, \end{cases} \tag{1.3.24}$$

where $\alpha\in(0,1)$, $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $*$, $D_{0^+}^{2\alpha}=D_{0^+}^{\alpha}D_{0^+}^{\alpha}$ is the sequential Riemann-Liouville fractional derivative, $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, and I_i, $\overline{I}_i:\mathbb{R}\to\mathbb{R}$ are continuous, m is a fixed positive integer, t_i $(i\in\mathbb{N}_1^m)$ are fixed points with $0<t_1<t_2<\cdots<t_m<1$. The main result (Theorem 3.6) in [12] is based upon the assumption that (1.3.24) has a couple of lower and upper solutions. Studies on solvability of IVPs or BVPs for higher order fractional differential equations are of importance and of interest.

In [79], authors, by utilizing boundedness, continuity, monotonicity and nonnegative of Mittag-Leffler functions and fixed-point theorem, studied the existence results of solutions for the following impulsive fractional Langevin equations with two different fractional derivatives

$$\begin{cases} {}^cD_{0,t}^{\alpha}({}^cD_{0,t}^{\beta}+\lambda)u(t)=f(t,u(t)), \quad t\in[0,1]\setminus\{t_1,t_2,\cdots,t_m\},\\ \Delta u(t_k)=u(t_k^+)-u(t_k^-)=I_k\in\mathbb{R}, \quad k\in\mathbb{N}_1^m,\\ u(0)=u(1)=0, \quad u(\eta_k)=0, \quad \eta_k\in(t_k,t_{k+1}), \quad k\in\mathbb{N}_0^m, \end{cases} \tag{1.3.25}$$

where $f\in C([0,1]\times\mathbb{R},\mathbb{R})$, $0<\alpha,\beta<1$, $\alpha+\beta<1$, $\lambda>0$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $u(t_k^+)$, $u(t_k^-)$ denote the right limit and the left limit of $u(t)$ at t_k, respectively.

In [88], Zhao studied existence and uniqueness of the solutions of the following two classes of impulsive boundary value problems for fractional differential equations sequential Caputo fractional derivatives

$$\begin{cases} {}^cD_{0,t}^{\alpha}({}^cD_{0,t}^{\beta}u(t))+\lambda u(t)=f(t,u(t)), \quad t\in[0,1]\setminus\{t_1,t_2,\cdots,t_m\},\\ \Delta u(t_k)=u(t_k^+)-u(t_k^-)=y_k, \quad k\in\mathbb{N}_1^m,\\ au(0)+bu(1)=c, \quad [{}^cD_{0,t}^{\beta}u]|_{t=t_k}=d_k, \quad k\in\mathbb{N}_0^m, \end{cases} \tag{1.3.26}$$

and

$$\begin{cases} {}^cD_{0,t}^{\alpha}({}^cD_{0,t}^{\beta}u(t))+\lambda u(t)=f(t,u(t)), & t\in[0,1]\setminus\{t_1,t_2,\cdots,t_m\},\\ \Delta u(t_k)=u(t_k^+)-u(t_k^-)=y_k, & k\in\mathbb{N}_1^m,\\ a[{}^cD_{0,t}^{\beta}u]|_{t=0}+b[{}^cD_{0,t}^{\beta}u]|_{t=t_m}=c, & u(t_k)=d_k,\quad k\in\mathbb{N}_0^m, \end{cases} \tag{1.3.27}$$

where $f\in C([0,1]\times\mathbb{R},\mathbb{R})$, $0<\alpha,\beta<1$, $y_k\in\mathbb{R}$, $\lambda>0$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $u(t_k^+),u(t_k^-)$ denote the right limit and the left limit of $u(t)$ at t_k, respectively, $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, $d_k\geqslant 0$ are real constants. In (1.2) on page 3 in [88], $a[{}^cD_{0,t}^{\beta}u]|_{t=0}+b[{}^cD_{0,t}^{\beta}u]|_{t=t_m}=c$ may be $a[{}^cD_{0,t}^{\beta}u]|_{t=0}+b[{}^cD_{0,t}^{\beta}u]|_{t=t_{m+1}}=c$ (this may be a type mistake). However we find that formulae of a solution for these problems (Lemma 2.9 and Lemma 2.10 in [88]) are wrong, see Remark 6.8 and Remark 6.9 in Section 6.

RESULT 1.1. BVPs for IFDEs with a single starting point are different from those with multiple starting points.

Consider the following two impulsive boundary value problems involving the Caputo fractional derivatives

$${}^cD_{t_i^+}^{\alpha}x(t)=1,\quad t\in(t_i,t_{i+1}],\quad i=0,1,\quad x(0)=-x(1),\quad x(1/2^+)=-x(1/2), \tag{I}$$

and

$${}^cD_{0^+}^{\alpha}x(t)=1,\quad t\in(t_i,t_{i+1}],\quad i=0,1,\quad x(0)=-x(1),\quad x(1/2^+)=-x(1/2), \tag{II}$$

where $0=t_0<t_1=\frac{1}{2}<t_2=1$, and $\alpha\in(0,1)$. It is easy to see that BVP(I) has infinitely many solutions $\{x_c:c\in\mathbb{R}\}$ defined by

$$x_c(t)=\begin{cases} c+\frac{t^{\alpha}}{\Gamma(\alpha+1)}, & t\in[0,1/2],\\ -c+\frac{1}{2^{\alpha}\Gamma(\alpha+1)}+\frac{(t-1/2)^{\alpha}}{\Gamma(\alpha+1)}, & t\in(t_1,1]. \end{cases}$$

While BVP(II) has no solution. In fact, suppose that x is a solution of BVP(II). By Corollary 3.3 in Section 3, there exists constants $a,b\in\mathbb{R}$ such that

$$x(t)=\begin{cases} a+\frac{t^{\alpha}}{\Gamma(\alpha+1)}, & t\in[0,1/2],\\ a+b+\frac{t^{\alpha}}{\Gamma(\alpha+1)}, & t\in(t_1,1]. \end{cases}$$

Then $a=-\left(a+b+\frac{1}{\Gamma(\alpha+1)}\right)$ and $a+b+\frac{(1/2)^{\alpha}}{\Gamma(\alpha+1)}=-\left(a+\frac{(1/2)^{\alpha}}{\Gamma(\alpha+1)}\right)$. Then $2a+b=-\frac{1}{\Gamma(\alpha+1)}$ and $2a+b=-2\frac{(1/2)^{\alpha}}{\Gamma(\alpha+1)}$, which means a contradiction.

Consider the following the following two impulsive boundary value problems involving the Riemman-Liouville fractional derivatives

$$\begin{aligned} &D_{t_i^+}^{\alpha}x(t)=1,\quad t\in(t_i,t_{i+1}],\quad i=0,1,\\ &\lim_{t\to 0^+}t^{1-\alpha}x(t)=-x(1),\quad \lim_{t\to(1/2)^+}(t-1/2)^{1-\alpha}x(t)=-2^{2\alpha-2}x(1/2), \end{aligned} \tag{III}$$

and

$$\begin{aligned} &D_{0^+}^{\alpha}x(t) = 1, \quad t \in (t_i, t_{i+1}], \quad i = 0, 1, \\ &\lim_{t \to 0^+} t^{1-\alpha}x(t) = -x(1), \qquad \lim_{t \to (1/2)^+} (t - 1/2)^{1-\alpha}x(t) = -2^{2\alpha-2}x(1/2), \end{aligned} \tag{IV}$$

where $0 = t_0 < t_1 = \frac{1}{2} < t_2 = 1$, and $\alpha \in (0,1)$. It is easy to show that $x(t) = -\frac{1}{2\Gamma(\alpha+1)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha+1)}t^{\alpha}$ is a solution of BVP(IV). While BVP(III) has no solution. In fact, suppose x is a solution of BVP(III). Then by Corollary 3.1 in Section 3, there exist constants $a, b \in \mathbb{R}$ such that

$$x(t) = \begin{cases} at^{\alpha-1}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, & t \in [0, 1/2], \\ b(t-1/2)^{\alpha-1} + \frac{(t-1/2)^{\alpha}}{\Gamma(\alpha+1)}, & t \in (1/2, 1]. \end{cases}$$

Then $a = -\left(b(1/2)^{\alpha-1} + \frac{(1/2)^{\alpha}}{\Gamma(\alpha+1)}\right)$ and $b = -2^{2\alpha-2}\left(a(1/2)^{\alpha-1}\frac{(1/2)^{\alpha}}{\Gamma(\alpha+1)}\right)$, which means a contradiction.

RESULT 1.2. Resonant and non-resonant BVPs caused by impulse effects.

Let $h \in L^1(0,1) \cap C(0,1)$, $\alpha \in (0,1)$, b_0, $c_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and there exist constant $k > -1$, $l \in (-1, 0]$ such that $k + l + 1 \geqslant 0$ and $|h(t)| \leqslant (t_i - t)^k (t_{i+1} - t)^l$ for all $t \in (t_i, t_{i+1})$. We know that both

$$D_{t_i^+}^{\alpha}x(t) = h(t), \text{ a.e., } t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m, \quad I_{0^+}^{1-\alpha}x(0) = k_1 I_{t_m^+}^{1-\alpha}x(1) \tag{V}$$

and

$$\begin{aligned} &D_{t_i^+}^{\alpha}x(t) = h(t), \text{ a.e., } t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m, \quad I_{0^+}^{1-\alpha}x(0) = k_1 I_{t_m^+}^{1-\alpha}x(1), \\ &\Delta I_{0^+}^{1-\alpha}x(t_i) = c_i I_{t_{i-1}^+}^{1-\alpha}(t_i), \quad i \in \mathbb{N}_1^m, \end{aligned} \tag{VI}$$

are impulsive boundary value problems. It is easy to see from Corollary 3.1.1 that BVP(V) has solutions $x(t) = \int_{t_i}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + C_i(t - t_i)^{\alpha-1}$, $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}_0^m$ with $C_0 = k_1 C_m$ and $C_i \in \mathbb{R}$. So BVP(V) is a resonant problem, while BVP(VI) is a resonant problem if and only if $1 - \Gamma(\alpha)k_1 \prod_{\sigma=1}^{m}(1 + c_{\sigma}) = 0$.

Let $h \in L^1(0,1) \cap C(0,1)$, $\alpha \in (0,1)$, b_0, $c_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and there exist constant $k > -1$, $l \in (-1, 0]$ such that $k + l + 1 \geqslant 0$ and $|h(t)| \leqslant t^k(1-t)^l$ for all $t \in (0,1)$. Consider the following boundary value problem

$$D_{0^+}^{\alpha}x(t) = h(t), \text{ a.e., } t \in (0, 1], \quad I_{0^+}^{1-\alpha}x(0) = k_1 I_{0^+}^{1-\alpha}x(1) \tag{VII}$$

and the impulsive boundary value problem

$$\begin{aligned} &D_{0^+}^{\alpha}x(t) = h(t), \text{ a.e., } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, \quad I_{0^+}^{1-\alpha}x(0) = k_1 I_{0^+}^{1-\alpha}x(1), \\ &\Delta I_{0^+}^{1-\alpha}x(t_i) = c_i I_{0^+}^{1-\alpha}(t_i), \quad i \in \mathbb{N}_1^m. \end{aligned} \tag{VIII}$$

It is easy to see from Corollary 3.1.1 that BVP(VII) has infinitely many solutions

$$x(t) = \int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + Ct^{\alpha-1}, \quad t \in (0,1]$$

with $C \in \mathbb{R}$ if and only if $k_1 = 1$. So BVP(VII) is a non-resonant problem if $k_1 \neq 1$, while BVP(VIII) is a resonant problem if and only if

$$1 - k_1 = k_1 \sum_{i=1}^m \left[\sum_{\sigma=1}^{i-2} \left((-1)^{i+\sigma}(1+c_i)(-1)^{i-\sigma} \prod_{u=\sigma+1}^{i-1} (1+c_i) \right) + (1+c_i) + 1 \right]. \quad \text{(IX)}$$

The reason is as follows: for BVP(VIII), from Corollary 3.1, we know there exist constants C_i $(i \in \mathbb{N}_0^m$ such that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \sum_{j=0}^i C_j(t-t_j)^{\alpha-1}, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m.$$

Then we get

$$I_{0^+}^{1-\alpha} x(t) = \int_0^t h(s)ds + \Gamma(\alpha) \sum_{j=0}^i C_j, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m.$$

Hence

$$\Gamma(\alpha)C_0 = k_1 \int_0^1 h(s)ds + \Gamma(\alpha)k_1 \sum_{j=0}^m C_j.$$

It follows that

$$(1-k_1)C_0 = \frac{k_1}{\Gamma(\alpha)} \int_0^1 h(s)ds + k_1 \sum_{i=1}^m C_i.$$

On the other hand, we have from $\Delta I_{0^+}^{1-\alpha} x(t_i) = c_i I_{0^+}^{1-\alpha}(t_i)$, $i \in \mathbb{N}_1^m$ that

$$\Gamma(\alpha)C_i = (1+c_i) \left[\int_0^{t_i} h(s)ds + \Gamma(\alpha) \sum_{j=0}^{i-1} C_j \right], \quad i \in \mathbb{N}_1^m.$$

It turns to

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1-c_2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1-c_i & -1-c_i & -1-c_i & \cdots & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \cdots \\ C_i \end{pmatrix} = \begin{pmatrix} C_0 + \frac{1+c_1}{\Gamma(\alpha)} \int_0^{t_1} h(s)ds \\ C_0 + \frac{1+c_2}{\Gamma(\alpha)} \int_0^{t_2} h(s)ds \\ \cdots \\ C_0 + \frac{1+c_i}{\Gamma(\alpha)} \int_0^{t_i} h(s)ds \end{pmatrix}.$$

Then

$$\begin{pmatrix} C_1 \\ C_2 \\ \cdots \\ C_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1-c_2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1-c_i & -1-c_i & -1-c_i & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} C_0 + \frac{1+c_1}{\Gamma(\alpha)} \int_0^{t_1} h(s)ds \\ C_0 + \frac{1+c_2}{\Gamma(\alpha)} \int_0^{t_2} h(s)ds \\ \cdots \\ C_0 + \frac{1+c_i}{\Gamma(\alpha)} \int_0^{t_i} h(s)ds \end{pmatrix}.$$

It follows that

$$\begin{aligned}
C_i &= \sum_{\sigma=1}^{i-2}\left((-1)^{i+\sigma}(1+c_i)(-1)^{i-\sigma}\prod_{u=\sigma+1}^{i-1}(1+c_i)\right)\left(C_0+\frac{1+c_\sigma}{\Gamma(\alpha)}\int_0^{t_\sigma}h(s)ds\right)\\
&\quad+(-1)^{i+i-1}(-1-c_i)\left(C_0+\frac{1+c_{i-1}}{\Gamma(\alpha)}\int_0^{t_{i-1}}h(s)ds\right)+(-1)^{i+i}\left(C_0+\frac{1+c_i}{\Gamma(\alpha)}\int_0^{t_i}h(s)ds\right)\\
&=\sum_{\sigma=1}^{i-2}\left((-1)^{i+\sigma}(1+c_i)(-1)^{i-\sigma}\prod_{u=\sigma+1}^{i-1}(1+c_i)\right)C_0+(1+c_i)C_0+C_0\\
&\quad+\sum_{\sigma=1}^{i-2}\left((-1)^{i+\sigma}(1+c_i)(-1)^{i-\sigma}\prod_{u=\sigma+1}^{i-1}(1+c_i)\right)\frac{1+c_\sigma}{\Gamma(\alpha)}\int_0^{t_\sigma}h(s)ds\\
&\quad+\frac{(1+c_i)(1+c_{i-1})}{\Gamma(\alpha)}\int_0^{t_{i-1}}h(s)ds+\frac{1+c_i}{\Gamma(\alpha)}\int_0^{t_i}h(s)ds,\quad i\in\mathbb{N}_1^m.
\end{aligned}$$

Hence

$$\begin{aligned}
(1-k_1)C_0 &= \frac{k_1}{\Gamma(\alpha)}\int_0^1 h(s)ds+k_1\sum_{i=1}^m\left[\sum_{\sigma=1}^{i-2}\left((-1)^{i+\sigma}(1+c_i)(-1)^{i-\sigma}\prod_{u=\sigma+1}^{i-1}(1+c_i)\right)C_0\right.\\
&\quad+(1+c_i)C_0+C_0\\
&\quad+\sum_{\sigma=1}^{i-2}\left((-1)^{i+\sigma}(1+c_i)(-1)^{i-\sigma}\prod_{u=\sigma+1}^{i-1}(1+c_i)\right)\frac{1+c_\sigma}{\Gamma(\alpha)}\int_0^{t_\sigma}h(s)ds\\
&\quad\left.+\frac{(1+c_i)(1+c_{i-1})}{\Gamma(\alpha)}\int_0^{t_{i-1}}h(s)ds+\frac{1+c_i}{\Gamma(\alpha)}\int_0^{t_i}h(s)ds\right].
\end{aligned}$$

Now one sees that BVP(VIII) has infinitely many solutions if and only if (IX) holds. □

1.4. Purposes of this paper

We note that in known papers existence of solutions of boundary value problems for the Caputo type fractional differential equations of lower order has been discussed deeply. Solvability of boundary value problems for impulsive higher order fractional differential equations has not been studied. The reasons are as follows: higher order fractional differential equations can not be converted to fractional differential systems with lower order since $D_{0^+}^{\alpha}D_{0^+}^{\beta}x(t)\neq D_{0^+}^{\alpha+\beta}x(t)\neq D_{0^+}^{\beta}D_{0^+}^{\alpha}x(t)$ [65]. It is also difficult to convert an impulsive higher order fractional differential equation with the Riemann-Liouville fractional derivatives to an equivalent integral equation. These results generalize Theorem 3.2, Theorem 3.4, Theorem 3.6 and Theorem 3.8 (on pages 436, 441, 446 and 452 in [**?**] respectively. The purposes of this paper are as follows:

Purpose (I) we establish a general method for converting an impulsive fractional differential equation with the Riemann-Liouville fractional derivatives to an equivalent integral equation. i.e., we establish explicit piecewise continuous solutions of the following four linear fractional differential equations (LFDEs):

$$D_{0^+}^{\alpha}x(t)-\lambda x(t)=h_1(t),\ \text{a.e.},\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m, \tag{1.4.1}$$

$${}^cD_{0^+}^{\alpha}x(t)-\lambda x(t)=h_2(t),\ \text{a.e.},\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m, \tag{1.4.2}$$

$$D_{0^+}^{\alpha}D_{0^+}^{\beta}x(t)-\lambda x(t)=h_3(t),\ \text{a.e.},\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m, \tag{1.4.3}$$

$$^cD_{0^+}^{\alpha}\,^cD_{0^+}^{\beta}x(t)-\lambda x(t)=h_4(t),\ \text{a.e.},\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m, \tag{1.4.4}$$

where $n-1<\alpha<n,\ r-1<\beta<r$ with r,n being positive integers, $D_{0^+}^{*}$ or $^cD_{0^+}^{*}$ stands the Riemann-Liouville fractional derivative or the Caputo fractional derivative with order $*$ and starting point $0,\ 0=t_0<t_1<\cdots<t_m<t_{m+1}=1,\ \lambda\in\mathbb{R},\ h_i\in L^(0,1),\ m$ is a positive integer, $\mathbb{N}_0^m=\{0,1,2,\cdots,m\}$ and $\mathbb{N}_1^m\{1,2,\cdots,m\},\ h_i\in L^1(t_i,t_{i+1})\cap C(t_i,t_{i+1}]\ (i\in\mathbb{N}_0^m)$.

(i) A function x with $x:(0,1]\to\mathbb{R}$ is said to be a *piecewise continuous solution* of (1.4.1) if x satisfies (1.4.1) and

$$x|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}],\quad \lim_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t)\quad\text{exists for all}\quad s\in\mathbb{N}_0^m.$$

(ii) A function x with $x:(0,1]\to\mathbb{R}$ is said to be a *piecewise continuous solution* of (1.4.2) (or (1.4.4)) if x satisfies (1.4.2) (or (1.4.4)) nd

$$x|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}],\quad \lim_{t\to t_s^+}x(t)\quad\text{exists for all}\quad s\in\mathbb{N}_0^m.$$

(iii) A function x with $x:(0,1]\to\mathbb{R}$ is said to be a *piecewise continuous solution* of (1.4.3) if x satisfies (1.4.3) and

$$x|_{(t_s,t_{s+1}]},D_{0^+}^{\beta}x|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}],\quad \lim_{t\to t_s^+}(t-t_s)^{r-\beta}x(t),\quad \lim_{t\to t_s^+}(t-t_s)^{n-\alpha}D_{0^+}^{\beta}x(t)$$

exists for all $s\in\mathbb{N}_0^m$.

Purpose (II) We establish existence results for solutions of the following eight classes of boundary value problems of fractional differential equations with impulse effects:

$$\begin{cases} D_{0^+}^{\alpha}x(t)-\lambda x(t)=f(t,x(t),D_{0^+}^{p}x(t)),\ \text{a.e.},\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ I_{0^+}^{2-\alpha}u(0)=I_{0^+}^{2-\alpha}u(1),\quad D_{0^+}^{\alpha-1}u(0)=D_{0^+}^{\alpha-1}u(1),\\ \lim\limits_{t\to t_s^+}(t-t_s)^{2-\alpha}u(t)=\frac{I(t_s,u(t_s),D_{0^+}^{p}u(t_s))}{\Gamma(\alpha-1)},\quad s\in\mathbb{N}_1^m,\\ \Delta D_{0^+}^{\alpha-1}u(t_s)=J(t_s,u(t_s),D_{0^+}^{p}u(t_s)),\quad s\in\mathbb{N}_1^m, \end{cases} \tag{1.4.5}$$

$$\begin{cases} D_{t_i^+}^{\alpha}x(t)-\lambda x(t)=f(t,x(t),D_{t_i^+}^{p}x(t)),\ \text{a.e.},\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ I_{0^+}^{2-\alpha}u(0)=I_{t_m^+}^{2-\alpha}u(1),\quad D_{0^+}^{\alpha-1}u(0)=D_{t_m^+}^{\alpha-1}u(1),\\ \lim\limits_{t\to t_s^+}(t-t_s)^{2-\alpha}u(t)=\frac{I(t_s,u(t_s),D_{t_{s-1}^+}^{p}x(t_s))}{\Gamma(\alpha-1)},\quad s\in\mathbb{N}_1^m,\\ D_{t_s^+}^{\alpha-1}u(t_s)=J(t_s,u(t_s),D_{t_{s-1}^+}^{p}x(t_s)),\sigma\in\mathbb{N}_1^{n-1},\quad s\in\mathbb{N}_1^m, \end{cases} \tag{1.4.6}$$

$$\begin{cases} ^cD_{0^+}^{\alpha}x(t)-\lambda x(t)=g(t,x(t),{}^cD_{0^+}^{p}x(t)),\ \text{a.e.},\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ u(0)=u(1),\quad u'(0)=u'(1),\\ \Delta u(t_s)=\overline{I}(t_s,u(t_s),{}^cD_{0^+}^{p}x(t_s)),\quad s\in\mathbb{N}_1^m,\\ \Delta u'(t_s)=\overline{J}(t_s,u(t_s),{}^cD_{0^+}^{p}x(t_s)),\quad s\in\mathbb{N}_1^m, \end{cases} \tag{1.4.7}$$

$$\begin{cases} {}^cD_{t_i^+}^{\alpha}x(t)-\lambda x(t)=g(t,x(t),\ \ {}^cD_{t_i^+}^{p}x(t)),\ \text{a.e.,}\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ u(0)=u(1),\quad u'(0)=u'(1),\\ u(t_s)=\overline{I}(t_s,u(t_s),{}^cD_{t_{s-1}^+}^{p}x(t_s)),\quad s\in\mathbb{N}_1^m,\\ u'(t_s)=\overline{J}(t_s,u(t_s),{}^cD_{t_{s-1}^+}^{p}x(t_s)),\quad s\in\mathbb{N}_1^m,\end{cases}\tag{1.4.8}$$

$$\begin{cases} D_{0^+}^{\alpha}D_{0^+}^{\beta}x(t)-\lambda x(t)=f_1(t,x(t),D_{0^+}^{q}x(t)),\ \text{a.e.,}\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ I_{0^+}^{1-\beta}x(0)=I_{0^+}^{1-\beta}x(1),\quad I_{0^+}^{1-\gamma}D_{0^+}^{\beta}x(0)=I_{0^+}^{1-\gamma}D_{0^+}^{\beta}x(1),\\ \lim\limits_{t\to t_s^+}(t-t_s)^{1-\beta}x(t)=\frac{I_1(t_s,x(t_s),D_{0^+}^{q}x(t_s))}{\Gamma(\beta)},\quad s\in\mathbb{N}_1^m,\\ \lim\limits_{t\to t_s^+}(t-t_s)^{1-\alpha}D_{0^+}^{\beta}x(t)=\frac{J_1(t_s,x(t_s),D_{0^+}^{q}x(t_s))}{\gamma(\alpha)},\quad s\in\mathbb{N}_1^m,\end{cases}\tag{1.4.9}$$

$$\begin{cases} D_{t_i^+}^{\alpha}D_{t_i^+}^{\beta}x(t)-\lambda x(t)=f_1(t,x(t),{}^cD_{t_i^+}^{q}x(t)),\ \text{a.e.,}\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ I_{0^+}^{1-\beta}x(0)=I_{t_m^+}^{1-\beta}x(1),\quad I_{0^+}^{1-\gamma}D_{0^+}^{\beta}x(0)=I_{t_m^+}^{1-\gamma}D_{t_m^+}^{\beta}x(1),\\ \lim\limits_{t\to t_s^+}(t-t_s)^{1-\beta}x(t)=\frac{I_1(t_s,x(t_s),D_{t_{s-1}^+}^{q}x(t_s))}{\Gamma(\beta)},\quad s\in\mathbb{N}_1^m,\\ \lim\limits_{t\to t_s^+}(t-t_s)^{1-\alpha}D_{0^+}^{\beta}x(t)=\frac{J_1(t_s,x(t_s),D_{t_{s-1}^+}^{q}x(t_s))}{\Gamma(\alpha)},\quad s\in\mathbb{N}_1^m,\end{cases}\tag{1.4.10}$$

$$\begin{cases} {}^cD_{0^+}^{\alpha}\,{}^cD_{0^+}^{\beta}x(t)-\lambda x(t)=g_1(t,x(t),{}^cD_{0^+}^{q}x(t)),\ \text{a.e.,}\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ u(0)=u(1),\quad {}^cD_{0^+}^{\beta}u(0)={}^cD_{0^+}^{\beta}u(1),\\ \Delta u(t_s)=\overline{I}_1(t_s,x(t_s),{}^cD_{0^+}^{q}x(t_s)),\quad s\in\mathbb{N}_1^m,\\ \Delta({}^cD_{0^+}^{\beta}x)(t_s)=\overline{J}_1(t_s,x(t_s),{}^cD_{0^+}^{q}x(t_s)),\quad s\in\mathbb{N}_1^m,\end{cases}\tag{1.4.11}$$

$$\begin{cases} {}^cD_{t_i^+}^{\alpha}\,{}^cD_{t_i^+}^{\beta}x(t)-\lambda x(t)=g_1(t,x(t),{}^cD_{t_i^+}^{q}x(t)),\ \text{a.e.,}\ t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ x(0)=x(1),\quad {}^cD_{0^+}^{\beta}x(0)={}^cD_{t_m^+}^{\beta}x(1),\\ x(t_s)=\overline{I}_1(t_s,x(t_s),{}^cD_{t_{s-1}^+}^{q}x(t_s)),\quad s\in\mathbb{N}_1^m,\\ {}^cD_{0^+}^{\beta}x(t_s)=\overline{J}_1(t_s,x(t_s),{}^cD_{t_{s-1}^+}^{q}x(t_s)),\quad s\in\mathbb{N}_1^m,\end{cases}\tag{1.4.12}$$

where

(a) $1<\alpha<2$ and $p\in(0,\alpha-1)$ in (1.4.5)–(1.4.8), $0<\beta<1$ and $0<\alpha<1$, $q\in(0,\beta)$ in (1.4.9)–(1.4.12), $D_{a^+}^{b}$ the Riemann-Liouville fractional derivatives of order b with starting point a and ${}^cD_{a^+}^{b}$ the Caputo fractional derivatives of order b with starting point a respectively with $a\geqslant 0$ and $b>0$,

(b) $\lambda\in\mathbb{R}$, m is a positive integer, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $\mathbb{N}_a^b=\{a,a+1,a+2,\cdots,b\}$ for every pair of integers $a<b$,

(c) f is a I-Carathéodory function (Definition 2.4) and g,g_1 are II-Carathdóry functions (Definition 2.5), f_1 is a III-Carathódory function (Definition 2.6),

(d) I,J are discrete I-Carathéodory functions (Definition 2.7), $\overline{I},\overline{J},\overline{I}_1,\overline{J}_1$ are discrete II-Carathéodory functions (Definition 2.8), I_1,J_1 are discrete III-Carathéodory functions (Definition 2.9).

Obviously, BVP(1.4.5)–BVP(1.4.12) generalize some know problems discussed in [12, 13, 14, 45, 52, 70, 78, 79, 88, 90]. The definitions of solutions of above problems are defined as follows:

(i) A function x with $x:(0,1]\to\mathbb{R}$ is said to be a *solution* of (1.4.5) (or BVP(1.4.6)) if $x|_{(t_s,t_{s+1}]}, D_{0^+}^{p}x|_{(t_s,t_{s+1}]}\in C^0(t_s,\ \ t_{s+1}]$, $\lim\limits_{t\to t_s^+}(t-t_s)^{2-\alpha}x(t)$, $\lim\limits_{t\to t_s^+}(t-t_s)^{2+p-\alpha}D_{0^+}^{p}x(t)$ exists for all $s\in\mathbb{N}_0^m$, $D_{0^+}^{\alpha}x$ are measurable on $(0,1]$ and x satisfies all equations in (1.4.5) or (1.4.6).

(ii) A function x with $x:(0,1]\to\mathbb{R}$ is said to be a *solution* of (1.4.7) (or BVP(1.4.8)) if $x|_{(t_s,t_{s+1}]}, {}^cD_{t_i^+}^{p}x|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]$, $\lim\limits_{t\to t_s^+}x(t)$, $\lim\limits_{t\to t_s^+}{}^cD_{t_i^+}^{p}x(t)$ exists for all $s\in\mathbb{N}_0^m$, $D_{t_i^+}^{\alpha}x$ are measurable on $(t_i,t_{i+1}]$, $i\in\mathbb{N}_0^m$ and x satisfies all equations in (1.4.7) or (1.4.8).

(iii) A function x with $x:(0,1]\to\mathbb{R}$ is said to be a *solution* of (1.4.9) (or BVP(1.4.10)) if $x|_{(t_s,t_{s+1}]}, D_{0^+}^{q}x|_{(t_s,t_{s+1}]}\in C^0(t_s,\ \ t_{s+1}]$, $\lim\limits_{t\to t_s^+}(t-t_s)^{1-\beta}x(t)$, $\lim\limits_{t\to t_s^+}(t-t_s)^{1+q-\beta}D_{0^+}^{p}x(t)$ exists for all $s\in\mathbb{N}_0^m$, $D_{0^+}^{\alpha}D_{0^+}^{\beta}x$ are measurable on $(0,1]$ and x satisfies all equations in (1.4.9) or (1.4.10).

(iv) A function x with $x:(0,1]\to\mathbb{R}$ is said to be a *solution* of (1.4.11) (or BVP(1.4.12)) if $x|_{(t_s,t_{s+1}]}, {}^cD_{t_i^+}^{q}x|_{(t_s,t_{s+1}]}\in C^0(t_s,\,t_{s+1}]$, $\lim\limits_{t\to t_s^+}x(t)$, $\lim\limits_{t\to t_s^+}{}^cD_{t_i^+}^{q}x(t)$ exist for all $s\in\mathbb{N}_0^m$, ${}^cD_{t_i^+}^{\alpha}\,{}^cD_{0^+}^{\beta}x$ is measurable on $(t_i,t_{i+1}]$, $i\in\mathbb{N}_0^m$ and x satisfies all equations in (1.4.11) or (1.4.12).

Purpose III We establish existence results for solutions of the following three BVPs for higher order IFDEs:

$$\begin{cases} {}^cD_{0^+}^{\beta}u(t)=f(t,u(t)),\ \text{a.e.},\ t\in(t_s,t_{s+1}], & s\in\mathbb{N}_0^m,\\ u^{(2i)}(0)=0, \quad i\in\mathbb{N}_0^{n-1},\\ u^{(2j)}(1)=0, \quad j\in\mathbb{N}_0^{n-1},\\ \Delta u^{(j)}(t_s)=I_j(t_s,u(t_s)), \quad j\in\mathbb{N}_0^{2n-1}, & s\in\mathbb{N}_1^m, \end{cases} \tag{1.4.13}$$

$$\begin{cases} D_{0^+}^{\alpha}x(t)=f(t,x(t)), \quad t\in(t_s,t_{s+1}], \quad s\in\mathbb{N}_0^m,\\ D_{0^+}^{\alpha-i}x(0)=0, \quad i\in\mathbb{N}_1^k,\\ D_{0^+}^{\alpha-j}x(1)=0, \quad j\in\mathbb{N}_k^{n-1},\\ \Delta I_{0^+}^{n-\alpha}x(t)=I_n(t_s,x(t_s)), \quad s\in\mathbb{N}_1^m,\\ \Delta D_{0^+}^{\alpha-j}x(t_s)=I_j(t_s,x(t_s)), \quad j\in\mathbb{N}_1^{n-1}, \quad s\in\mathbb{N}_1^m, \end{cases} \tag{1.4.14}$$

and

$$\begin{cases} {}^cD_{0+}^{\alpha}x(t)=f(t,x(t)), & t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m, \\ \Delta x^{(j)}(t_i)=I_j(t_i,x(t_i)), & i\in\mathbb{N}_1^m, \quad j\in\mathbb{N}_0^{n-1}, \\ x^{(i)}(0)-k_ix^{(i)}(1)=0, & i\in\mathbb{N}_0^{n-1}, \end{cases} \tag{1.4.15}$$

where $n-1\leqslant\alpha<n$, $2n-1\leqslant\beta<2n$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $k_i\in[0,1)$, f,I_i satisfy some suitable assumptions given in Section 6. Obviously, BVP(1.4.13) generalizes BVP(1.1.1), BVP(1.4.14) is a generalized form of BVP(1.1.10), BVP(1.4.15) is a generalization of BVP(1.1.8).

We firstly present explicit piecewise continuous solution of (1.4.1), (1.4.2), (1.4.3) and (1.4.4) respectively. Then we obtain existence results for solutions of BVP(1.4.5)–BVP(1.4.12) respectively. Finally we establish existence results for solutions of BVP(1.4.13)–BVP(1.4.15). The methods used in this paper are based upon the standard fixed point theorems.

Our results are new and naturally complement the literature on boundary value problems for higher order impulsive fractional differential equations. This paper may be the first one concerned with the solvability of boundary value problems for higher order singular fractional differential equations involving the operators $D_{*+}^{\alpha}-\lambda x$ and $D_{*+}^{\beta}D_{*+}^{\beta}-\lambda x$ with impulse effects and the Riemann-Liouville and Caputo fractional derivatives.

The paper is outlined as follows. Section 2 contains some preliminary definitions and results. In Section 3, we establish explicit piecewise continuous solutions of equations (1.4.1)–(1.4.4) respectively. In Section 4, existence results for BVP(1.4.5)–BVP(1.4.12) are given respectively. In Section 5, existence results for BVP(1.4.13)–BVP(1.4.15) are proved. Some remarks are given in Section 6 to illustrate some unsuitable results in known papers by wrong concepts and careless computations. Examples are presented in Section 7. Possible trends of researches are given at the end of the paper.

2. Related definitions and preliminary results

For the convenience of the readers, we shall state the necessary definitions from fractional calculus theory.

For $\phi\in L^1(0,1)$, denote $\|\phi\|_1=\int_0^1|\phi(s)|ds$. Let the Gamma, beta functions and Mittag-Leffler function ($\Gamma(\alpha)(\alpha>0)$, $\mathbf{B}(a,b)$ $(a>0,\ b>0)$ and $\mathbf{E}_{\alpha,\beta}(x)$) be defined by

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx,\quad \mathbf{B}(a,b)=\int_0^1x^{a-1}(1-x)^{b-1}dx,\quad \mathbf{E}_{\alpha,\beta}(x)=\sum_{i=0}^{\infty}\frac{x^i}{\Gamma(\alpha i+\beta)}.$$

DEFINITION 2.1. ([65]) The left *Riemann-Liouville fractional integral of order* $\alpha>0$ of a function $g:(0,\infty)\to\mathbb{R}$ is given by $I_{0+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds$, $t>0$ provided that the right-hand side exists.

DEFINITION 2.2. ([65]) (i) Let $a>0$. The left *Riemann-Liouville fractional derivative of order* $\alpha>0$ of a function $g:(0,a]\to\mathbb{R}$ is given by

$$D_{0^+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{g(s)}{(t-s)^{\alpha-n+1}}ds,\quad t\in(0,a]$$

where $n-1<\alpha<n$, provided that the right-hand side exists.

(ii) The Caputo fractional derivative of order $\alpha>0$ of a function $g:(0,a]\to\mathbb{R}$ with $g\in AC^n(0,a]$ is defined by

$$^cD_{0^+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha<n$, provided that the right-hand side exists.

DEFINITION 2.4. Set $k>-1$ and $l\in(-1,0]$. We say $K:(0,1)\times\mathbb{R}\to\mathbb{R}$ is a I-Carathéodory function if it satisfies the followings:

(i) $t\to K\left(t,(t-t_s)^{\alpha-2}x,(t-t_s)^{\alpha-p-2}y\right)$ is integral on $(t_s,t_{s+1}](s\in\mathbb{N}_0^m)$ for every $x,y\in\mathbb{R}$,

(ii) $(x,y)\to K\left(t,(t-t_s)^{\alpha-2}x,(t-t_s)^{\alpha-p-2}y\right)$ is continuous on $\mathbb{R}^2$ for all $t\in(t_s,t_{s+1}]$ $(s\in\mathbb{N}_0^m)$;

(iii) for each $r>0$ there exists a constant $A_{r,f}\geqslant 0$ satisfying

$$\left|K\left(t,(t-t_s)^{\alpha-2}x,(t-t_s)^{\alpha-p-2}y\right)\right|\leqslant A_{r,K}t^k(1-t)^l$$

holds for $t\in(t_s,t_{s+1}]$, $s\in\mathbb{N}_0^m$, $|x|,|y|\leqslant r$.

DEFINITION 2.5. Set $k>-1$ and $l\in(-1,0]$. We say $K:(0,1)\times\mathbb{R}\to\mathbb{R}$ is a II-Carathéodory function if it satisfies the followings:

(i) $t\to K(t,x,y)$ is integral on $(t_s,t_{s+1})(s\in\mathbb{N}_0^m)$ for every $x,y\in\mathbb{R}$,

(ii) $(x,y)\to K(t,x,y)$ is continuous on $\mathbb{R}^2$ for all $t\in(t_s,t_{s+1})$ $(s\in\mathbb{N}_0^m)$;

(iii) for each $r>0$ there exists a constant $A_{r,f}\geqslant 0$ satisfying

$$|K(t,x,y)|\leqslant A_{r,K}(t-t_s)^k(t_{s+1}-t)^l$$

holds for $t\in(t_s,t_{s+1})$, $s\in\mathbb{N}_0^m$, $|x|,|y|\leqslant r$.

DEFINITION 2.6. Set $k>-1$ and $l\in(-1,0]$. We say $K:(0,1)\times\mathbb{R}\to\mathbb{R}$ is a III-Carathéodory function if it satisfies the followings:

(i) $t\to K\left(t,(t-t_s)^{\beta-1}x,(t-t_s)^{\beta-q-1}y\right)$ is integral on $(t_s,t_{s+1}](s\in\mathbb{N}_0^m)$ for every $x,y\in\mathbb{R}$,

(ii) $(x,y)\to K\left(t,(t-t_s)^{\beta-1}x,(t-t_s)^{\beta-q-1}y\right)$ is continuous on $\mathbb{R}^2$ for all $t\in(t_s,t_{s+1}]$ $(s\in\mathbb{N}_0^m)$;

(iii) for each $r>0$ there exists a constant $A_{r,f}\geqslant 0$ satisfying

$$\left|K\left(t,(t-t_s)^{\beta-1}x,(t-t_s)^{\beta-q-1}y\right)\right|\leqslant A_{r,K}t^k(1-t)^l$$

holds for $t\in(t_s,t_{s+1}]$, $s\in\mathbb{N}_0^m$, $|x|,|y|\leqslant r$.

DEFINITION 2.7. $G:\{t_s:s\in\mathbb{N}_1^m\}\times\mathbb{R}\to\mathbb{R}$ is called a discrete I-Carathéodory function if

(i) $(x,y)\to G\left(t_s,(t_s-t_{s-1})^{\alpha-2}x,(t_s-t_{s-1})^{\alpha-p-2}y\right)$ is continuous on $\mathbb{R}^2$ for each $s\in\mathbb{N}_1^m$,

(ii) for each $r>0$ there exists $A_{r,G,s}\geqslant 0$ such that

$$\left|G\left(t_s,(t_s-t_{s-1})^{\alpha-2}x,(t_s-t_{s-1})^{\alpha-p-2}y\right)\right|\leqslant A_{r,G,s}$$

holds for $|x|,|y|\leqslant r$, $s\in\mathbb{N}_1^m$.

DEFINITION 2.8. $G:\{t_s:s\in\mathbb{N}_1^m\}\times\mathbb{R}\to\mathbb{R}$ is called a discrete II-Carathéodory function if

(i) $(x,y)\to G(t_s,x,y)$ is continuous on $\mathbb{R}^2$ for each $s\in\mathbb{N}_1^m$,

(ii) for each $r>0$ there exists $A_{r,G,s}\geqslant 0$ such that

$$|G(t_s,x,y)|\leqslant A_{r,G,s}$$

holds for $|x|,|y|\leqslant r$, $s\in\mathbb{N}_1^m$.

DEFINITION 2.9. $G:\{t_s:s\in\mathbb{N}_1^m\}\times\mathbb{R}\to\mathbb{R}$ is called a discrete III-Carathéodory function if

(i) $(x,y)\to G\left(t_s,(t_s-t_{s-1})^{\beta-1}x,(t_s-t_{s-1})^{\beta-q-1}y\right)$ is continuous on $\mathbb{R}^2$ for each $s\in\mathbb{N}_1^m$,

(ii) for each $r>0$ there exists $A_{r,G,s}\geqslant 0$ such that

$$\left|G\left(t_s,(t_s-t_{s-1})^{\beta-1}x,(t_s-t_{s-1})^{\beta-q-1}y\right)\right|\leqslant A_{r,G,s}$$

holds for $|x|,|y|\leqslant r$, $s\in\mathbb{N}_1^m$.

DEFINITION 2.10. ([60]) Let E and Z be Banach spaces. $L:D(L)\subset E\to Z$ is called a Fredholm operator of index zero if $\mathrm{Im}L$ is closed in E and $\dim\mathrm{Ker}L=\mathrm{co}\dim\mathrm{Im}L<+\infty$.

It is easy to see that if L is a Fredholm operator of index zero, then there exist the projectors $P:E\to E$, and $Q:Z\to Z$ such that

$$\mathrm{Im}\,P=\mathrm{Ker}\,L,\ \mathrm{Ker}\,Q=\mathrm{Im}\,L,\ X=\mathrm{Ker}\,L\oplus\mathrm{Ker}\,P,\ Y=\mathrm{Im}\,L\oplus\mathrm{Im}\,Q.$$

If $L:D(L)\subset E\to Z$ is called a Fredholm operator of index zero, the inverse of

$$L|_{D(L)\cap\mathrm{Ker}\,P}:D(L)\cap\mathrm{Ker}\,P\to\mathrm{Im}\,L$$

is denoted by K_p.

DEFINITION 2.11. ([60]) Suppose that $L : D(L) \subset E \to Z$ is a Fredholm operator of index zero. For nonempty open bounded subset Ω of E satisfying $D(L)\cap\overline{\Omega} \neq \emptyset$, the continuous map $N : \overline{\Omega} \to Z$ is called L-compact if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N(\overline{\Omega})$ is bounded and relatively compact.

To obtain the main results, we need the Leray-Schauder nonlinear alternative.

LEMMA 2.1. ([60]) *Let X be a Banach space and $T : X \to X$ be a completely continuous operator. Suppose $\overline{\Omega}$ is a nonempty open closed subset of X of homeomorphism to a unit ball and $T(\overline{\Omega}) \subseteq \overline{\Omega}$. Then T has fixed point $x \in \overline{\Omega}$.*

LEMMA 2.2. ([60]) *Let L be a Fredholm operator of index zero and N be L-compact on each open nonempty set Ω centered at zero. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [D(L)\setminus KerL)\cap\partial\Omega]\times(0,1)$;

(ii) $Nx \notin ImL$ for every $x \in KerL\cap\partial\Omega$;

(iii) $deg(\wedge^{-1}QN\big|_{KerL}, \Omega\cap KerL, 0) \neq 0$, where $\wedge^{-1} : Y/ImL \to KerL$ is the inverse of the isomorphism $\wedge : KerL \to Y/ImL$.

Then the equation $Lx = Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$.

LEMMA 2.3. ([15, 42]) *Suppose that $\alpha \in [n-1,n)$, $h \in L^1(a,b)\bigcap C^0(a,b]$. Then x is a solution of $D^{\alpha}_{a^+}x(t) - \lambda x(t) = h(t)$, $t \in (a,b]$ if and only if there exist constants $c_{v0} \in \mathbb{R}$ such that*

$$x(t) = \sum_{v=1}^{n} c_{v0}(t-a)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-a)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds,$$
$$t \in (a,b].$$

COROLLARY 2.1. *Suppose that $\alpha \in [n-1,n)$, $h \in L^1(a,b)\bigcap C^0(a,b]$. Then x is a solution of $D^{\alpha}_{a^+}x(t) = h(t)$, a.e., $t \in (a,b]$ if and only if there exist constants $c_{v0} \in \mathbb{R}$ such that*

$$x(t) = \sum_{v=1}^{n} \frac{c_{v0}}{\Gamma(\alpha-v+1)}(t-a)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad t \in (a,b].$$

Proof. Choose $\lambda = 0$ in Lemma 2.3, we get this corollary. □

3. Exact piecewise continuous solutions of LFDEs

In this section, we establish explicit piecewise continuous solutions of (1.4.1), (1.4.2), (1.4.3) and (1.4.4) respectively.

3.1. Exact solution of (1.4.1)

We firstly give exact expression of solution of (1.4.1) by using Lemma 2.3 and Corollary 2.1.

THEOREM 3.1.1. *Suppose* $h_1 \in L^1(0,1)\cap C(0,1)$ *and there exist constants* $k > -1$, $l \in (-1,0]$ *such that* $|h_1(t)| \leqslant t^l(1-t)^l$. *Then* x *is a piecewise continuous solution of (1.4.1) if and only if there exist constants* $c_{vj} \in \mathbb{R}$ $(v \in \mathbb{N}_1^n, j \in \mathbb{N}_0^m)$ *such that*

$$x(t) = \sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}(t-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_j)^{\alpha}) + \int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h_1(s)ds, \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{3.1.1}$$

Proof. Step 1. We prove that x satisfies (3.1.1) if x is a piecewise continuous solution of (1.4.1).

We note $h \in L(0,t_1)\cap C(0,t_1]$. By Lemma 2.3 and (1.4.1), we know that there exist constants $c_{v0} \in \mathbb{R}$ such that

$$x(t) = \sum_{v=1}^{n} c_{v0}t^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda t^{\alpha}) + \int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h_1(s)ds, \quad t \in (0,t_1].$$

Then (3.1.1) holds for $i=0$. Now we suppose that (3.1.1) holds for $i = 0,1,2,\cdots,j$, i.e.,

$$x(t) = \sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}(t-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_j)^{\alpha}) + \int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h_1(s)ds, \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^j. \tag{3.1.2}$$

We prove that (3.1.1) holds for $i = j+1$. By mathematical induction method, we know that (3.1.1) holds for all $i \in \mathbb{N}_0^m$. Suppose that

$$x(t) = \Phi(t) + \sum_{\sigma=0}^{j}\sum_{v=1}^{n} c_{vj}(t-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_j)^{\alpha}) + \int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h_1(s)ds, \quad t \in (t_{j+1},t_{j+2}]. \tag{3.1.3}$$

Using (3.1.2), we know for $t \in (t_{j+1},t_{j+2}]$ that

$$\begin{aligned}
&h_1(t)+\lambda x(t) = D_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left[\int_0^t(t-s)^{n-\alpha-1}x(s)ds\right]^{(n)}\\
&= \frac{\left[\sum_{\tau=0}^{j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}x(s)ds\right]^{(n)} + \left[\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}x(s)ds\right]^{(n)}}{\Gamma(n-\alpha)}\\
&= \frac{\left[\sum_{\tau=0}^{j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum_{\sigma=0}^{\tau}\sum_{v=1}^{n}c_{v\sigma}(s-t_\sigma)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^{\alpha})+\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)du\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}\\
&+\frac{\left[\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}\left(\Phi(s)+\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{v\sigma}(s-t_\sigma)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^{\alpha})+\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)du\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}\\
&= D_{t_{j+1}^+}^{\alpha}\Phi(t) + \frac{\left[\sum_{\tau=0}^{j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{\tau}\sum_{v=1}^{n}c_{v\sigma}(s-t_\sigma)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^{\alpha})ds\right]^{(n)}}{\Gamma(n-\alpha)}
\end{aligned}$$

$$+\frac{\left[\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{v\sigma}(s-t_\sigma)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^{\alpha})ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)duds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$=D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\frac{\left[\sum_{\tau=0}^{j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{\tau}\sum_{v=1}^{n}c_{v\sigma}(s-t_\sigma)^{\alpha-v}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}(s-t_\sigma)^{\alpha\omega}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{v\sigma}(s-t_\sigma)^{\alpha-v}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}(s-t_\sigma)^{\alpha\omega}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-u)^{\alpha-1}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}(s-u)^{\alpha\omega}h_1(u)duds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$=D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\frac{\left[\sum_{\tau=0}^{j}\sum_{\sigma=0}^{\tau}\sum_{v=1}^{n}c_{v\sigma}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_\sigma)^{\alpha(\omega+1)-v}ds\right]^{(n)}}{\Gamma(n-\alpha)}\text{ use }w=\frac{s-t_\sigma}{t-t_\sigma}$$

$$+\frac{\left[\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{v\sigma}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}(s-t_\sigma)^{\alpha(\omega+1)-v}ds\right]^{(n)}}{\Gamma(n-\alpha)}\text{ use }w=\frac{s-t_\sigma}{t-t_\sigma}$$

$$+\frac{\left[\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-u)^{\alpha(\omega+1)-1}h_1(u)duds\right]^{(n)}}{\Gamma(n-\alpha)}\text{ re-order the integrals}$$

$$=D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\frac{\left[\sum_{\tau=0}^{j}\sum_{\sigma=0}^{\tau}\sum_{v=1}^{n}c_{v\sigma}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}(t-t_\sigma)^{n+\alpha\omega-v}\int_{\frac{t_\tau-t_\sigma}{t-t_\sigma}}^{\frac{t_{\tau+1}-t_\sigma}{t-t_\sigma}}(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-v}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

re-order the sums of τ,σ

$$+\frac{\left[\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{v\sigma}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}(t-t_\sigma)^{n+\alpha\omega-v}\int_{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-v}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t\int_s^t(t-s)^{n-\alpha-1}(s-u)^{\alpha(\omega+1)-1}dsh_1(u)du\right]^{(n)}}{\Gamma(n-\alpha)}\text{ use }w=\frac{s-u}{t-u}$$

$$=D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\frac{\left[\sum_{\sigma=0}^{j}\sum_{\tau=\sigma}^{j}\sum_{v=1}^{n}c_{v\sigma}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}(t-t_\sigma)^{n+\alpha\omega-v}\int_{\frac{t_\tau-t_\sigma}{t-t_\sigma}}^{\frac{t_{\tau+1}-t_\sigma}{t-t_\sigma}}(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-v}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{\sigma=0}^{j}\sum_{v=1}^{n}c_{v\sigma}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}(t-t_\sigma)^{n+\alpha\omega-v}\int_{\frac{t_{j+1}-t_\sigma}{t-t_\sigma}}^{1}(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-v}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-u)^{n+\alpha\omega-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-1}dwh_1(u)du\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= D^{\alpha}_{t^+_{j+1}}\Phi(t) + \frac{\left[\sum\limits_{\sigma=0}^{j}\sum\limits_{v=1}^{n} c_{v\sigma}\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}(t-t_\sigma)^{n+\alpha\omega-v}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-v}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-u)^{n+\alpha\omega-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-1}dwh_1(u)du\right]^{(n)}}{\Gamma(n-\alpha)} \quad \text{use} \quad \mathbf{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$= D^{\alpha}_{t^+_{j+1}}\Phi(t) + \left[\sum_{\sigma=0}^{j}\sum_{v=1}^{n} c_{v\sigma}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(n+\alpha\omega-v+1)}(t-t_\sigma)^{n+\alpha\omega-v}\right]^{(n)}$$

$$+\left[\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+n))}\int_0^t(t-u)^{n+\alpha\omega-1}h_1(u)du\right]^{(n)}$$

$$= \left[\sum_{\sigma=0}^{j}\sum_{v=1}^{n} c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(n+\alpha\omega-v+1)}(t-t_\sigma)^{n+\alpha\omega-v}+\sum_{\sigma=0}^{j}\sum_{\tau=\sigma}^{j}\sum_{v=1}^{n}c_{v\sigma}\frac{1}{\Gamma(n-v+1)}(t-t_\sigma)^{n-v}\right]^{(n)}$$

$$+\left[\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+n))}\int_0^t(t-u)^{n+\alpha\omega-1}h_1(u)du+\frac{1}{\Gamma(n)}\int_0^t(t-u)^{n-1}h_1(u)du\right]^{(n)}+D^{\alpha}_{t^+_{j+1}}\Phi(t)$$

$$= \sum_{\sigma=0}^{j}\sum_{v=1}^{n} c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega-v+1)}(t-t_\sigma)^{\alpha\omega-v}+\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega))}\int_0^t(t-u)^{\alpha\omega-1}h_1(u)du$$
$$+h_1(t)+D^{\alpha}_{t^+_{j+1}}\Phi(t)$$

$$= \lambda\left[\sum_{\sigma=0}^{j}\sum_{v=1}^{n} c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega-1}}{\Gamma(\alpha\omega-v+1)}(t-t_\sigma)^{\alpha\omega-v}+\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega-1}}{\Gamma(\alpha\omega))}\int_0^t(t-u)^{\alpha\omega-1}h_1(u)du\right]$$
$$+h_1(t)+D^{\alpha}_{t^+_{j+1}}\Phi(t)$$

$$= \lambda\left[\sum_{\sigma=0}^{j}\sum_{v=1}^{n} c_{v\sigma}(t-t_\sigma)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_\sigma)^{\alpha})+\int_0^t(t-u)^{\alpha-1}\mathbf{E}\alpha,\alpha(\lambda(t-s)^{\alpha})h_1(u)du\right]$$
$$+h_1(t)+D^{\alpha}_{t^+_{j+1}}\Phi(t)$$
$$= \lambda\left[x(t)-\Phi(t)\right]+h_1(t)+D^{\alpha}_{t^+_{j+1}}\Phi(t), \quad t\in(t_{j+1},t_{j+2}].$$

It follows that $D^{\alpha}_{t^+_{j+1}}\Phi(t)-\lambda\Phi(t)=0$ on $(t_{j+1},t_{j+2}]$. By Lemma 2.3, we know that there exists constants c_{vj+1} $(v\in\mathbb{N}_1^n)$ such that

$$\Phi(t)=\sum_{v=1}^{n}c_{vj+1}(t-t_{j+1})^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(t-t_{j+1})^{\alpha}).$$

Substituting Φ into (3.1.3). We know that (3.1.1) holds for $i=j+1$. By mathematical induction method, we know that (3.1.1) holds for $i\in\mathbb{N}_0^m$.

Step 2. We prove that x is a piecewise continuous solution of (1.4.1) if x satisfies (3.1.1).

Since x satisfies (3.1.1), we know that $x|_{(t_i,t_{i+1}]}\in C(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m)$ and $\lim\limits_{t\to t_i^+}(t-t_i)^{n-\alpha}x(t)$ exists and is finite for all $i\in\mathbb{N}_0^m$. We will prove that $D^{\alpha}_{0^+}x(t)-\lambda x=h_1(t)$ for all $t\in(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m)$.

In fact, we have for $t \in (t_i, t_{i+1}]$ that

$$D_{0^+}^{\alpha} x(t) = \frac{1}{(n-\alpha)} \left[\sum_{\tau=0}^{i-1} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} x(s)ds + \int_{t_i}^{t} (t-s)^{n-\alpha-1} x(s)ds \right]^{(n)}$$

$$= \frac{\left[\sum_{\tau=0}^{i-1} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} \left(\sum_{\sigma=0}^{\tau} \sum_{v=1}^{n} c_{v\sigma}(s-t_\sigma)^{\alpha-v} \mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha) + \int_0^s (s-v)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) h_1(u)du \right) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_{t_i}^{t} (t-s)^{n-\alpha-1} \left(\sum_{\sigma=0}^{i} \sum_{v=1}^{n} c_{v\sigma}(s-t_\sigma)^{\alpha-v} \mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha) + \int_0^s (s-v)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) h_1(u)du \right) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum_{\tau=0}^{i-1} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} \sum_{\sigma=0}^{\tau} \sum_{v=1}^{n} c_{v\sigma}(s-t_\sigma)^{\alpha-v} \mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_{t_i}^{t} (t-s)^{n-\alpha-1} \sum_{\sigma=0}^{i} \sum_{v=1}^{n} c_{v\sigma}(s-t_\sigma)^{\alpha-v} \mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_\sigma)^\alpha) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_0^t (t-s)^{n-\alpha-1} \int_0^s (s-v)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) h_1(u)duds\right]^{(n)}}{\Gamma(n-\alpha)}.$$

Similarly to Step 1 we can get that

$$D_{0^+}^{\alpha} x(t) = h_1(t) + \lambda x(t), \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m.$$

So x is a piecewise continuous solution of (1.4.1). The proof is complete. □

COROLLARY 3.1.1. *In Theorem 3.1.1, when choose $\lambda = 0$, we get that x is a piecewise continuous solution of equation*

$$D_{0^+}^{\alpha} x(t) = h(t), \quad t \in (t_i, t_{i+1}] \quad (i \in \mathbb{N}_0^m)$$

if and only if there exist constants $c_{vj} \in \mathbb{R}$ $(v \in \mathbb{N}_1^n, j \in \mathbb{N}_0^m)$ such that

$$x(t) = \sum_{j=0}^{i} \sum_{v=1}^{n} \frac{c_{vj}}{\Gamma(\alpha-v+1)} (t-t_j)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m.$$

COROLLARY 3.1.2. *Suppose x is a piecewise continuous solution defined by (3.1.1). Then*

$$I_{0^+}^{n-\alpha} x(t) = \sum_{j=0}^{i} \sum_{v=1}^{n} c_{vj}(t-t_j)^{n-v} \mathbf{E}_{\alpha,n-v+1}(\lambda(t-t_j)^\alpha)$$
$$+ \int_0^t (t-s)^{n-1} \mathbf{E}_{\alpha,n}(\lambda(t-s)^\alpha) h_1(s)ds, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m$$

and

$$D_{0^+}^{\alpha-\sigma} x(t) = \sum_{j=0}^{i} \sum_{v=1}^{\sigma} c_{vj}(t-t_j)^{\sigma-v} \mathbf{E}_{\alpha,\sigma-v+1}(\lambda(t-t_j)^\alpha)$$

$$+\lambda\sum_{j=0}^{i}\sum_{v=\sigma+1}^{n}c_{vj}(t-t_j)^{\alpha+\sigma-v}\mathbf{E}_{\alpha,\alpha+\sigma-v+1}(\lambda(t-t_j)^{\alpha})$$
$$+\int_0^t(t-u)^{\sigma-1}\mathbf{E}_{\alpha,\sigma}(\lambda(t-u)^{\alpha})h_1(u)du,\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\ \sigma\in\mathbb{N}_1^{n-1}.$$

Proof. Firstly by Definition 2.1, we have for $t\in(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m)$ that

$$I_{0^+}^{n-\alpha}x(t)=\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}x(s)ds=\frac{\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}x(s)ds+\int_{t_i}^{t}(t-s)^{n-\alpha-1}x(s)ds}{\Gamma(n-\alpha)}$$

$$=\frac{\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})+\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)du\right)ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_{t_i}^{t}(t-s)^{n-\alpha-1}\left(\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})+\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)du\right)ds}{\Gamma(n-\alpha)}$$

$$=\frac{\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_{t_i}^{t}(t-s)^{n-\alpha-1}\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)duds}{\Gamma(n-\alpha)}$$

$$=\frac{\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha\omega+\alpha-v}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{t_i}^{t}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-v}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t\int_s^t(t-s)^{n-\alpha-1}(s-u)^{\alpha\omega+\alpha-1}h_1(u)duds}{\Gamma(n-\alpha)}$$

$$=\frac{\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{\frac{t_\tau-t_j}{t-t_j}}^{\frac{t_{\tau+1}-t_j}{t-t_j}}(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-v}dw}{\Gamma(n-\alpha)}$$
$$+\frac{\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{\frac{t_i-t_j}{t-t_j}}^{1}(1-w)^{n-\alpha-1}w^{\alpha-v}dw}{\Gamma(n-\alpha)}$$
$$+\frac{\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-u)^{\alpha\omega+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-1}dwh_1(u)du}{\Gamma(n-\alpha)}$$

$$=\frac{\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{\frac{t_\tau-t_j}{t-t_j}}^{\frac{t_{\tau+1}-t_j}{t-t_j}}(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-v}dw}{\Gamma(n-\alpha)}$$
$$+\frac{\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{\frac{t_i-t_j}{t-t_j}}^{1}(1-w)^{n-\alpha-1}w^{\alpha-v}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-u)^{\alpha\omega+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-1}dwh_1(u)du}{\Gamma(n-\alpha)}$$

$$=\frac{\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-v}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-u)^{\alpha\omega+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-1}dwh_1(u)du}{\Gamma(n-\alpha)}$$

$$=\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+n-v+1)}+\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+n)}\int_0^t(t-u)^{\alpha\omega+n-1}h_1(u)du$$

$$=\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(t-t_j)^{n-v}\mathbf{E}_{\alpha,n-v+1}(\lambda(t-t_j)^{\alpha})$$

$$+\int_0^t(t-s)^{n-1}\mathbf{E}_{\alpha,n}(\lambda(t-s)^{\alpha})h_1(s)ds,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m$$

Secondly for $\sigma\in\mathbb{N}_1^{n-1}$ and $t\in(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m)$, by Definition 2.2, we have

$$D_{0^+}^{\alpha-\sigma}x(t)=\frac{\left[\int_0^t(t-s)^{n-\alpha-1}x(s)ds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}x(s)ds+\int_{t_i}^t(t-s)^{n-\alpha-1}x(s)ds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})+\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)du\right)ds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_{t_i}^t(t-s)^{n-\alpha-1}\left(\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})+\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)du\right)ds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})ds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_{t_i}^t(t-s)^{n-\alpha-1}\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})ds+\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h_1(u)duds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{vj}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})ds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}\int_{t_i}^t(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-v}\mathbf{E}_{\alpha,\alpha-v+1}(\lambda(s-t_j)^{\alpha})ds+\int_0^t\int_s^t(t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})dsh_1(u)du\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha\omega+\alpha-v}ds\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{t_i}^t(t-s)^{n-\alpha-1}(s-t_j)^{\alpha\omega+\alpha-v}ds+\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t\int_s^t(t-s)^{n-\alpha-1}(s-u)^{\alpha\omega+\alpha-1}dsh_1(u)du\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}\sum_{v=1}^{n} c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{\frac{t_\tau-t_j}{t-t_j}}^{\frac{t_{\tau+1}-t_j}{t-t_j}}(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-v}dw\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$
$$+\frac{\left[\sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_{\frac{t_i-t_j}{t-t_j}}^{1}(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-v}dw\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$
$$+\frac{\left[\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-u)^{\alpha\omega+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-1}dwh_1(u)du\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$
$$= \frac{\left[\sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+\alpha-v+1)}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-v}dw\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$
$$+\frac{\left[\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1))}\int_0^t(t-u)^{\alpha\omega+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega+\alpha-1}dwh_1(u)du\right]^{(n-\sigma)}}{\Gamma(n-\alpha)}$$
$$= \left[\sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+n-v}}{\Gamma(\alpha\omega+n-v+1)}\right]^{(n-\sigma)} + \left[\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+n)}\int_0^t(t-u)^{\alpha\omega+n-1}h_1(u)du\right]^{(n-\sigma)}$$
$$= \left[\sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}\frac{(t-t_j)^{n-v}}{\Gamma(n-v+1)}\right]^{(n-\sigma)} + \sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+\sigma-v}}{\Gamma(\alpha\omega+\sigma-v+1)}$$
$$+\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\sigma)}\int_0^t(t-u)^{\alpha\omega+\sigma-1}h_1(u)du$$
$$= \left[\sum_{j=0}^{i}\sum_{v=1}^{\sigma} c_{vj}\frac{(t-t_j)^{n-v}}{\Gamma(n-v+1)} + \sum_{j=0}^{i}\sum_{v=\sigma+1}^{n} c_{vj}\frac{(t-t_j)^{n-v}}{\Gamma(n-v+1)}\right]^{(n-\sigma)}$$
$$+\sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+\sigma-v}}{\Gamma(\alpha\omega+\sigma-v+1)} + \sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\sigma)}\int_0^t(t-u)^{\alpha\omega+\sigma-1}h_1(u)du$$
$$= \sum_{j=0}^{i}\sum_{v=1}^{\sigma} c_{vj}\frac{(t-t_j)^{\sigma-v}}{\Gamma(\sigma-v+1)} + \sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}(t-t_j)^{\alpha\omega+\sigma-v}}{\Gamma(\alpha\omega+\sigma-v+1)}$$
$$+\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+\sigma)}\int_0^t(t-u)^{\alpha\omega+\sigma-1}h_1(u)du$$
$$= \sum_{j=0}^{i}\sum_{v=1}^{\sigma} c_{vj}(t-t_j)^{\sigma-v}\mathbf{E}_{\alpha,\sigma-v+1}(\lambda(t-t_j)^{\alpha})$$
$$+\lambda\sum_{j=0}^{i}\sum_{v=\sigma+1}^{n} c_{vj}(t-t_j)^{\alpha+\sigma-v}\mathbf{E}_{\alpha,\alpha+\sigma-v+1}(\lambda(t-t_j)^{\alpha})$$
$$+\int_0^t(t-u)^{\sigma-1}\mathbf{E}_{\alpha,\sigma}(\lambda(t-u)^{\alpha})h_1(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m, \quad \sigma\in\mathbb{N}_1^{n-1}.$$

Thirdly, we can get

$$D_{0^+}^p x(t) = \sum_{j=0}^{i}\sum_{v=1}^{n} c_{vj}(t-t_j)^{\alpha-p-v}\mathbf{E}_{\alpha,\alpha-p-v+1}(\lambda(t-t_j)^{\alpha})$$

$$+\int_0^t (t-s)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-s)^\alpha)h_1(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.$$

The proof is complete. □

3.2. Exact solution of (1.4.2)

To get explicit piecewise continuous solutions of (1.4.2), fix $\eta_i\in\mathbb{R}$ $(i\in\mathbb{N}_0^{n-1})$ and $h\in C(0,1)$ satisfying $|h(t)|\leqslant t^k(1-t)^l$ for all $t\in(0,1)$ with $k>-1$, $l\in(-1,0]$, we choose the following Picard function sequence:

$$\phi_0(t)=\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^v+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad t\in[0,1],$$

$$\phi_i(t)=\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-1}(s)ds, \quad t\in[0,1], \quad i=1,2,\cdots.$$

LEMMA 3.2.1. *ϕ_i is continuous on $[0,1]$.*

Proof. We have

$$\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right|\leqslant\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^lds\leqslant t^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^kds$$
$$=t^{\alpha+k+l}\int_0^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^kdw=t^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

One sees easily that ϕ_0 is continuous on $[0,1]$. It follows that ϕ_1 is continuous on $[0,1]$. By mathematical induction method, we can prove that ϕ_i is continuous on $[0,1]$. □

LEMMA 3.2.2. *ϕ_i is convergent uniformly on $[0,1]$.*

Proof. In fact, similarly to Lemma 3.2.1, we have for $t\in[0,1]$ that

$$|\phi_1(t)-\phi_0(t)|=\left|\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_0(s)ds\right|\leqslant|\lambda|\frac{||\phi_0||}{\Gamma(\alpha+1)}t^\alpha.$$

So

$$|\phi_2(t)-\phi_1(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda[\phi_1(s)-\phi_0(s)]ds\right|$$
$$\leqslant|\lambda|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(|\lambda|||\phi_0||\frac{s^\alpha}{\Gamma(\alpha+1)}\right)ds=|\lambda|^2\frac{||\phi_0||}{\Gamma(\alpha+1)}\frac{\mathbf{B}(\alpha,\alpha+1)}{\Gamma(\alpha)}t^{2\alpha}.$$

By using the mathematical induction method, we get for every $i=1,2,\cdots$ that

$$|\phi_{i+1}(t)-\phi_i(t)|\leqslant|\lambda|^{i+1}\frac{||\phi_0||}{\Gamma(\alpha+1)}\left(\prod_{j=1}^i\frac{\mathbf{B}(\alpha,j\alpha+1)}{\Gamma(\alpha)}\right)t^{(i+1)\alpha}, \quad t\in[0,1].$$

It follows that

$$|\phi_{i+1}(t)-\phi_i(t)|\leqslant|\lambda|^{i+1}\frac{||\phi_0||}{\Gamma(\alpha+1)}\prod_{j=1}^i\frac{\mathbf{B}(\alpha,j\alpha+1)}{\Gamma(\alpha)}, \quad t\in[0,1].$$

Consider

$$\sum_{i=1}^{+\infty} u_i =: \sum_{i=1}^{+\infty} |\lambda|^{i+1} \frac{||\phi_0||}{\Gamma(\alpha+1)} \prod_{j=1}^{i} \frac{\mathbf{B}(\alpha, j\alpha+1)}{\Gamma(\alpha)}.$$

One sees for sufficiently large i that

$$\begin{aligned}
\frac{u_{i+1}}{u_i} &= |\lambda| \frac{\mathbf{B}(\alpha,(i+1)\alpha+1)}{\Gamma(\alpha)} = |\lambda| \int_0^1 (1-x)^{\alpha-1} x^{(i+1)\alpha} dx \\
&\leqslant |\lambda| \int_0^{\delta} (1-x)^{\alpha-1} x^{(i+1)\alpha} dx + |\lambda| \int_{\delta}^1 (1-x)^{\alpha-1} dx \quad \text{with} \quad \delta \in (0,1) \\
&\leqslant |\lambda| \int_0^{\delta} (1-x)^{\alpha-1} dx \delta^{(i+1)\alpha} + \frac{|\lambda|}{\alpha} \delta^{\alpha} \leqslant \frac{|\lambda|(1-(1-\delta)^{\alpha})}{\alpha} \delta^{(i+1)\alpha} + \frac{|\lambda|}{\alpha} \delta^{\alpha}.
\end{aligned}$$

For any $\varepsilon > 0$, it is easy to see that there exists $\delta \in (0,1)$ such that $\frac{|\lambda|}{\alpha}\delta^{\alpha} < \frac{\varepsilon}{2}$. For this δ, there exists an integer $N > 0$ sufficiently large such that $\frac{|\lambda|(1-(1-\delta)^{\alpha}}{\alpha}\delta^{(i+1)\alpha} < \frac{\varepsilon}{2}$ for all $i > N$. So $0 < \frac{u_{i+1}}{u_i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $i > N$. It follows that $\lim_{i \to +\infty} \frac{u_{i+1}}{u_i} = 0$. Then $\sum_{i=1}^{+\infty} u_i$ is convergent. Similarly we get $\sum_{i=1}^{+\infty} v_i$ is convergent. Hence dominant control convergence theorem implies that

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_i(t) - \phi_{i-1}(t)] + \cdots, \quad t \in [0,1]$$

is uniformly convergent. Then ϕ_i is convergent uniformly on $[0,1]$. □

LEMMA 3.2.3. *$\phi(t) = \lim_{i \to +\infty} \phi_i(t)$ defined on $[0,1]$ is a unique continuous solution of the integral equation*

$$x(t) = \sum_{v=0}^{n-1} \frac{\eta_v}{\Gamma(v+1)} t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\lambda x(s) + h(s)] ds, \quad t \in [0,1]. \tag{3.2.1}$$

Proof. By Lemma 3.2.1 and Lemma 3.2.2, we have ϕ_i is convergent uniformly on $[0,1]$ to a function. Suppose that $\phi(t) = \lim_{i \to +\infty} \phi_i(t)$. We see $\phi(t)$ is continuous on $[0,1]$. We have from Lemma 3.1.2 that

$$\begin{aligned}
\phi(t) &= \lim_{i \to +\infty} \phi_i(t) = \lim_{i \to +\infty} \left[\sum_{v=0}^{n-1} \frac{\eta_v}{\Gamma(v+1)} t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\lambda \phi_{i-1}(s) + h(s)] ds \right] \\
&= \sum_{v=0}^{n-1} \frac{\eta_v}{\Gamma(v+1)} t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\lambda \lim_{i \to +\infty} \phi_{i-1}(s) + h(s) \right] ds \\
&= \sum_{v=0}^{n-1} \frac{\eta_v}{\Gamma(v+1)} t^v + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\lambda \phi(s) + h(s)] ds.
\end{aligned}$$

Then ϕ is a continuous solution of (3.2.1) defined on $[0,1]$.

Suppose that ψ continuous $[0,1]$ is also a solution of (3.2.1). Then

$$\psi(t) = \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha-v+1)} t^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\lambda \psi(s) + h(s)] ds, \quad t \in [0,1].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $[0,1]$. Similarly to Lemma 3.2.2, we get

$$|\psi(t)-\phi_0(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\lambda\psi_0(s)+h(s)]ds\right| \leqslant |\lambda|||\phi_0||\frac{t^\alpha}{\Gamma(\alpha+1)} + t^{\alpha+k+l}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}.$$

Similarly to Lemma 3.2.2, by the mathematical induction method, we get

$$|\psi(t)-\phi_i(t)| \leqslant |\lambda|^{i+1}\frac{||\phi_0||}{\Gamma(\alpha+1)}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha,j\alpha+1)}{\Gamma(\alpha)} + |\lambda|^i\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha,j\alpha+k+l+1)}{\Gamma(\alpha)},\ t\in[0,1].$$

Then $\lim_{i\to+\infty}\phi_i(t) = \psi(t)$ uniformly on $[0,1]$. Then $\phi(t)\equiv\psi(t)$ on $[0,1]$. Then (3.4) has a unique solution ϕ. The proof is complete. □

LEMMA 3.2.4. *x is a solution of*

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) - \lambda x(t) = h(t), & t\in(0,1], \\ x^{(i)}(0) = \eta_i, & i\in\mathbb{N}_0^{n-1} \end{cases} \tag{3.2.2}$$

if and only if x is a solution of the integral equation (3.2.1).

Proof. Suppose that x is a solution of IVP(3.2.2). By Definition 2.2, we know that $x^{(i)}(t)$ is well defined on $[0,1]$. Then

$$\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\lambda x(s)+h(s)]ds = I_{0^+}^{\alpha}[\lambda x(t)+h(t)] = I_{0^+}^{\alpha}\,{}^cD_{0^+}^{\alpha}x(t)\\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\Gamma(n-\alpha)}\int_0^s (s-u)^{n-\alpha-1}x^{(n)}(u)duds\\
&= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^t\int_s^t (t-s)^{\alpha-1}(s-u)^{n-\alpha-1}dsx^{(n)}(u)du \quad \text{by changing the order of integrals}\\
&= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^t (t-u)^{n-1}\int_0^1(1-w)^{\alpha-1}w^{n-\alpha-1}dwx^{(n)}(u)du \quad \text{by} \quad w=\frac{s-u}{t-u}\\
&= \frac{1}{\Gamma(n)}\int_0^t (t-u)^{n-1}x^{(n)}(u)du\\
&= \frac{1}{\Gamma(n)}(t-u)^{n-1}x^{(n-1)}(u)\Big|_0^t + \frac{1}{\Gamma(n-1)}\int_0^t (t-u)^{n-2}x^{(n-1)}(u)du\\
&= -\frac{\eta_{n-1}}{\Gamma(n)}t^{n-1} + \frac{1}{\Gamma(n-1)}\int_0^t (t-u)^{n-2}x^{(n-1)}(u)du = \cdots\cdots = -\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^v + x(t).
\end{aligned}$$

Then we get (3.2.1).

If (3.2.1) holds, we can prove that x is a solution of (3.2.2) by direct computation. □

LEMMA 3.2.5. *x is a solution of (3.2.2) if and only if there exist constants $c_{v0}\in\mathbb{R}$ ($v\in\mathbb{N}_0^{n-1}$) such that*

$$x(t) = \sum_{v=0}^{n-1} c_{v0}t^v\mathbf{E}_{\alpha,v+1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)h(s)ds. \tag{3.2.3}$$

Proof. (i) From Lemma 3.2.3 and Lemma 3.2.4, we know that (3.2.1) has a unique solution which is the unique solution of (3.2.2).

(ii) We get from the Picard function sequence that

$$\begin{aligned}
\phi_i(t) &= \phi_0(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-1}(s)ds\\
&= \phi_0(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\phi_0(s)+\lambda\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-2}(u)du\right)ds\\
&= \phi_0(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_0(s)ds+\lambda^2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\phi_{i-2}(u)duds\\
&= \phi_0(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_0(s)ds+\lambda^2\int_0^t\int_s^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}ds\phi_{i-2}(u)du\\
&= \phi_0(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_0(s)ds+\lambda^2\int_0^t(t-u)^{2\alpha-1}\int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dw\phi_{i-2}(u)du\\
&= \phi_0(t)+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_0(s)ds+\lambda^2\int_0^t \frac{(t-u)^{2\alpha-1}}{\Gamma(2\alpha)}\phi_{i-2}(u)du\\
&\qquad\vdots\\
&= \phi_0(t)+\sum_{\sigma=1}^{i-1}\lambda^\sigma\int_0^t \frac{(t-s)^{(\sigma-1)\alpha-1}}{\Gamma((\sigma-1)\alpha)}\phi_0(s)ds+\lambda^i\int_0^t \frac{(t-u)^{i\alpha-1}}{\Gamma(i\alpha)}\phi_0(u)du\\
&= \sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^v+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\\
&\quad+\sum_{\sigma=1}^{i-1}\lambda^\sigma\int_0^t \frac{(t-s)^{(\sigma-1)\alpha-1}}{\Gamma((\sigma-1)\alpha)}\left(\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}s^v+\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)du\right)ds\\
&\quad+\lambda^i\int_0^t \frac{(t-s)^{i\alpha-1}}{\Gamma(i\alpha)}\left(\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}s^v+\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)du\right)ds\\
&= \sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^v+\sum_{\sigma=1}^{i-1}\lambda^\sigma\int_0^t \frac{(t-s)^{(\sigma-1)\alpha-1}}{\Gamma((\sigma-1)\alpha)}\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}s^vds+\lambda^i\int_0^t \frac{(t-s)^{i\alpha-1}}{\Gamma(i\alpha)}\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}s^vds\\
&\quad+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\sum_{\sigma=1}^{i-1}\lambda^\sigma\int_0^t \frac{(t-s)^{(\sigma-1)\alpha-1}}{\Gamma((\sigma-1)\alpha)}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)duds\\
&\quad+\lambda^i\int_0^t \frac{(t-u)^{i\alpha-1}}{\Gamma(i\alpha)}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)duds\\
&= \sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^v+\sum_{\sigma=1}^{i-1}\lambda^\sigma\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^{(\sigma-1)\alpha+v}\int_0^1 \frac{(1-w)^{(\sigma-1)\alpha-1}}{\Gamma((\sigma-1)\alpha)}w^vdw\\
&\quad+\lambda^i\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^{i\alpha+v}\int_0^1 \frac{(1-w)^{i\alpha-1}}{\Gamma(i\alpha)}w^vdw\\
&\quad+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\sum_{\sigma=1}^{i-1}\lambda^\sigma\int_0^t\int_s^t \frac{(t-s)^{(\sigma-1)\alpha-1}}{\Gamma((\sigma-1)\alpha)}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dsh(u)du\\
&\quad+\lambda^i\int_0^t\int_s^t \frac{(t-u)^{i\alpha-1}}{\Gamma(i\alpha)}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dsh(u)du\\
&= \sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(v+1)}t^v+\sum_{\sigma=1}^{i-1}\lambda^\sigma\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma((\sigma-1)\alpha+v+1)}t^{(\sigma-1)\alpha+v}+\lambda^i\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(i\alpha+v+1)}t^{i\alpha+v}\\
&\quad+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\sum_{\sigma=1}^{i-1}\lambda^\sigma\int_0^t(t-u)^{\sigma\alpha-1}\int_0^1 \frac{(1-w)^{(\sigma-1)\alpha-1}}{\Gamma((\sigma-1)\alpha)}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwh(u)du\\
&\quad+\lambda^i\int_0^t(t-u)^{(i+1)\alpha-1}\int_0^1 \frac{(1-w)^{i\alpha-1}}{\Gamma(i\alpha)}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwh(u)du
\end{aligned}$$

$$= \sum_{v=0}^{n-1} \eta_v \sum_{\sigma=0}^{i} \frac{\lambda^\sigma}{\Gamma(\sigma\alpha+v+1)} t^{\sigma\alpha+v} + \int_0^t \sum_{\sigma=0}^{i} \frac{\lambda^\sigma (t-u)^{\sigma\alpha+\alpha-1}}{\Gamma((\sigma+1)\alpha)} h(u)du$$
$$\to \sum_{v=0}^{n-1} \eta_v t^v E_{\alpha,v+1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) h(s)ds, \quad i \to +\infty.$$

Then $x(t) = \lim_{i\to+\infty} \phi_i(t)$ is a unique solution of (3.2.2). We know that x is just (3.2.3). The proof is complete. □

THEOREM 3.2.1. *x is a solution of (1.4.2) if and only if there exist constants $c_{vj} \in \mathbb{R}$ $(v \in \mathbb{N}_0^{n-1},\ j \in \mathbb{N}_0^m)$ such that*

$$x(t) = \sum_{j=0}^{i} \sum_{v=0}^{n-1} c_{vj}(t-t_j)^v \mathbf{E}_{\alpha,v+1}(\lambda(t-t_j)^\alpha)$$
$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) h_2(s)ds, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{3.2.4}$$

Proof. If x is a piecewise continuous solution of (1.4.2), by Lemma 3.2.5, we know there exist constants $c_{v0} \in \mathbb{R}$ $(v \in \mathbb{N}_0^{n-1})$ such that

$$x(t) = \sum_{v=0}^{n-1} c_{v0} t^v \mathbf{E}_{\alpha,v+1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) h_2(s)ds, \quad t \in (0, t_1].$$

Hence (3.2.4) holds for $i = 0$. Now suppose that (3.2.4) holds for $i = 0, 1, \cdots, j$, i.e.,

$$x(t) = \sum_{j=0}^{i} \sum_{v=0}^{n-1} c_{vj}(t-t_j)^v \mathbf{E}_{\alpha,v+1}(\lambda(t-t_j)^\alpha)$$
$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) h_2(s)ds, \quad t \in (t_i, t_{i+1}], \quad i = 0, 1, \cdots, j.$$

We will prove that (3.2.4) holds for $i = j+1$. Then by mathematical induction method, (3.2.4) holds for all $i \in \mathbb{N}_0^m$. We suppose

$$x(t) = \Phi(t) + \sum_{\sigma=0}^{j} \sum_{v=0}^{n-1} c_{v\sigma}(t-t_\sigma)^v \mathbf{E}_{\alpha,v+1}(\lambda(t-t_\sigma)^\alpha)$$
$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) h_2(s)ds, \quad t \in (t_{j+1}, t_{j+2}]. \tag{3.2.5}$$

We know for $t \in (t_{j+1}, t_{j+2}]$ that

$$h_1(t) + \lambda x(t) = {}^cD_{0^+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)ds$$
$$= \frac{\sum_{\tau=0}^{j} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} x^{(n)}(s)ds + \int_{t_{j+1}}^t (t-s)^{n-\alpha-1} x^{(n)}(s)ds}{\Gamma(n-\alpha)}$$
$$= \frac{\sum_{\tau=0}^{j} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} \left(\sum_{\sigma=0}^{\tau} \sum_{v=0}^{n-1} c_{v\sigma}(s-t_\sigma)^v \mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha) + \int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) h_2(u)du \right)^{(n)} ds}{\Gamma(n-\alpha)}$$
$$+ \frac{\int_{t_{j+1}}^t (t-s)^{n-\alpha-1} \left(\Phi(s) + \sum_{\sigma=0}^{j} \sum_{v=0}^{n-1} c_{v\sigma}(s-t_\sigma)^v \mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha) + \int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) h_2(u)du \right)^{(n)} ds}{\Gamma(n-\alpha)}$$

$$= {}^cD^{\alpha}_{t^+_{j+1}}\Phi(t) + \frac{\sum\limits_{\tau=0}^{j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum\limits_{\sigma=0}^{\tau}\sum\limits_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}\left(\sum\limits_{\sigma=0}^{j}\sum\limits_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h_2(u)du\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$= {}^cD^{\alpha}_{t^+_{j+1}}\Phi(t) + \frac{\sum\limits_{\tau=0}^{j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum\limits_{\sigma=0}^{\tau}\sum\limits_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega+v+1)}(s-t_\sigma)^{\alpha\omega}\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}\left(\sum\limits_{\sigma=0}^{j}\sum\limits_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega+v+1)}(s-t_\sigma)^{\alpha\omega}\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s(s-u)^{\alpha-1}\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha(\omega+1))}(s-u)^{\alpha\omega}h_2(u)du\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$= {}^cD^{\alpha}_{t^+_{j+1}}\Phi(t) + \frac{\sum\limits_{\tau=0}^{j}\sum\limits_{\sigma=0}^{\tau}\sum\limits_{v=0}^{n-1}c_{v\sigma}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega+v+1)}(s-t_\sigma)^{\alpha\omega+v}\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\sum\limits_{\sigma=0}^{j}\sum\limits_{v=0}^{n-1}c_{v\sigma}\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}\left(\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega+v+1)}(s-t_\sigma)^{\alpha\omega+v}\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha(\omega+1))}(s-u)^{\alpha(\omega+1)-1}h_2(u)du\right)^{(n)}ds}{\Gamma(n-\alpha)}$$
$$= {}^cD^{\alpha}_{t^+_{j+1}}\Phi(t) + \frac{\sum\limits_{\tau=0}^{j}\sum\limits_{\sigma=0}^{\tau}\sum\limits_{v=0}^{n-1}c_{v\sigma}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\sum\limits_{\omega=1}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega-n+v+1)}(s-t_\sigma)^{\alpha\omega-n+v}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\sum\limits_{\sigma=0}^{j}\sum\limits_{v=0}^{n-1}c_{v\sigma}\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}\sum\limits_{\omega=1}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega-n+v+1)}(s-t_\sigma)^{\alpha\omega-n+v}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha(\omega+1)-n+1)}(s-u)^{\alpha(\omega+1)-n}h_2(u)du\right)'ds}{\Gamma(n-\alpha)}$$
$$= {}^cD^{\alpha}_{t^+_{j+1}}\Phi(t) + \frac{\sum\limits_{\tau=0}^{j}\sum\limits_{\sigma=0}^{\tau}\sum\limits_{v=0}^{n-1}c_{v\sigma}\sum\limits_{\omega=1}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega-n+v+1)}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_\sigma)^{\alpha\omega-n+v}ds}{\Gamma(n-\alpha)}$$
$$+\frac{\sum\limits_{\sigma=0}^{j}\sum\limits_{v=0}^{n-1}c_{v\sigma}\sum\limits_{\omega=1}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega-n+v+1)}\int_{t_{j+1}}^{t}(t-s)^{n-\alpha-1}(s-t_\sigma)^{\alpha\omega-n+v}ds}{\Gamma(n-\alpha)} \quad \text{use} \quad w=\frac{s-t_\sigma}{t-t_\sigma}$$
$$+\frac{\left[\int_0^t(t-s)^{n-\alpha}\left(\int_0^s\sum\limits_{\omega=0}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha(\omega+1)-n+1)}(s-u)^{\alpha(\omega+1)-n}h_2(u)du\right)'ds\right]'}{(n-\alpha)\Gamma(n-\alpha)}$$
$$= {}^cD^{\alpha}_{t^+_{j+1}}\Phi(t) + \frac{\sum\limits_{\tau=0}^{j}\sum\limits_{\sigma=0}^{\tau}\sum\limits_{v=0}^{n-1}c_{v\sigma}\sum\limits_{\omega=1}^{+\infty}\frac{\lambda^\omega}{\Gamma(\alpha\omega-n+v+1)}(t-t_\sigma)^{\alpha(\omega-1)+v}\int_{\frac{t_\tau-t_\sigma}{t-t_\sigma}}^{\frac{t_{\tau+1}-t_\sigma}{t-t_\sigma}}(1-w)^{n-\alpha-1}w^{\alpha\omega-n+v}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega-n+v+1)}(t-t_{\sigma})^{\alpha(\omega-1)+v}\int_{\frac{t_{j+1}-t_{\sigma}}{t-t_{\sigma}}}^{1}(1-w)^{n-\alpha-1}w^{\alpha\omega-n+v}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\left[(t-s)^{n-\alpha}\left(\int_0^s\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1)-n+1)}(s-u)^{\alpha(\omega+1)-n}h_2(u)du\right)\Big|_0^t\right]'}{(n-\alpha)\Gamma(n-\alpha)}$$

$$+\frac{\left[(n-\alpha)\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1)-n+1)}(s-u)^{\alpha(\omega+1)-n}h_2(u)du\right)ds\right]'}{(n-\alpha)\Gamma(n-\alpha)}$$

$$={}^{c}D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\frac{\sum_{\sigma=0}^{j}\sum_{\tau=\sigma}^{j}\sum_{v=0}^{n-1}c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega-n+v+1)}(t-t_{\sigma})^{\alpha(\omega-1)+v}\int_{\frac{t_{\tau}-t_{\sigma}}{t-t_{\sigma}}}^{\frac{t_{\tau+1}-t_{\sigma}}{t-t_{\sigma}}}(1-w)^{n-\alpha-1}w^{\alpha\omega-n+v}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega-n+v+1)}(t-t_{\sigma})^{\alpha(\omega-1)+v}\int_{\frac{t_{j+1}-t_{\sigma}}{t-t_{\sigma}}}^{1}(1-w)^{n-\alpha-1}w^{\alpha\omega-n+v}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1)-n+1)}(s-u)^{\alpha(\omega+1)-n}h_2(u)du\right)ds\right]'}{\Gamma(n-\alpha)}$$

$$={}^{c}D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\frac{\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega-n+v+1)}(t-t_{\sigma})^{\alpha(\omega-1)+v}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha\omega-n+v}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t\int_s^t(t-s)^{n-\alpha-1}\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1)-n+1)}(s-u)^{\alpha(\omega+1)-n}dsh_2(u)du\right]'}{\Gamma(n-\alpha)}$$

$$={}^{c}D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega-1)+v+1)}(t-t_{\sigma})^{\alpha(\omega-1)+v}$$

$$+\frac{\left[\int_0^t\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha(\omega+1)-n+1)}(t-u)^{\alpha\omega}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha(\omega+1)-n}dwh_2(u)du\right]'}{\Gamma(n-\alpha)}$$

$$={}^{c}D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\lambda\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega-1}}{\Gamma(\alpha(\omega-1)+v+1)}(t-t_{\sigma})^{\alpha(\omega-1)+v}$$

$$+\left[\int_0^t\sum_{\omega=0}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega+1)}(t-u)^{\alpha\omega}h_2(u)du\right]'$$

$$={}^{c}D_{t_{j+1}^{+}}^{\alpha}\Phi(t)+\lambda\sum_{\sigma=0}^{j}\sum_{v=0}^{n-1}c_{v\sigma}\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega-1}}{\Gamma(\alpha(\omega-1)+v+1)}(t-t_{\sigma})^{\alpha(\omega-1)+v}$$

$$+\int_0^t\sum_{\omega=1}^{+\infty}\frac{\lambda^{\omega}}{\Gamma(\alpha\omega)}(t-u)^{\alpha\omega-1}h_2(u)du+h_2(t)$$

$$=\lambda\left[x(t)-\Phi(t)\right]+h_2(t)+{}^{c}D_{t_{j+1}^{+}}^{\alpha}\Phi(t),\quad t\in(t_{j+1},t_{j+2}].$$

It follows that ${}^{c}D_{t_{j+1}^{+}}^{\alpha}\Phi(t)-\lambda\Phi(t)=0$ on $(t_{j+1},t_{j+2}]$. By Lemma 3.2.5, we know that there exists constants c_{vj+1} $(v\in\mathbb{N}_0^{n-1})$ such that

$$\Phi(t)=\sum_{v=0}^{n-1}c_{vj+1}(t-t_{j+1})^{v}\mathbf{E}_{\alpha,v+1}(\lambda(t-t_{j+1})^{\alpha}).$$

Substituting Φ into (3.2.5). We know that (3.2.4) holds for $i = j+1$. By mathematical induction method, we know that (3.2.4) holds for $i \in \mathbb{N}_0^m$.

Step 2. We prove that x is a piecewise continuous solution of (1.4.2) if x satisfies (3.2.4).

Since x satisfies (3.2.4), we know that $x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}]$ $(i \in \mathbb{N}_0^m)$ and $\lim_{t\to t_i^+} x(t)$ exists and is finite for all $i \in \mathbb{N}_0^m$. We will prove that ${}^cD_{0^+}^{\alpha}x(t) - \lambda x = h_2(t)$ for all $t \in (t_i,t_{i+1}]$ $(i \in \mathbb{N}_0^m)$.

In fact, we have for $t \in (t_i,t_{i+1}]$ that

$$
\begin{aligned}
{}^cD_{0^+}^{\alpha}x(t) &= \frac{1}{(n-\alpha)}\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}x^{(n)}(s)ds + \int_{t_i}^{t}(t-s)^{n-\alpha-1}x^{(n)}(s)ds\right]\\
&= \frac{\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum_{\sigma=0}^{\tau}\sum_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h_2(u)du\right)^{(n)}ds}{\Gamma(n-\alpha)}\\
&\quad+\frac{\int_{t_i}^{t}(t-s)^{n-\alpha-1}\left(\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h_2(u)du\right)^{(n)}ds}{\Gamma(n-\alpha)}\\
&= \frac{\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\sum_{\sigma=0}^{\tau}\sum_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)\right)^{(n)}ds}{\Gamma(n-\alpha)}\\
&\quad+\frac{\int_{t_i}^{t}(t-s)^{n-\alpha-1}\left(\sum_{\sigma=0}^{i}\sum_{v=0}^{n-1}c_{v\sigma}(s-t_\sigma)^v\mathbf{E}_{\alpha,v+1}(\lambda(s-t_\sigma)^\alpha)\right)^{(n)}ds}{\Gamma(n-\alpha)}\\
&\quad+\frac{\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h_2(u)du\right)^{(n)}ds}{\Gamma(n-\alpha)}.
\end{aligned}
$$

Similarly to Step 1 we can get that

$$
{}^cD_{0^+}^{\alpha}x(t) = h_2(t) + \lambda x(t), \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m.
$$

So x is a piecewise continuous solution of (1.4.2). The proof is complete. □

COROLLARY 3.2.1. *Suppose that* $h \in L^1(0,1)$ *and there exist* $k > -1$, $l \in (-1,0]$ *such that* $|h(t)| \leqslant t^k(1-t)^l$, $t \in (0,1)$. *Then* x *is a piecewise solution of* ${}^cD_{0^+}^{\alpha}x(t) = h(t)$, $\alpha \in [n-1,n)$ *if and only if there exist constants* $c_{vj} \in \mathbb{R}$ $(v \in \mathbb{N}_0^{n-1},\ j \in \mathbb{N}_0^m)$ *such that*

$$
x(t) = \sum_{j=0}^{i}\sum_{v=0}^{n-1}\frac{c_{vj}}{\Gamma(v+1)}(t-t_j)^v + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{3.7}
$$

Proof. In Theorem 3.2.1, we choose $\lambda = 0$. The result follows. □

REMARK 3.2.1. In Lemma 2.3 in [70], authors gave an expression of solutions of the impulsive fractional differential equation ${}^cD_{0^+}^{\alpha}y(x) = h(x)$, $x \in [0,T]$, $n-1 < \alpha <$

n. It was proved that $y \in A^n([0,T],\mathbb{R})$ is a solution of ${}^cD_{0+}^{\alpha}y(x) = h(x)$ if and only if $y \in PC^{n-1}([0,T],\mathbb{R})$ and

$$y(x) = \sum_{j=0}^{n-1}\frac{y^{(j)}(0)}{j!}x^j + \sum_{j=0}^{n-1}\left(\sum_{0<x_k<x}\frac{\Delta y^{(j)}(x_k)}{j!}(x-x_k)^j\right) + \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad x \in [0,T],$$

where $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = T$, and $h \in PC([0,T],\mathbb{R})$. Their result is just Corollary 3.3 but the method of the proof is different.

3.3. Exact solution of (1.4.3)

To get explicit piecewise continuous solutions of (1.4.3), fix $\xi_i (i \in \mathbb{N}_1^m)$, $\eta_i \in \mathbb{R}$ $(i \in \mathbb{N}_1^n)$ and $h \in C(0,1)$ satisfying $|h(t)| \leqslant t^k(1-t)^l$ for all $t \in (0,1)$ with $k > -1$ and $l \in (-1,0]$, we choose the following Picard function sequence:

$$\phi_0(t) = \sum_{u=1}^{r}\frac{\xi_u}{\Gamma(\beta-u+1)}t^{\beta-u} + \sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha+\beta-v+1)}t^{\alpha+\beta-v} + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds, \quad t \in (0,1],$$
$$\phi_i(t) = \phi_0(t) + \lambda\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_{i-1}(s)ds, \quad t \in (0,1], \quad i = 1,2,\cdots.$$

LEMMA 3.3.1. *$t \to t^{r-\beta}\phi_i$ is continuous on $[0,1]$.*

Proof. We have

$$\left|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds\right| \leqslant \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}s^k(1-s)^l ds$$
$$\leqslant \int_0^t \frac{(t-s)^{\alpha+\beta+l-1}}{\Gamma(\alpha+\beta)}s^k ds = t^{\alpha+\beta+k+l}\frac{\mathbf{B}(\alpha+\beta+l,k+1)}{\Gamma(\alpha+\beta)}.$$

One sees easily that

$$t^{r-\beta}\phi_0(t) = \phi_0(t) = \sum_{u=1}^{m}\frac{\xi_u}{\Gamma(\beta-u+1)}t^{r-u} + \sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha+\beta-v+1)}t^{\alpha+r-v} + t^{r-\beta}\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds.$$

Then $t \to t^{r-\beta}\phi_0$ is continuous on $[0,1]$. It follows that $t \to t^{r-\beta}\phi_1(t)$ is continuous on $[0,1]$. By mathematical induction method, we can prove that $t \to t^{r-\beta}\phi_i(t)$ is continuous on $[0,1]$. □

LEMMA 3.3.2. *$\{\phi_i(t)\}$ is convergent uniformly on $(0,1]$.*

Proof. In fact, similarly to Lemma 3.3.1, we have for $t \in (0,1]$ that

$$|\phi_1(t) - \phi_0(t)| = \left|\lambda\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_0(s)ds\right| \leqslant |\lambda|\frac{||\phi_0||}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}.$$

So

$$|\phi_2(t) - \phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\lambda[\phi_1(s) - \phi_0(s)]ds\right|$$
$$\leqslant |\lambda|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|\lambda|||\phi_0||\frac{s^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}ds$$

$$= |\lambda|^2 \frac{||\phi_0||}{\Gamma(\alpha+\beta+1)} \frac{\mathbf{B}(\alpha+\beta,\alpha+\beta+1)}{\Gamma(\alpha+\beta)} t^{2\alpha+2\beta}.$$

By using the mathematical induction method, we get for every $i = 1,2,\cdots$ that

$$|\phi_{i+1}(t) - \phi_i(t)| \leqslant |\lambda|^{i+1} \frac{||\phi_0||}{\Gamma(\alpha+\beta+1)} \left(\prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+\beta,j\alpha+j\beta+1)}{\Gamma(\alpha+\beta)} \right) t^{(i+1)[\alpha+\beta]}, \quad t \in (0,1].$$

It follows that

$$|\phi_{i+1}(t) - \phi_i(t)| \leqslant |\lambda|^{i+1} \frac{||\phi_0||}{\Gamma(\alpha+\beta+1)} \left(\prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+\beta,j\alpha+j\beta+1)}{\Gamma(\alpha+\beta)} \right), \quad t \in (0,1].$$

Consider

$$\sum_{i=1}^{+\infty} u_i =: \sum_{i=1}^{+\infty} |\lambda|^{i+1} \frac{||\phi_0||}{\Gamma(\alpha+\beta+1)} \left(\prod_{j=1}^{i} \frac{\mathbf{B}(\alpha+\beta,j\alpha+j\beta+1)}{\Gamma(\alpha+\beta)} \right).$$

One sees for sufficiently large i that

$$\begin{aligned}
\frac{u_{i+1}}{u_i} &= |\lambda| \frac{\mathbf{B}(\alpha+\beta,(i+1)[\alpha+\beta]+1)}{\Gamma(\alpha+\beta)} = |\lambda| \int_0^1 (1-x)^{\alpha+\beta-1} x^{(i+1)[\alpha+\beta]} dx \\
&\leqslant |\lambda| \int_0^\delta (1-x)^{\alpha+\beta-1} x^{(i+1)[\alpha+\beta]} dx + |\lambda| \int_\delta^1 (1-x)^{\alpha+\beta-1} dx \quad \text{with} \quad \delta \in (0,1) \\
&\leqslant |\lambda| \int_0^\delta (1-x)^{\alpha+\beta-1} dx \delta^{(i+1)[\alpha+\beta]} + \frac{|\lambda|}{\alpha+\beta} \delta^{\alpha+\beta} \\
&\leqslant \frac{|\lambda|(1-(1-\delta)^{\alpha+\beta})}{\alpha+\beta} \delta^{(i+1)[\alpha+\beta]} + \frac{|\lambda|}{\alpha+\beta} \delta^{\alpha+\beta}.
\end{aligned}$$

For any $\varepsilon > 0$, it is easy to see that there exists $\delta \in (0,1)$ such that $\frac{|\lambda|}{\alpha+\beta}\delta^{\alpha+\beta}$. For this δ, there exists an integer $N > 0$ sufficiently large such that $\frac{|\lambda|(1-(1-\delta)^{\alpha+\beta})}{\alpha+\beta}\delta^{(i+1)[\alpha+\beta]} < \frac{\varepsilon}{2}$ for all $i > N$. So $0 < \frac{u_{i+1}}{u_i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $i > N$. It follows that $\lim_{i\to+\infty} \frac{u_{i+1}}{u_i} = 0$. Then $\sum_{i=1}^{+\infty} u_i$ is convergent. Hence dominant control convergence theorem implies that

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_i(t) - \phi_{i-1}(t)] + \cdots, \quad t \in (0,1]$$

is uniformly convergent. Then $\{t \to \phi_i(t)\}$ is convergent uniformly on $(0,1]$. □

LEMMA 3.3.3. *$\phi(t) = \lim_{i\to+\infty} \phi_i(t)$ defined on $(0,1]$ is a unique continuous solution of the integral equation*

$$x(t) = \sum_{u=1}^{r} \frac{\xi_u}{\Gamma(\beta-u+1)} t^{\beta-u} + \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha+\beta-v+1)} t^{\alpha+\beta-v} + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} [\lambda x(s) + h(s)] ds,\ t \in (0,1]. \tag{3.3.1}$$

Proof. By Lemma 3.3.1 and Lemma 3.3.2, we have $\{\phi_i(t)\}$ is convergent uniformly on $(0,1]$ to a function. Suppose that $\phi(t) = \lim_{i\to+\infty} \phi_i(t)$. We see $\phi(t)$ is continuous on $(0,1]$. We have from Lemma 3.3.2 that

$$\phi(t) = \lim_{i\to+\infty} \phi_i(t)$$

$$= \lim_{i\to+\infty}\left[\sum_{u=1}^{r}\frac{\xi_u t^{\beta-u}}{\Gamma(\beta-u+1)}+\sum_{v=1}^{n}\frac{\eta_v t^{\alpha+\beta-v}}{\Gamma(\alpha+\beta-v+1)}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_{i-1}(s)ds\right]$$
$$= \sum_{u=1}^{r}\frac{\xi_u t^{\beta-u}}{\Gamma(\beta-u+1)}+\sum_{v=1}^{n}\frac{\eta_v t^{\alpha+\beta-v}}{\Gamma(\alpha+\beta-v+1)}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\lim_{i\to+\infty}\phi_{i-1}(s)ds$$
$$= \sum_{u=1}^{r}\frac{\xi_u t^{\beta-u}}{\Gamma(\beta-u+1)}+\sum_{v=1}^{n}\frac{\eta_v t^{\alpha+\beta-v}}{\Gamma(\alpha+\beta-v+1)}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi(s)ds$$
$$= \sum_{u=1}^{r}\frac{\xi_u t^{\beta-u}}{\Gamma(\beta-u+1)}+\sum_{v=1}^{n}\frac{\eta_v t^{\alpha+\beta-v}}{\Gamma(\alpha+\beta-v+1)}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[\lambda\phi(s)+h(s)]ds.$$

Then ϕ is a continuous solution of (3.3.1) defined on $(0,1]$.

Suppose that ψ continuous $(0,1]$ is also a solution of (3.3.1). Then

$$\psi(t)=\sum_{u=1}^{r}\frac{\xi_u t^{\beta-u}}{\Gamma(\beta-u+1)}+\sum_{v=1}^{n}\frac{\eta_v t^{\alpha+\beta-v}}{\Gamma(\alpha+\beta-v+1)}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[\lambda\psi(s)+h(s)]ds,\quad t\in(0,1].$$

We need to prove that $\phi(t)\equiv\psi(t)$ on $(0,1]$. Similarly to Lemma 3.3.2, we get

$$|\psi(t)-\phi_0(t)|=\left|\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\psi_0(s)ds\right|\leqslant|\lambda|||\phi_0||\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

Similarly to Lemma 3.3.2, by the mathematical induction method, we get

$$|\psi(t)-\phi_i(t)|\leqslant|\lambda|^{i+1}\frac{||\phi_0||}{\Gamma(\alpha+\beta+1)}\prod_{j=1}^{i}\frac{\mathbf{B}(\alpha+\beta,j\alpha+j\beta)}{\Gamma(\alpha+\beta)},\quad t\in(0,1].$$

Then $\lim_{i\to+\infty}\phi_i(t)=\psi(t)$ uniformly on $(0,1]$. Then $\phi(t)\equiv\psi(t)$ on $(0,1]$. Then (3.8) has a unique solution ϕ. The proof is complete. □

LEMMA 3.3.4. *x is a solution of*

$$\begin{cases} D_{0^+}^{\alpha}D_{0^+}^{\beta}x(t)-\lambda x(t)=h(t),\quad t\in(0,1],\\ I_{0^+}^{m-\beta}x(0)=\xi_m,\ D_{0^+}^{\beta-i}x(0)=\xi_i,\quad i\in\mathbb{N}_1^{m-1},\\ I_{0^+}^{n-\alpha}D_{0^+}^{\beta}x(0)=\eta_n,\ D_{0^+}^{\alpha-i}D_{0^+}^{\beta}x(0)=\eta_i,\quad i\in\mathbb{N}_1^{n-1}\end{cases}\tag{3.3.2}$$

if and only if x is a solution of the integral equation (3.3.1).

Proof. Similar to Lemma 3.2.4 and the details are omitted. □

LEMMA 3.3.5. *x is a solution of (3.3.2) if and only if there exist constants $c_{v0}\in\mathbb{R}$ $(v\in\mathbb{N}_1^m)$, $d_{v0}\in\mathbb{R}$ $(v\in\mathbb{N}_1^n)$ such that*

$$x(t)=\sum_{u=1}^{r}\xi_u t^{\beta-u}\mathbf{E}_{\alpha+\beta,\beta-u+1}(\lambda t^{\alpha+\beta})+\sum_{v=1}^{n}\eta_v t^{\alpha+\beta-v}\mathbf{E}_{\alpha+\beta,\alpha+\beta-v+1}(\lambda t^{\alpha+\beta})$$
$$+\int_0^t(t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})h(u)du.\tag{3.3.3}$$

Proof. (i) From Lemma 3.3.3 and Lemma 3.3.4, we know that (3.3.1) has a unique solution which is the unique solution of (3.3.2).

(ii) We get from the Picard function sequence at the beginning of this subsection that

$$\phi_0(t)=\sum_{u=1}^{r}\frac{\xi_u}{\Gamma(\beta-u+1)}t^{\beta-u}+\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha+\beta-v+1)}t^{\alpha+\beta-v}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds,\quad t\in(0,1]$$

and

$$\begin{aligned}
&\phi_j(t)=\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_{j-1}(s)ds\\
&=\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(\phi_0(s)+\lambda\int_0^s\frac{(s-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_{j-2}(u)du\right)ds\\
&=\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_0(s)ds+\lambda^2\int_0^t\int_s^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\frac{(s-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\phi_{j-2}(u)du\\
&=\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_0(s)ds\\
&\quad+\lambda^2\int_0^t(t-u)^{2\alpha+2\beta-1}\int_0^1\frac{(1-w)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\frac{w^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}dw\phi_{i-2}(u)du\\
&=\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_0(s)ds+\lambda^2\int_0^t\frac{(t-u)^{2\alpha+2\beta-1}}{\Gamma(2(\alpha+\beta))}\phi_{j-2}(u)du\\
&\qquad\vdots\\
&=\phi_0(t)+\sum_{i=1}^{j}\lambda^i\int_0^t\frac{(t-s)^{i(\alpha+\beta)-1}}{\Gamma(i(\alpha+\beta))}\phi_0(s)ds+\lambda^{j+1}\int_0^t\frac{(t-u)^{(j+1)(\alpha+\beta)-1}}{\Gamma((j+1)(\alpha+\beta))}\phi_0(u)du\\
&=\sum_{u=1}^{r}\frac{\xi_u}{\Gamma(\beta-u+1)}t^{\beta-u}+\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha+\beta-v+1)}t^{\alpha+\beta-v}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds\\
&\quad+\sum_{i=1}^{j}\lambda^i\int_0^t\frac{(t-s)^{i(\alpha+\beta)-1}}{\Gamma(i(\alpha+\beta))}\left[\sum_{u=1}^{r}\frac{\xi_u s^{\beta-u}}{\Gamma(\beta-u+1)}+\sum_{v=1}^{n}\frac{\eta_v s^{\alpha+\beta-v}}{\Gamma(\alpha+\beta-v+1)}+\int_0^s\frac{(s-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(u)du\right]ds\\
&\quad+\lambda^{j+1}\int_0^t\frac{(t-s)^{(j+1)(\alpha+\beta)-1}}{\Gamma((j+1)(\alpha+\beta))}\left[\sum_{u=1}^{r}\frac{\xi_u s^{\beta-u}}{\Gamma(\beta-u+1)}+\sum_{v=1}^{n}\frac{\eta_v s^{\alpha+\beta-v}}{\Gamma(\alpha+\beta-v+1)}+\int_0^s\frac{(s-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(u)du\right]ds\\
&=\sum_{u=1}^{r}\frac{\xi_u}{\Gamma(\beta-u+1)}t^{\beta-u}+\sum_{v=1}^{n}\frac{\eta_v}{\Gamma(\alpha+\beta-v+1)}t^{\alpha+\beta-v}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds\\
&\quad+\sum_{i=1}^{j}\lambda^i\sum_{u=1}^{r}\frac{\xi_u t^{i(\alpha+\beta)+\beta-u}}{\Gamma(\beta-u+1)}\int_0^1\frac{(1-w)^{i(\alpha+\beta)-1}w^{\beta-u}dw}{\Gamma(i(\alpha+\beta))}\\
&\quad+\sum_{i=1}^{j}\lambda^i\sum_{v=1}^{n}\frac{\eta_v t^{(i+1)(\alpha+\beta)-v}}{\Gamma(\alpha+\beta-v+1)}\int_0^1\frac{(1-w)^{i(\alpha+\beta)-1}w^{\alpha+\beta-v}dw}{\Gamma(i(\alpha+\beta))}\\
&\quad+\sum_{i=1}^{j}\lambda^i\int_0^t\int_s^t\frac{(t-s)^{i(\alpha+\beta)-1}}{\Gamma(i(\alpha+\beta))}\frac{(s-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}dsh(u)du\\
&\quad+\lambda^{j+1}\sum_{u=1}^{r}\frac{\xi_u t^{(j+1)(\alpha+\beta)+\beta-u}}{\Gamma(\beta-u+1)}\int_0^1\frac{(1-w)^{(j+1)(\alpha+\beta)-1}w^{\beta-u}dw}{\Gamma((i+1)(\alpha+\beta))}\\
&\quad+\lambda^{j+1}\sum_{v=1}^{n}\frac{\eta_v t^{(j+2)(\alpha+\beta)-v}}{\Gamma(\alpha+\beta-v+1)}\int_0^1\frac{(1-w)^{(j+1)(\alpha+\beta)-1}w^{\alpha+\beta-v}dw}{\Gamma((j+1)(\alpha+\beta))}\\
&\quad+\lambda^{j+1}\int_0^t\int_s^t\frac{(t-s)^{(j+1)(\alpha+\beta)-1}}{\Gamma((j+1)(\alpha+\beta))}\frac{(s-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}dsh(u)du
\end{aligned}$$

$$= \sum_{u=1}^{r} \frac{\xi_u}{\Gamma(\beta-u+1)} t^{\beta-u} + \sum_{i=1}^{j} \lambda^i \sum_{u=1}^{r} \frac{\xi_u t^{i(\alpha+\beta)+\beta-u}}{\Gamma(i(\alpha+\beta)+\beta-u+1)} + \lambda^{j+1} \sum_{u=1}^{r} \frac{\xi_u t^{(j+1)(\alpha+\beta)+\beta-u}}{\Gamma((j+1)(\alpha+\beta)+\beta-u+1)}$$
$$+ \sum_{v=1}^{n} \frac{\eta_v}{\Gamma(\alpha+\beta-v+1)} t^{\alpha+\beta-v} + \sum_{i=1}^{j} \lambda^i \sum_{v=1}^{n} \frac{\eta_v t^{(i+1)(\alpha+\beta)-v}}{\Gamma((i+1)(\alpha+\beta)-v+1)} + \lambda^{j+1} \sum_{v=1}^{n} \frac{\eta_v t^{(j+2)(\alpha+\beta)-v}}{\Gamma((j+2)(\alpha+\beta)-v+1)}$$
$$+ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds + \sum_{i=1}^{j} \lambda^i \int_0^t (t-u)^{(i+1)(\alpha+\beta)-1} \int_0^1 \frac{(1-w)^{i(\alpha+\beta)-1}}{\Gamma(i(\alpha+\beta))} \frac{w^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} dw h(u)du$$
$$+\lambda^{j+1} \int_0^t (t-u)^{(i+2)(\alpha+\beta)-1} \int_0^1 \frac{(1-w)^{(j+1)(\alpha+\beta)-1}}{\Gamma((j+1)(\alpha+\beta))} \frac{w^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} dw h(u)du$$
$$= \sum_{i=0}^{j+1} \lambda^i \sum_{u=1}^{r} \frac{\xi_u t^{i(\alpha+\beta)+\beta-u}}{\Gamma(i(\alpha+\beta)+\beta-u+1)} + \sum_{i=0}^{j+1} \lambda^i \sum_{v=1}^{n} \frac{\eta_v t^{(i+1)(\alpha+\beta)-v}}{\Gamma((i+1)(\alpha+\beta)-v+1)}$$
$$+ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)ds + \sum_{i=1}^{j} \lambda^i \int_0^t \frac{(t-u)^{(i+1)(\alpha+\beta)-1}}{\Gamma((i+1)(\alpha+\beta))} h(u)du + \lambda^{j+1} \int_0^t \frac{(t-u)^{(j+2)(\alpha+\beta)-1}}{\Gamma((j+2)(\alpha+\beta))} h(u)du$$
$$= \sum_{i=0}^{j+1} \lambda^i \sum_{u=1}^{m} \frac{\xi_u t^{i(\alpha+\beta)+\beta-u}}{\Gamma(i(\alpha+\beta)+\beta-u+1)} + \sum_{i=0}^{j+1} \lambda^i \sum_{v=1}^{n} \frac{\eta_v t^{(i+1)(\alpha+\beta)-v}}{\Gamma((i+1)(\alpha+\beta)-v+1)}$$
$$+ \sum_{i=0}^{j+1} \lambda^i \int_0^t \frac{(t-u)^{(i+1)(\alpha+\beta)-1}}{\Gamma((i+1)(\alpha+\beta))} h(u)du$$
$$= \sum_{u=1}^{r} \xi_u \sum_{i=0}^{j+1} \frac{\lambda^i t^{i(\alpha+\beta)} t^{\beta-u}}{\Gamma(i(\alpha+\beta)+\beta-u+1)} + \sum_{v=1}^{n} \eta_v \sum_{i=0}^{j+1} \frac{\lambda^i t^{i(\alpha+\beta)} t^{\alpha+\beta-v}}{\Gamma(i(\alpha+\beta)+\alpha+\beta-v+1)}$$
$$+ \int_0^t \sum_{i=0}^{j+1} \frac{\lambda^i (t-u)^{(i+1)(\alpha+\beta)-1}}{\Gamma((i+1)(\alpha+\beta))} h(u)du$$
$$\to \sum_{u=1}^{r} \xi_u t^{\beta-u} \mathbf{E}_{\alpha+\beta,\beta-u+1}(\lambda t^{\alpha+\beta}) + \sum_{v=1}^{n} \eta_v t^{\alpha+\beta-v} \mathbf{E}_{\alpha+\beta,\alpha+\beta-v+1}(\lambda t^{\alpha+\beta})$$
$$+ \int_0^t (t-u)^{\alpha+\beta-1} \mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda (t-u)^{\alpha+\beta}) h(u)du \quad \text{as} \quad j \to \infty.$$

Then $x(t) = \lim_{i\to+\infty} \phi_i(t)$ is a unique solution of (3.3.2). We know that x is just (3.3.3). The proof is complete. □

THEOREM 3.3.1. *x is a solution of (1.4.3) if and only if there exist constants c_{vj}, $d_{uj} \in \mathbb{R}$ $(v \in \mathbb{N}_1^n,\ u \in \mathbb{N}_1^m,\ j \in \mathbb{N}_0^m)$ such that*

$$x(t) = \sum_{j=0}^{i} \sum_{u=1}^{m} d_{uj}(t-t_j)^{\beta-u} \mathbf{E}_{\alpha+\beta,\beta-u+1}(\lambda (t-t_j)^{\alpha+\beta})$$
$$+ \sum_{j=0}^{i} \sum_{v=1}^{n} c_{vj}(t-t_j)^{\alpha+\beta-v} \mathbf{E}_{\alpha+\beta,\alpha+\beta-v+1}(\lambda (t-t_j)^{\alpha+\beta})$$
$$+ \int_0^t (t-u)^{\alpha+\beta-1} \mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda (t-u)^{\alpha+\beta}) h_3(u)du, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{3.3.4}$$

Proof. The proof is similar to that of Theorem 3.2.1. □

COROLLARY 3.3.1. *Suppose that* $h\in L^1(0,1)$ *and there exist* $k>-1$, $l\in(-1,0]$ *such that* $|h(t)|\leqslant t^k(1-t)^l$, $t\in(0,1)$. *Then* x *is a piecewise solution of*

$$^cD_{0^+}^{\alpha}D_{0^+}^{\beta}x(t)=h(t),\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m$$

if and only if there exist constants $c_{vj},d_{uj}\in\mathbb{R}$ $(v\in\mathbb{N}_1^n,\ u\in\mathbb{N}_1^m,\ j\in\mathbb{N}_0^m)$ *such that*

$$\begin{aligned}x(t)=&\sum_{j=0}^{i}\sum_{u=1}^{r}\frac{d_{uj}}{\Gamma(\beta-u+1)}(t-t_j)^{\beta-u}+\sum_{j=0}^{i}\sum_{v=1}^{n}\frac{c_{vj}}{\Gamma(\alpha+\beta-v+1)}(t-t_j)^{\alpha+\beta-v}\\&+\int_0^t\frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h_3(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}\tag{3.3.5}$$

Proof. In Theorem 3.3.1, we choose $\lambda=0$. The result follows. □

COROLLARY 3.3.2. *Suppose* x *is a piecewise continuous solution of (1.4.3). Then*

$$\begin{aligned}I_{0^+}^{r-\beta}x(t)=&\sum_{j=0}^{i}\sum_{u=1}^{r}d_{uj}(t-t_j)^{m-u}\mathbf{E}_{\alpha+\beta,r-u+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(t-t_j)^{\alpha+r-v}\mathbf{E}_{\alpha+\beta,\alpha+r-v+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\int_0^t(t-u)^{\alpha+r-1}\mathbf{E}_{\alpha+\beta,\alpha+r}(\lambda(t-u)^{\alpha+\beta})h_3(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}$$

$$\begin{aligned}D_{0^+}^{\beta-\sigma}x(t)=&\sum_{j=0}^{i}\sum_{u=1}^{\sigma}d_{uj}(t-t_j)^{\sigma-u}\mathbf{E}_{\alpha+\beta,\sigma-u+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(t-t_j)^{\alpha+\sigma-v}\mathbf{E}_{\alpha+\beta,\alpha+\sigma-v+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\int_0^t(t-u)^{\alpha+\sigma-1}\mathbf{E}_{\alpha+\beta,\alpha+\sigma}(\lambda(t-u)^{\alpha+\beta})h_3(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}$$

$$\begin{aligned}D_{0^+}^{\beta}x(t)=&\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(t-t_j)^{\alpha-v}\mathbf{E}_{\alpha+\beta,\alpha-v+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})h_3(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}$$

$$\begin{aligned}I_{0^+}^{n-\alpha}D_{0^+}^{\beta}x(t)=&\sum_{j=0}^{i}\sum_{v=1}^{n}c_{vj}(t-t_j)^{n-v}\mathbf{E}_{\alpha+\beta,n-v+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\int_0^t(t-u)^{n-1}\mathbf{E}_{\alpha+\beta,n}(\lambda(t-u)^{\alpha+\beta})h_3(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}$$

$$\begin{aligned}D_{0^+}^{\alpha-\sigma}D_{0^+}^{\beta}x(t)=&\sum_{j=0}^{i}\sum_{v=1}^{\sigma}c_{vj}(t-t_j)^{\sigma-v}\mathbf{E}_{\alpha+\beta,\sigma-v+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\int_0^t(t-u)^{\sigma-1}\mathbf{E}_{\alpha+\beta,\sigma}(\lambda(t-u)^{\alpha+\beta})h_3(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}$$

Proof. By Theorem 3.3.1, we know x satisfies (3.3.4). Use the method in the proof of Corollary 3.2.1, by direct computation, we can derive above mentioned equations. □

3.4. Exact solution of (1.4.4)

In this subsection, we establish exact piecewise continuous solution of (1.4.4). To get explicit piecewise continuous solutions of (1.4.4), fix $\xi_i\ (i\in\mathbb{N}_0^{m-1}),\ \eta_i\in\mathbb{R}$ $(i\in\mathbb{N}_0^{n-1})$ and $h\in C(0,1)$ satisfying $|h(t)|\leqslant t^k(1-t)^l$ for all $t\in(0,1)$ with $k>-1$ and $l\in(-1,0]$, we choose the following Picard function sequence:

$$\phi_0(t)=\sum_{u=0}^{r-1}\frac{\xi_u}{\Gamma(u+1)}t^u+\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(\beta+v+1)}t^{\beta+v}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(s)ds,\quad t\in[0,1],$$

$$\phi_i(t)=\phi_0(t)+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\phi_{i-1}(s)ds,\quad t\in[0,1],\quad i=1,2,\cdots.$$

LEMMA 3.4.1. *ϕ_i is continuous on $[0,1]$.*

Proof. It similar to the proof of Lemma 3.2.1. □

LEMMA 3.4.2. *ϕ_i is convergent uniformly on $[0,1]$.*

Proof. It is similar to the proof of Lemma 3.2.2. □

LEMMA 3.3.3. *$\phi(t)=\lim_{i\to+\infty}\phi_i(t)$ defined on $[0,1]$ is a unique continuous solution of the integral equation*

$$x(t)=\sum_{u=0}^{r-1}\frac{\xi_u}{\Gamma(u+1)}t^u+\sum_{v=0}^{n-1}\frac{\eta_v}{\Gamma(\beta+v+1)}t^{\beta+v}+\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[\lambda x(s)+h(s)]ds,\quad t\in[0,1].\tag{3.4.1}$$

Proof. It is similar to the proof of Lemma 3.2.4. □

LEMMA 3.4.5. *x is a solution of (3.4.1) if and only if there exist constants c_{v0}, $d_{u0}\in\mathbb{R}$ $(v\in\mathbb{N}_0^{n-1},\ u\in\mathbb{N}_0^{m-1})$ such that*

$$\begin{aligned}x(t)=&\sum_{u=0}^{r-1}d_{u0}t^u\mathbf{E}_{\alpha+\beta,u+1}(\lambda t^{\alpha+\beta})+\sum_{v=0}^{n-1}\eta_v t^{\beta-v}\mathbf{E}_{\alpha+\beta,\beta-v+1}(\lambda t^{\alpha+\beta})\\&+\int_0^t(t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})h(u)du,\quad t\in[0,1].\end{aligned}\tag{3.4.2}$$

Proof. The proof is similar to that of Lemma 3.2.5. □

THEOREM 3.4.1. *x is a solution of (1.4.4) if and only if there exist constants c_{vj}, $d_{uj}\in\mathbb{R}$ $(v\in\mathbb{N}_0^{n-1},\ u\in\mathbb{N}_0^{m-1},\ j\in\mathbb{N}_0^{m})$ such that*

$$\begin{aligned}x(t)=&\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}(t-t_j)^u\mathbf{E}_{\alpha+\beta,u+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}(t-t_j)^{\beta+v}\mathbf{E}_{\alpha+\beta,\beta+v+1}(\lambda(t-t_j)^{\alpha+\beta})\\&+\int_0^t(t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})h(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}\tag{3.4.3}$$

Proof. The proof is similar to that of Lemma 3.2.2. □

COROLLARY 3.4.1. *Suppose that $h\in L^1(0,1)$ and there exist $k>-1$, $l\in(-1,0]$ such that $|h(t)|\leqslant t^k(1-t)^l$, $t\in(0,1)$. Then x is a piecewise solution of*

$$ {}^cD_{0^+}^{\alpha}\,{}^cD_{0^+}^{\beta}x(t)=h(t),\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\quad \alpha\in[n-1,n),\quad \beta\in[r-1,r) $$

if and only if there exist constants c_{vj}, $c_{uj}\in\mathbb{R}$ $(v\in\mathbb{N}_0^{n-1},\ u\in\mathbb{N}_0^{m-1},\ j\in\mathbb{N}_0^m)$ such that

$$ \begin{aligned} x(t)=&\sum_{j=0}^{i}\sum_{u=0}^{r-1}\frac{d_{uj}}{\Gamma(u+1)}(t-t_j)^u+\sum_{j=0}^{i}\sum_{v=0}^{n-1}\frac{c_{vj}}{\Gamma(\beta-v+1)}(t-t_j)^{\beta+v}\\ &+\int_0^t\frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m. \end{aligned}\tag{3.4.4} $$

Proof. In Theorem 3.4.1, we choose $\lambda=0$. The result follows. $\square$

COROLLARY 3.4.2. *Suppose that x is a solution of (1.4.4) given by (3.4.3). Then*

$$ \begin{aligned} {}^cD_{0^+}^{\beta}x(t)=&\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}(t-t_j)^{v}\mathbf{E}_{\alpha+\beta,v+1}(\lambda(t-t_j)^{\alpha+\beta})\\ &+\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})h(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m. \end{aligned} $$

Proof. By Definition 2.2, we have for $t\in(t_i,t_{i+1}]$ that

$$ \begin{aligned} {}^cD_{0^+}^{\beta}x(t)=&\frac{\int_0^t(t-s)^{r-\beta-1}x^{(r)}(s)ds}{\Gamma(r-\beta)}=\frac{\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}x^{(r)}(s)ds+\int_{t_i}^t(t-s)^{r-\beta-1}x^{(r)}(s)ds}{\Gamma(r-\beta)}\\ =&\frac{1}{\Gamma(r-\beta)}\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{\tau}\sum_{u=0}^{r-1}d_{uj}(s-t_j)^u\mathbf{E}_{\alpha+\beta,u+1}(\lambda(s-t_j)^{\alpha+\beta})\right.\right.\\ &+\int_0^s(s-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(s-u)^{\alpha+\beta})h(u)du\\ &\left.\left.+\sum_{j=0}^{\tau}\sum_{v=0}^{n-1}c_{vj}(s-t_j)^{\beta+v}\mathbf{E}_{\alpha+\beta,\beta+v+1}(\lambda(s-t_j)^{\alpha+\beta})\right)^{(r)}ds\right]\\ &+\frac{1}{\Gamma(r-\beta)}\left[\int_{t_i}^t(t-s)^{r-\beta-1}\left(\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}(s-t_j)^u\mathbf{E}_{\alpha+\beta,u+1}(\lambda(s-t_j)^{\alpha+\beta})\right.\right.\\ &+\int_0^s(s-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(s-u)^{\alpha+\beta})h(u)du\\ &\left.\left.+\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}(s-t_j)^{\beta+v}\mathbf{E}_{\alpha+\beta,\beta+v+1}(\lambda(s-t_j)^{\alpha+\beta})\right)^{(m)}ds\right]\\ =&\frac{1}{\Gamma(r-\beta)}\left[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{\tau}\sum_{u=0}^{r-1}d_{uj}(s-t_j)^u\mathbf{E}_{\alpha+\beta,u+1}(\lambda(s-t_j)^{\alpha+\beta})\right)^{(r)}ds\right.\\ &+\int_0^t(t-s)^{r-\beta-1}\left(\int_0^s(s-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(s-u)^{\alpha+\beta})h(u)du\right)^{(r)}ds \end{aligned} $$

$$+\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{\tau}\sum_{v=0}^{n-1}c_{vj}(s-t_j)^{\beta+v}\mathbf{E}_{\alpha+\beta,\beta+v+1}(\lambda(s-t_j)^{\alpha+\beta})\right)^{(r)}ds\Bigg]$$

$$+\frac{1}{\Gamma(r-\beta)}\Bigg[\int_{t_i}^{t}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}(s-t_j)^{u}\mathbf{E}_{\alpha+\beta,u+1}(\lambda(s-t_j)^{\alpha+\beta})\right)^{(r)}$$

$$+\int_{t_i}^{t}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}(s-t_j)^{\beta+v}\mathbf{E}_{\alpha+\beta,\beta+v+1}(\lambda(s-t_j)^{\alpha+\beta})\right)^{(r)}ds\Bigg]$$

$$=\frac{1}{\Gamma(r-\beta)}\Bigg[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{\tau}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma(\Gamma((\alpha+\beta)\sigma+u+1)}(s-t_j)^{(\alpha+\beta)\sigma+u}\right)^{(r)}ds$$

$$+\int_0^t(t-s)^{r-\beta-1}\left(\int_0^s\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)(\sigma+1))}(s-u)^{(\alpha+\beta)\sigma+\alpha+\beta-1}h(u)du\right)^{(r)}ds$$

$$+\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{\tau}\sum_{v=0}^{n-1}c_{vj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\beta+v+1)}(s-t_j)^{(\alpha+\beta)\sigma+\beta+v}\right)^{(r)}ds\Bigg]$$

$$+\int_{t_i}^{t}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+u+1)}(s-t_j)^{(\alpha+\beta)\sigma+u}\right)^{(r)}ds$$

$$+\int_{t_i}^{t}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\beta+v+1)}(s-t_j)^{(\alpha+\beta)\sigma+\beta+v}\right)^{(r)}ds\Bigg]$$

$$=\frac{1}{\Gamma(r-\beta)}\Bigg[\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{\tau}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^\sigma}{\Gamma(\Gamma((\alpha+\beta)\sigma+u-r+1)}(s-t_j)^{(\alpha+\beta)\sigma+u-r}\right)ds$$

$$+\int_0^t(t-s)^{r-\beta-1}\left(\int_0^s\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)(\sigma+1)-r+1)}(s-u)^{(\alpha+\beta)\sigma+\alpha+\beta-r}h(u)du\right)'ds$$

$$+\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{\tau}\sum_{v=0}^{n-1}c_{vj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\beta-r+v+1)}(s-t_j)^{(\alpha+\beta)\sigma+\beta-r+v}\right)ds\Bigg]$$

$$+\int_{t_i}^{t}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma-r+u+1)}(s-t_j)^{(\alpha+\beta)\sigma-r+u}\right)ds$$

$$+\int_{t_i}^{t}(t-s)^{r-\beta-1}\left(\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\beta-r+v+1)}(s-t_j)^{(\alpha+\beta)\sigma+\beta-r+v}\right)ds\Bigg]$$

$$=\frac{1}{\Gamma(r-\beta)}\Bigg[\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^\sigma}{\Gamma(\Gamma((\alpha+\beta)\sigma+u-r+1)}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}(s-t_j)^{(\alpha+\beta)\sigma+u-r}ds$$

$$+\frac{1}{r-\beta}\left(\int_0^t(t-s)^{r-\beta}\left(\int_0^s\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)(\sigma+1)-r+1)}(s-u)^{(\alpha+\beta)\sigma+\alpha+\beta-r}h(u)du\right)'ds\right)'$$

$$+\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}\sum_{v=0}^{n-1}c_{vj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\beta-r+v+1)}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{r-\beta-1}(s-t_j)^{(\alpha+\beta)\sigma+\beta-r+v}ds\Bigg]$$

$$+\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma-r+u+1)}\int_{t_i}^{t}(t-s)^{r-\beta-1}(s-t_j)^{(\alpha+\beta)\sigma-r+u})ds$$

$$+\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\beta-r+v+1)}\int_{t_i}^{t}(t-s)^{r-\beta-1}(s-t_j)^{(\alpha+\beta)\sigma+\beta-r+v}ds\Bigg]$$

$$=\frac{1}{\Gamma(r-\beta)}[M_1+M_2+M_3+M_4+M_5].$$

One sees that

$$\begin{aligned}
M_2 =& (t-s)^{r-\beta}\left(\int_0^s\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)(\sigma+1)-r+1)}(s-u)^{(\alpha+\beta)\sigma+\alpha+\beta-r}h(u)du\right)\Bigg|_0^t\\
&=(r-\beta)\left[\int_0^t(t-s)^{r-\beta-1}\int_0^s\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)(\sigma+1)-r+1)}(s-u)^{(\alpha+\beta)\sigma+\alpha+\beta-r}h(u)duds\right]'\\
&=(r-\beta)\left[\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)(\sigma+1)-r+1)}\int_0^t\int_s^t(t-s)^{r-\beta-1}(s-u)^{(\alpha+\beta)\sigma+\alpha+\beta-r}dsh(u)du\right]'\\
&=(r-\beta)\left[\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)(\sigma+1)-r+1)}\int_0^t(t-u)^{(\alpha+\beta)\sigma+\alpha}\right.\\
&\quad\left.\times\int_0^1(1-w)^{r-\beta-1}w^{(\alpha+\beta)\sigma+\alpha+\beta-r}dwh(u)du\right]'\\
&=\Gamma(r-\beta+1)\left[\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\alpha+1)}\int_0^t(t-u)^{(\alpha+\beta)\sigma+\alpha}h(u)du\right]'\\
&=\Gamma(r-\beta+1)\sum_{\sigma=0}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma+\alpha)}\int_0^t(t-u)^{(\alpha+\beta)\sigma+\alpha-1}h(u)du\\
&=\Gamma(r-\beta+1)\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})h(u)du,
\end{aligned}$$

$$\begin{aligned}
M_1+M_4 =& \sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^\sigma}{\Gamma(\Gamma((\alpha+\beta)\sigma+u-r+1)}(t-t_j)^{(\alpha+\beta)\sigma+u-\beta}\\
&\times\int_{\frac{t_\tau-t_j}{t-t_j}}^{\frac{t_{\tau+1}-t_j}{(t-t_j}}(1-w)^{r-\beta-1}w^{(\alpha+\beta)\sigma+u-r}dw\\
&+\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^\sigma}{\Gamma((\alpha+\beta)\sigma-r+u+1)}(t-t_j)^{(\alpha+\beta)\sigma+u-\beta}\\
&\times\int_{\frac{t_i-t_j}{t-t_j}}^{1}(1-w)^{r-\beta-1}w^{(\alpha+\beta)\sigma-r+u})dw\\
=&\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^\sigma}{\Gamma(\Gamma((\alpha+\beta)\sigma+u-r+1)}(t-t_j)^{(\alpha+\beta)\sigma+u-\beta}\\
&\times\int_{\frac{t_\tau-t_j}{t-t_j}}^{\frac{t_{\tau+1}-t_j}{(t-t_j}}(1-w)^{r-\beta-1}w^{(\alpha+\beta)\sigma+u-r}dw
\end{aligned}$$

$$+\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^{\sigma}}{\Gamma((\alpha+\beta)\sigma-r+u+1)}(t-t_j)^{(\alpha+\beta)\sigma+u-\beta}$$
$$\times\int_{\frac{t_i-t_j}{t-t_j}}^{1}(1-w)^{r-\beta-1}w^{(\alpha+\beta)\sigma-r+u})dw$$
$$=\sum_{j=0}^{i}\sum_{u=0}^{r-1}d_{uj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^{\sigma}}{\Gamma((\alpha+\beta)\sigma-r+u+1)}(t-t_j)^{(\alpha+\beta)\sigma+u-\beta}$$
$$\times\int_{0}^{1}(1-w)^{r-\beta-1}w^{(\alpha+\beta)\sigma-r+u})dw.$$

Similarly we have

$$M_3+M_5=\sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}\sum_{\sigma=1}^{+\infty}\frac{\lambda^{\sigma}}{\Gamma((\alpha+\beta)\sigma-r+u+1)}(t-t_j)^{(\alpha+\beta)\sigma+u-\beta}$$
$$\times\int_{0}^{1}(1-w)^{r-\beta-1}w^{(\alpha+\beta)\sigma-r+u})dw.$$

Hence

$$\begin{aligned}{}^{c}D_{0^+}^{\beta}x(t) &= \frac{1}{\Gamma(r-\beta)}[M_1+M_2+M_3+M_4+M_5]\\ &= \sum_{j=0}^{i}\sum_{v=0}^{n-1}c_{vj}(t-t_j)^{v}\mathbf{E}_{\alpha+\beta,v+1}(\lambda(t-t_j)^{\alpha+\beta})\\ &\quad+\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})h(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}$$

The proof is complete. □

4. Existence results for BVPs of IFDEs

In this section, we use Lemma 2.1 and Lemma 2.2 to establish new existence results for solutions of BVP(1.4.5)–BVP(1.4.12) respectively. We firstly define Banach spaces, then we convert these boundary value problems for impulsive fractional differential equation with the Riemann-Liouville fractional derivatives to equivalent integral equations. Finally we establish existence results by fixed point theorems (Lemma 2.1 and Lemma 2.2 in Section 2).

Define

$$X=\left\{x:(0,1]\to\mathbb{R}:\begin{array}{c}x|_{(t_s,t_{s+1}]},D_{0^+}^{p}x|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]\quad(s\in\mathbb{N}_0^m),\\ \lim_{t\to t_s^+}(t-t_s)^{n-\alpha}x(t),\qquad \lim_{t\to t_s^+}(t-t_s)^{n+p-\alpha}D_{0^+}^{p}x(t)\\ \text{exist and are finite}\quad(s\in\mathbb{N}_0^m)\end{array}\right\}.$$

For $x\in X$, define the norms by

$$\|x\|=\|x\|_X=\max\left\{\sup_{t\in(t_s,t_{s+1}]}(t-t_s)^{n-\alpha}|x(t)|,\ \sup_{t\to t_s^+}(t-t_s)^{n+p-\alpha}|D_{0^+}^{p}x(t)|:s\in\mathbb{N}_0^m\right\}.$$

Define

$$Y=\left\{x:(0,1]\to\mathbb{R}:\begin{array}{l} x|_{(t_s,t_{s+1}]},{}^cD_{0^+}^px|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}]\quad (s\in\mathbb{N}_0^m),\\ \lim\limits_{t\to t_s^+}x(t),\quad \lim\limits_{t\to t_s^+}{}^cD_{0^+}^px(t)\\ \text{exist and are finite}\quad (s\in\mathbb{N}_0^m)\end{array}\right\}.$$

For $x\in Y$, define the norms by

$$\|x\|=\|x\|_X=\max\left\{\sup_{t\in(t_s,t_{s+1}]}|x(t)|,\sup_{t\to t_s^+}|{}^cD_{0^+}^px(t)|:s\in\mathbb{N}_0^m\right\}.$$

LEMMA 4.1. *Both X and Y are Banach spaces.*

Proof. The proof is standard and omitted. □

LEMMA 4.2. *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied:*

(i) both $\{t\to(t-t_s)^{n-\alpha}x(t):x\in M\}$ and $\left\{t\to(t-t_s)^{n+p-\alpha}D_{0^+}^px(t):x\in M\right\}$ are uniformly bounded,

(ii) both $\{t\to(t-t_s)^{n-\alpha}x(t):x\in M\}$ and $\left\{t\to(t-t_s)^{n+p-\alpha}D_{0^+}^px(t):x\in M\right\}$ are equicontinuous in any interval $(t_s,t_{s+1}]$ $(s\in\mathbb{N}_0^m)$.

Proof. The proof is standard and omitted. □

LEMMA 4.3. *Let M be a subset of Y. Then M is relatively compact if and only if the following conditions are satisfied:*

(i) both $\{t\to x(t):x\in M\}$ and $\left\{t\to{}^cD_{0^+}^px(t):x\in M\right\}$ are uniformly bounded,

(ii) both $\{t\to x(t):x\in M\}$ and $\left\{t\to{}^cD_{0^+}^px(t):x\in M\right\}$ are equicontinuous in any interval $(t_s,t_{s+1}]$ $(s\in\mathbb{N}_0^m)$.

Proof. The proof is standard and omitted. □

REMARK 4.1. For a boundary value problem, the initial data at 0 may be $x(0)=A$, or $\lim\limits_{t\to0}t^{n-\alpha}x(t)=B$ or $I_{0^+}^{n-\alpha}x(0)=C$. In BVP(1.4.5), $x(0)=A$ is unsuitable since each solutions of (1.4.5) may be un-continuous at 0 unless $A=0$ which is always replaced by the second one [48]. Suppose that $x\in C^0(0,t_1]$, $\alpha\in(n-1,n)$, A is a constant. Then $\lim\limits_{t\to0^+}t^{n-\alpha}x(t)=A$ implies $\lim\limits_{t\to0^+}I_{0^+}^{n-\alpha}x(t)=\Gamma(\alpha-n+1)A$. In fact, for each $\varepsilon>0$, by $\lim\limits_{t\to0^+}t^{n-\alpha}x(t)=A$, there exists $\delta\in(0,t_1]$ such that $A-\varepsilon<t^{n-\alpha}x(t)<A+\varepsilon$, $t\in(0,\delta)$. Note

$$\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}s^{\alpha-n}ds=\int_0^1\frac{(1-w)^{n-\alpha-1}}{\Gamma(n-\alpha)}w^{\alpha-n}dw=\Gamma(\alpha-n+1).$$

Then for $t\in(0,\delta)$, we have

$$\begin{aligned}|I_{0^+}^{n-\alpha}x(t)-\Gamma(\alpha-n+1)A|&=\left|\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}x(s)ds-\Gamma(\alpha-n+1)A\right|\\&\leqslant\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}s^{\alpha-n}\,|s^{n-\alpha}x(s)-A|\,ds<\varepsilon\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}s^{\alpha-n}\\&=\Gamma(\alpha-n+1)\varepsilon.\end{aligned}$$

This is $\lim_{t\to 0^+} I_{0^+}^{n-\alpha}x(t)=\Gamma(\alpha-n+1)A$. However we do not know that does $\lim_{t\to 0^+} I_{0^+}^{n-\alpha}x(t) =\Gamma(\alpha-n+1)A$ implies $\lim_{t\to 0^+} t^{n-\alpha}x(t)=A$?

4.1. Solvability of BVP(1.4.5)

In this subsection, we establish existence result for solutions of BVP(1.4.5) by using Lemma 2.1. For ease expression, we denote

$$\begin{aligned}
&I_x(t_s)=I(t_s,x(t_s),D_{0^+}^p x(t_s)), \quad s\in \mathbb{N}_1^m,\\
&J_x(t_s)=J(t_s,x(t_s),D_{0^+}^p x(t_s)), \quad s\in \mathbb{N}_1^m,\\
&f_x(t)=f(t,x(t),D_{0^+}^p x(t), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\
&M=\lambda\mathbf{E}_{\alpha,2}(\lambda)\mathbf{E}_{\alpha,\alpha}(\lambda)-(1-\mathbf{E}_{\alpha,1}(\lambda))^2,\\
&e_{a,b}^{c,d}(t,s)=(t-s)^c\mathbf{E}_{a,b}(\lambda(t-s)^d), \quad 1\geqslant t\geqslant s\geqslant 0,\ d>0,\ c\in\mathbb{R},\ a>0, \quad b>0.
\end{aligned}$$

REMARK 4.1.1. It is easy to see that

$$\begin{aligned}
&0\leqslant e_{a,b}^{c,d}(t,s)\leqslant \mathbf{E}_{a,b}(|\lambda|),\ 0\leqslant s\leqslant t\leqslant 1,\ c\geqslant 0,\ d\geqslant 0,\ a>0,\ b>0,\\
&(t-u)^{-c}e_{a,b}^{c,d}(t,v)\leqslant \mathbf{E}_{a,b}(|\lambda|),\ c<0,\ a>0,\ b>0,\ 0\leqslant v\leqslant\leqslant u\leqslant t\leqslant 1.
\end{aligned}$$

LEMMA 4.1.1. *x is a solution of BVP(1.4.5) if and only if*

$$\begin{aligned}
x(t)=&-\sum_{j=1}^m\Bigg[\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,t_j)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)]\\
&+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,t_j)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)]\Bigg]J_x(t_j)\\
&-\sum_{j=1}^m\Bigg[\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\\
&+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\Bigg]I_x(t_j)\\
&-\Bigg[\int_0^1\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,s)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,s)]\\
&+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,s)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,s)]\Bigg]f_x(s)ds\\
&+\sum_{j=1}^i e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,t_j)J_x(t_j)+\sum_{j=1}^i e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,t_j)I_x(t_j)\\
&+\int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)f_x(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \qquad (4.1.1)
\end{aligned}$$

Proof. Suppose that x is a solution of BVP(1.4.5). By Theorem 3.1.1, there exist constants $c_{vj} \in \mathbb{R}$ $(v \in \mathbb{N}_1^2,\ j \in \mathbb{N}_0^m)$ such that

$$x(t) = \sum_{j=0}^{i} \sum_{v=1}^{2} c_{vj} e_{\alpha,\alpha-v+1}^{\alpha-v,\alpha}(t,t_j) + \int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s) f_x(s) ds, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{4.1.2}$$

Use Corollary 3.1.2, we get

$$I_{0^+}^{2-\alpha} x(t) = \sum_{j=0}^{i} \sum_{v=1}^{2} c_{vj} e_{\alpha,3-v}^{2-v,\alpha}(t,t_j) + \int_0^t e_{\alpha,2}^{1,\alpha}(t,s) f_x(s) ds, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m \tag{4.1.3}$$

and

$$D_{0^+}^{\alpha-1} x(t) = \sum_{j=0}^{i} c_{1j} e_{\alpha,1}^{0,\alpha}(t,t_j) + \lambda \sum_{j=0}^{i} c_{2j} e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,t_j) + \int_0^t e_{\alpha,1}^{0,\alpha}(t,s) f_x(s) ds,$$
$$t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{4.1.4}$$

(i) From (4.1.2) and $\lim_{t \to t_s^+} (t-t_s)^{2-\alpha} x(t) = \frac{I_x(t_s)}{\Gamma(\alpha-1)}$, $s \in \mathbb{N}_1^m$, we have

$$lc_{2s} = I_x(t_s), \quad s \in \mathbb{N}_1^m. \tag{4.1.5}$$

(ii) From (4.1.3) and $I_{0^+}^{2-\alpha} x(0) = I_{0^+}^{2-\alpha} x(1)$, we have

$$c_{20} = \sum_{j=0}^{m} \sum_{v=1}^{2} c_{vj} e_{\alpha,3-v}^{2-v,\alpha}(1,t_j) + \int_0^1 e_{\alpha,2}^{1,\alpha}(1,s) f_x(s) ds.$$

(iii) From (4.1.4) and $D_{0^+}^{\alpha-1} x(0) = D_{0^+}^{\alpha-1} x(1)$, we have

$$c_{10} = \sum_{j=0}^{m} c_{1j} e_{\alpha,1}^{0,\alpha}(1,t_j) + \lambda \sum_{j=0}^{m} c_{2j} e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j) + \int_0^1 e_{\alpha,1}^{0,\alpha}(1,s) f_x(s) ds.$$

(iv) From (4.1.4) and $\Delta D_{0^+}^{\alpha-1} x(t_s) = J_x(t_s)$, $s \in \mathbb{N}_1^m$, we get

$$lc_{1s} = J_x(t_s), \quad s \in \mathbb{N}_1^m. \tag{4.1.6}$$

Substituting (iv) and (i) into (ii) and (iii), we have

$$\begin{aligned} c_{20} &= \sum_{v=1}^{2} c_{v0} \mathbf{E}_{\alpha,2-v+1}(\lambda) + \sum_{j=1}^{m} J_x(t_j) e_{\alpha,2}^{1,\alpha}(1,t_j) \\ &\quad + \sum_{j=1}^{m} I_x(t_j) e_{\alpha,1}^{0,\alpha}(1,t_j) + \int_0^1 e_{\alpha,2}^{1,\alpha}(1,s) f_x(s) ds, \\ c_{10} &= c_{10} \mathbf{E}_{\alpha,1}(\lambda) + \sum_{j=1}^{m} J_x(t_j) e_{\alpha,1}^{0,\alpha}(1,t_j) + \lambda c_{20} \mathbf{E}_{\alpha,\alpha}(\lambda) \\ &\quad + \lambda \sum_{j=1}^{m} I_x(t_j) e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j) + \int_0^1 e_{\alpha,1}^{0,\alpha}(1,s) f_x(s) ds. \end{aligned}$$

It follows that

$$c_{10} = \frac{1}{M}\Big[-\sum_{j=1}^{m}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,t_j)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)]J_x(t_j)$$
$$-\sum_{j=1}^{m}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]I_x(t_j)$$
$$-\int_0^1[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,s)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,s)]f_x(s)ds\Big], \quad (4.1.7)$$

and

$$c_{20} = \frac{1}{M}\Big[-\sum_{j=1}^{m}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,t_j)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)]J_x(t_j)$$
$$-\sum_{j=1}^{m}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]I_x(t_j)$$
$$-\int_0^1[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,s)f_x(s)ds+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,s)]f_x(s)ds\Big]. \quad (4.1.8)$$

Substituting (4.1.5), (4.1.6), (4.1.7) and (4.1.8) into (4.1.2), we get (4.1.1). On the other hand, if x satisfies (4.1.1), we can prove that x is a solution of (1.4.5) by direct computation. We omit the details. □

Define the nonlinear operator T on X by

$$(Tx)(t) = -\sum_{j=1}^{m}\Bigg[\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,t_j)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)]$$
$$+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,t_j)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)]\Bigg]J_x(t_j)$$
$$-\sum_{j=1}^{m}\Bigg[\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]$$
$$+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\Bigg]I_x(t_j)$$
$$-\Bigg[\int_0^1\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,s)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,s)]$$
$$+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,s)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,s)]\Bigg]f_x(s)ds$$
$$+\sum_{j=1}^{i}e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,t_j)J_x(t_j)+\sum_{j=1}^{i}e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,t_j)I_x(t_j)$$
$$+\int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)f_x(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.$$

Denote

$$M_1 = \Big[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\mathbf{E}_{\alpha,2}(\lambda)}{M}+m\Big]A_J$$
$$+\Big[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\lambda\mathbf{E}_{\alpha,2}(\lambda)}{M}+m\Big]A_I$$

$$+\left[\frac{\mathbf{E}_{\alpha,\alpha}(\lambda)+(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{(1-\mathbf{E}_{\alpha,1}(\lambda))+\mathbf{E}_{\alpha,2}(\lambda)}{M}+\mathbf{B}(l+1,k+1)\right]\mathbf{B}(l+1,k+1)a_f,$$

$$M_2=\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\mathbf{E}_{\alpha,2}(\lambda)}{M}+m\right]B_J$$
$$+\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\lambda\mathbf{E}_{\alpha,2}(\lambda)}{M}+m\right]B_I$$
$$+\left[\frac{\mathbf{E}_{\alpha,\alpha}(\lambda)+(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{(1-\mathbf{E}_{\alpha,1}(\lambda))+\mathbf{E}_{\alpha,2}(\lambda)}{M}+\mathbf{B}(l+1,k+1)\right]\mathbf{B}(l+1,k+1)b_f,$$

$$M_3=\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\mathbf{E}_{\alpha,2}(\lambda)}{M}+m\right]C_J$$
$$+\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\lambda\mathbf{E}_{\alpha,2}(\lambda)}{M}+m\right]C_I$$
$$+\left[\frac{\mathbf{E}_{\alpha,\alpha}(\lambda)+(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{(1-\mathbf{E}_{\alpha,1}(\lambda))+\mathbf{E}_{\alpha,2}(\lambda)}{M}+\mathbf{B}(l+1,k+1)\right]\mathbf{B}(l+1,k+1)c_f.$$

THEOREM 4.1.1. *Suppose that there exist nonnegative constants* μ, a_f, b_f, c_f, A_I, B_I, C_I, A_J, B_J, C_J *such that*

$$\left|f\left(t,\frac{u}{(t-t_s)^{2-\alpha}},\frac{v}{(t-t_s)^{2+p-\alpha}}\right)\right|\leqslant[a_f+b_f|u|^\mu+c_f|v|^\mu]t^k(1-t)^l,$$
$$t\in(t_s,t_{s+1}],\ \ s\in\mathbb{N}_0^m,\ \ u,v\in\mathbb{R},$$
$$\left|I\left(t_s,\frac{u}{(t_s-t_{s-1})^{2-\alpha}},\frac{v}{(t_s-t_{s-1})^{2+p-\alpha}}\right)\right|\leqslant A_I+B_I|u|^\mu+C_I|v|^\mu,\ \ s\in\mathbb{N}_1^m,\ \ u,v\in\mathbb{R},$$
$$\left|J\left(t_s,\frac{u}{(t_s-t_{s-1})^{2-\alpha}},\frac{v}{(t_s-t_{s-1})^{2+p-\alpha}}\right)\right|\leqslant A_J+B_J|u|^\mu+C_J|v|^\mu,\ \ s\in\mathbb{N}_1^m,\ \ u,v\in\mathbb{R}.\tag{4.1.9}$$

Then BVP(1.4.5) has at least one solution if

$$\begin{aligned}&\mu<1\quad or\quad \mu=1\quad with\quad M_2+M_3<1\quad or\\ &\mu>1\quad with\quad M_1+(M_2+M_3)\left(\frac{M_1}{(\mu-1)(M_2+M_3)}\right)^\mu\leqslant\frac{M_1}{(\mu-1)(M_2+M_3)}.\end{aligned}\tag{4.1.10}$$

Proof. Let T be defined above. By a standard proof, we know that $T:X\to X$ is well defined and T is completely continuous. By Lemma 4.1.1, we know that $x\in X$ is a solution of BVP(1.4.5) if and only if $x=Tx$. So we will seek some fixed point of T in X by using Schauder's fixed point theorem.

By the definition of T and and the method used in the proof of Corollary 3.2.1, we have for $t\in(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m)$ that

$$D_{0^+}^p(Tx)(t)=-\sum_{j=1}^m\left[\frac{e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,t_j)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)]\right.$$
$$\left.+\frac{e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,t_j)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)]\right]J_x(t_j)$$
$$-\sum_{j=1}^m\left[\frac{e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\right.$$

$$
\begin{aligned}
&+\frac{e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\Bigg]I_x(t_j)\\
&-\int_0^1\left[\frac{e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,s)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,s)]\right.\\
&\left.+\frac{e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,s)f_x(s)ds+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,s)]\right]f_x(s)ds\\
&+\sum_{j=1}^{i}e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,t_j)J_x(t_j)+\sum_{j=1}^{i}e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,t_j)I_x(t_j)\\
&+\int_0^t e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,s)f_x(s)ds. \qquad (4.1.11)
\end{aligned}
$$

By assumption in this theorem, we know

$$
\begin{aligned}
\left|f\left(t,x(t),D_{0+}^p x(t)\right)\right| &= \left|f\left(t,\frac{(t-t_s)^{2-\alpha}x(t)}{(t-t_s)^{2-\alpha}},\frac{(t-t_s)^{2+p-\alpha}D_{0+}^p x(t)}{(t-t_s)^{2+p-\alpha}}\right)\right|\\
&\leqslant \left[a_f+b_f\left|(t-t_s)^{2-\alpha}x(t)\right|^{\mu}+c_f\left|(t-t_s)^{2+p-\alpha}D_{0+}^p x(t)\right|^{\mu}\right]t^k(1-t)^l\\
&\leqslant \left[a_f+b_f||x||^{\mu}+c_f||x||^{\mu}\right]t^k(1-t)^l, \quad t\in(t_s,t_{s+1}], \quad s\in\mathbb{N}_0^m,\\
\left|I\left(t_s,x(t_s),D_{0+}^p x(t_s)\right)\right| &\leqslant A_I+B_I||x||^{\mu}+C_I||x||^{\mu}, \quad s\in\mathbb{N}_1^m,\\
\left|J\left(t_s,x(t_s),D_{0+}^p x(t_s)\right)\right| &\leqslant A_J+B_J||x||^{\mu}+C_J||x||^{\mu}, \quad s\in\mathbb{N}_1^m. \qquad (4.1.12)
\end{aligned}
$$

Hence for $t\in(t_i,t_{i+1}]$, use Remark 4.1, and (4.1.12), we have

$$
\begin{aligned}
&(t-t_i)^{2-\alpha}|(Tx)(t)|\\
&=(t-t_i)^{2-\alpha}\Bigg|-\sum_{j=1}^{m}\left[\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,t_j)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)]\right.\\
&\left.+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,t_j)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)]\right]J_x(t_j)\\
&-\sum_{j=1}^{m}\left[\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)+\sum_{j=1}^{m}\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\right.\\
&\left.+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\right]I_x(t_j)\\
&-\left[\int_0^1\frac{e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,s)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,s)]\right.\\
&\left.+\frac{e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,s)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,s)]\right]f_x(s)ds\\
&+\sum_{j=1}^{i}e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,t_j)J_x(t_j)+\sum_{j=1}^{i}e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,t_j)I_x(t_j)+\int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)f_x(s)ds\Bigg|\\
&\leqslant\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\mathbf{E}_{\alpha,2}(\lambda)}{M}\right][A_J+B_J||x||^{\mu}+C_J||x||^{\mu}]\\
&+\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\lambda\mathbf{E}_{\alpha,2}(\lambda)}{M}\right][A_I+B_I||x||^{\mu}+C_I||x||^{\mu}]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\int_0^1 \frac{1}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)+(1-\mathbf{E}_{\alpha,1}(\lambda))]+\frac{1}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))+\mathbf{E}_{\alpha,2}(\lambda)]\right] s^k(1-s)^l ds \\
&+m[A_J+B_J||x||^\mu+C_J||x||^\mu]+m[A_I+B_I||x||^\mu+C_I||x||^\mu] \\
&+\int_0^1 \left[a_f+b_f||x||^\mu+c_f||x||^\mu\right] s^k(1-s)^l ds \\
=&\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\mathbf{E}_{\alpha,2}(\lambda)}{M}\right][A_J+B_J||x||^\mu+C_J||x||^\mu] \\
&+\left[\frac{m\mathbf{E}_{\alpha,\alpha}(\lambda)+m\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{m(1-\mathbf{E}_{\alpha,1}(\lambda))+m\lambda\mathbf{E}_{\alpha,2}(\lambda)}{M}\right][A_I+B_I||x||^\mu+C_I||x||^\mu] \\
&+\left[\frac{\mathbf{E}_{\alpha,\alpha}(\lambda)+(1-\mathbf{E}_{\alpha,1}(\lambda))}{M}+\frac{(1-\mathbf{E}_{\alpha,1}(\lambda))+\mathbf{E}_{\alpha,2}(\lambda)}{M}\right]\mathbf{B}(l+1,k+1)\left[a_f+b_f||x||^\mu+c_f||x||^\mu\right] \\
&+m[A_J+B_J||x||^\mu+C_J||x||^\mu]+m[A_I+B_I||x||^\mu+C_I||x||^\mu] \\
&+\left[a_f+b_f||x||^\mu+c_f||x||^\mu\right]\mathbf{B}(l+1,k+1) \\
=&M_1+M_2||x||^\mu+M_3||x||^\mu.
\end{aligned}
$$

Similarly for $t\in(t_i,t_{i+1}]$, using Remark 4.1, we have

$$
\begin{aligned}
&(t-t_i)^{2+p-\alpha}|D_{0^+}^p(Tx)(t)| \\
\leqslant&(t-t_i)^{2+p-\alpha}\Bigg|-\sum_{j=1}^m\left[\frac{e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,t_j)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)]\right. \\
&\left.+\frac{e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,t_j)+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)]\right]J_x(t_j) \\
&-\sum_{j=1}^m\left[\frac{e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\right. \\
&\left.+\frac{e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,t_j)+\lambda\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)]\right]I_x(t_j) \\
&-\int_0^1\left[\frac{e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,0)}{M}[\mathbf{E}_{\alpha,\alpha}(\lambda)e_{\alpha,2}^{1,\alpha}(1,s)+(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,1}^{0,\alpha}(1,s)]\right. \\
&\left.+\frac{e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,0)}{M}[(1-\mathbf{E}_{\alpha,1}(\lambda))e_{\alpha,2}^{1,\alpha}(1,s)f_x(s)ds+\mathbf{E}_{\alpha,2}(\lambda)e_{\alpha,1}^{0,\alpha}(1,s)]\right]f_x(s)ds \\
&+\sum_{j=1}^i e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,t_j)J_x(t_j)+\sum_{j=1}^i e_{\alpha,\alpha-p-1}^{\alpha-p-2,\alpha}(t,t_j)I_x(t_j)+\int_0^t e_{\alpha,\alpha-p}^{\alpha-p-1,\alpha}(t,s)f_x(s)ds\Bigg| \\
\leqslant&N_1+N_2||x||^\mu+N_3||x||^\mu.
\end{aligned}
$$

It follows that

$$||Tx||\leqslant M_1+M_2||x||^\mu+M_3||x||^\mu. \tag{4.1.13}$$

We consider the inequality $M_1+M_2r^\mu+M_3r^\mu\leqslant r$.

(i) If $\mu\in[0,1)$, it is easy to see that there exists sufficiently small $r_0>0$ such that $M_1+M_2r_0^\mu+M_3r_0^\mu\leqslant r_0$. Let $\Omega_0=\{x\in X:||x||\leqslant r_0\}$. It is easy to see that $T\Omega_0\subseteq\Omega_0$. Hence Schauder's fixed point theorem (Lemma 2.1)implies that T has fixed point in X. This fixed point x is a solution of BVP(1.4.5).

(ii) If $\mu = 1$, we choose $r_0 > \frac{M_1}{1-M_2-M_3}$. Let $\Omega_0 = \{x \in X : ||x|| \leqslant r_0\}$. It is easy to see that $T\Omega_0 \subseteq \Omega_0$. Hence Schauder's fixed point theorem (Lemma 2.1)implies that T has fixed point in X. This fixed point x is a solution of BVP(1.4.5).

(iii) If $\mu > 1$, we choose $r_0 = \frac{M_1}{(\mu-1)(M_2+M_3)}$. Let $\Omega_0 = \{x \in X : ||x|| \leqslant r_0\}$. By

$$M_1 + (M_2+M_3)\left(\frac{M_1}{(\mu-1)(M_2+M_3)}\right)^{\mu} \leqslant \frac{M_1}{(\mu-1)(M_2+M_3)},$$

it is easy to see that $T\Omega_0 \subseteq \Omega_0$. Hence Schauder's fixed point theorem (Lemma 2.1) implies that T has fixed point in X. This fixed point x is a solution of BVP(1.4.5). The proof is complete. □

THEOREM 4.1.2. *Suppose that there exist constants $M_f \geqslant 0$, $M_I \geqslant 0$, $M_J \geqslant 0$ such that*

$$\left|f\left(t, \frac{u}{(t-t_s)^{2-\alpha}}, \frac{v}{(t-t_s)^{2+p-\alpha}}\right)\right| \leqslant M_f, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m, \quad u, v \in \mathbb{R},$$

$$\left|I\left(t_s, \frac{u}{(t_s-t_{s-1})^{2-\alpha}}, \frac{v}{(t_s-t_{s-1})^{2+p-\alpha}}\right)\right| \leqslant M_I, \quad s \in \mathbb{N}_1^m, \quad u, v \in \mathbb{R},$$

$$\left|J\left(t_s, \frac{u}{(t_s-t_{s-1})^{2-\alpha}}, \frac{v}{(t_s-t_{s-1})^{2+p-\alpha}}\right)\right| \leqslant M_J, \quad s \in \mathbb{N}_1^m, \quad u, v \in \mathbb{R}.$$

Then BVP(1.4.5) has at least one solution.

Proof. In Theorem 4.1.1, choose $a_f = M_f$, $b_f = c_f = 0$, $A_I = M_I$, $B_I = C_I = 0$, $A_J = M_J$, $B_J = C_J = 0$ and $\mu = 0$. Then assumptions in Theorem 4.1.1 hold. The result follows from Theorem 4.1.1 directly. □

4.2. Solvability of BVP(1.4.6)

In this subsection, we establish existence result for solutions of BVP(1.4.6) by using Lemma 2.1. Denote

$$\begin{aligned} f_x(t) &= f(t, x(t), D_{t_i^+}^p x(t)), \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m, \\ I_x(t_s) &= I(t_s, u(t_s), D_{t_{s-1}^+}^p x(t_s)), \quad s \in \mathbb{N}_1^m, \\ J_x(t_s) &= J(t_s, u(t_s), D_{t_{s-1}^+}^p x(t_s)), \quad s \in \mathbb{N}_1^m. \end{aligned}$$

LEMMA 4.2.1. *x is a solution of BVP(1.4.6) if and only if*

$$x(t) = \begin{cases} \left[e_{\alpha,1}^{0,\alpha}(1,t_m)J_x(t_m) + \int_{t_m}^1 e_{\alpha,1}^{0,\alpha}(1,s)f_x(s)ds\right] e_{\alpha,\alpha-2}^{\alpha-1,\alpha}(t,0) \\ \left[e_{\alpha,2}^{1,\alpha}(1,t_m)J_x(t_m) + e_{\alpha,1}^{0,\alpha}(1,t_m)I_x(t_m) + \int_{t_m}^1 e_{\alpha,2}^{1,\alpha}(1,s)f_x(s)ds\right] e_{\alpha,\alpha-3}^{\alpha-2,\alpha}(t,0) \\ \qquad + \int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)f_x(s)ds, \quad t \in (t_0,t_1], \\ e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,t_i)I_x(t_i) + e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,t_i)J_x(t_i) \\ \qquad + \int_{t_i}^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)f_x(s)ds, \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_1^m. \end{cases} \tag{4.2.1}$$

Proof. Suppose that x is a solution of BVP(1.4.6). By Theorem 3.1.1, there exist constants $c_{vj} \in \mathbb{R}$ $(v \in \mathbb{N}_1^n,\ j \in \mathbb{N}_0^m)$ such that

$$x(t) = \sum_{v=1}^{n} c_{vi} e_{\alpha,\alpha-v+1}^{\alpha-v,\alpha}(t,t_i) + \int_{t_i}^{t} e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s) f_x(s)ds, \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m. \quad (4.2.2)$$

By Corollary 3.1.2, we have

$$D_{t_i^+}^{\alpha-\sigma} x(t) = \sum_{v=1}^{\sigma} c_{vi} e_{\alpha,\sigma-v+1}^{\sigma-v,\alpha}(t,t_i) + \int_{t_i}^{t} e_{\alpha,\sigma}^{\sigma-1,\alpha}(t,s) f_x(s)ds, \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m \quad (4.2.3)$$

and

$$I_{t_i^+}^{n-\alpha} x(t) = \sum_{v=1}^{n} c_{vi} e_{\alpha,3-v}^{2-v,\alpha}(t,t_i) + \int_{t_i}^{t} e_{\alpha,2}^{1,\alpha}(t,s) f_x(s)ds, t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m. \quad (4.2.4)$$

(i) From $I_{0^+}^{2-\alpha} x(0) = I_{t_m^+}^{2-\alpha} x(1)$ and (4.2.3), we have

$$c_{20} = \sum_{v=1}^{2} c_{vm} e_{\alpha,3-v}^{2-v,\alpha}(1,t_m) + \int_{t_m}^{1} e_{\alpha,2}^{1,\alpha}(1,s) f_x(s)ds. \quad (4.2.5)$$

(ii) From $D_{0^+}^{\alpha-1} x(0) = D_{t_m^+}^{\alpha-1} x(1)$ and (4.2.2), we have

$$c_{10} = c_{1m} e_{\alpha,1}^{0,\alpha}(1,t_m) + \int_{t_m}^{1} e_{\alpha,1}^{0,\alpha}(1,s) f_x(s)ds. \quad (4.2.6)$$

(iii) From $\lim_{t \to t_s^+} (t-t_s)^{2-\alpha} x(t) = \frac{I_{x,n}(t_s)}{\Gamma(\alpha-1)}$, $s \in \mathbb{N}_1^m$ and (4.2.2), we have

$$c_{2s} = I_x(t_s), \quad s \in \mathbb{N}_1^m. \quad (4.2.7)$$

(iv) From $D_{t_s^+}^{\alpha-1} x(t_s) = J_x(t_s)$, $s \in \mathbb{N}_1^m$ and (4.2.2), we have

$$c_{1s} = J_x(t_s)(s \in \mathbb{N}_1^m). \quad (4.2.8)$$

Substituting (4.2.7)–(4.2.8) into (4.2.5) and (4.2.6), we have

$$c_{20} = J_x(t_m) e_{\alpha,\alpha}^{1,\alpha}(1,t_m) + I_x(t_m) e_{\alpha,\alpha-1}^{0,\alpha}(1,t_m) + \int_{t_m}^{1} e_{\alpha,2}^{1,\alpha}(1,s) f_x(s)ds,$$
$$c_{10} = J_x(t_m) e_{\alpha,1}^{0,\alpha}(1,t_m) + \int_{t_m}^{1} e_{\alpha,1}^{0,\alpha}(1,s) f_x(s)ds.$$

Substituting all c_{1s} into (4.2.2), we get (4.2.1). One the other hand, we can prove that x is a solution of (1.4.6) if x satisfies (4.2.1). □

One can define nonlinear operator on X and establish existence result for BVP(1.4.6) by using Lemma 2.1. The details are omitted.

4.3. Solvability of BVP(1.4.7)

In this subsection, we establish existence result for solutions of BVP(1.4.7) by using Lemma 2.1. Denote

$$\begin{aligned} g_x(t) &= g(t,x(t),{}^cD_{0+}^p x(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\ \overline{I}_x(t_s) &= \overline{I}(t_s,x(t_s),{}^cD_{0+}^p x(t_s)), \quad s\in\mathbb{N}_1^m,\\ \overline{J}_x(t_s) &= \overline{J}(t_s,x(t_s),{}^cD_{0+}^p x(t_s)), \quad s\in\mathbb{N}_1^m. \end{aligned}$$

LEMMA 4.3.1. *x is a solution of BVP(1.4.7) if and only if*

$$\begin{aligned} x(t) =& \sum_{j=0}^m \left[e_{\alpha,1}^{0,\alpha}(1,t_j)\overline{J}_x(t_j)+e_{\alpha,2}^{1,\alpha}(1,t_j)\overline{I}_x(t_j)+\int_0^1 e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,s)g_x(s)ds\right]e_{\alpha,1}^{0,\alpha}(t,0)\\ &+\sum_{j=0}^m \left[e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)\overline{J}_x(t_j)+e_{\alpha,1}^{0,\alpha}(1,t_j)\overline{I}_x(t_j)\right]+\int_0^1 e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(1,s)g_x(s)ds\Big]e_{\alpha,2}^{1,\alpha}(t,0)\\ &+\sum_{j=1}^i \left[e_{\alpha,1}^{0,\alpha}(t,t_j)\overline{J}_x(t_j)+e_{\alpha,2}^{1,\alpha}(t,t_j)\overline{I}_x(t_j)\right]+\int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)g_x(s)ds,\\ &t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned} \tag{4.3.1}$$

Proof. One sees for $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}_0^m$, $\sigma\in\mathbb{N}_0^{n-1}$ that

$$\begin{aligned} &[(t-t_j)^\nu\mathbf{E}_{\alpha,\nu+1}(\lambda(t-t_j)^\alpha)]^{(\sigma)} = \left[\sum_{\tau=0}^{+\infty}\frac{\lambda^\tau(t-t_j)^{\alpha\tau+\nu}}{\Gamma(\alpha\tau+\nu+1)}\right]^{(\sigma)}\\ &=\begin{cases}(t-s)^{\nu-\sigma}\mathbf{E}_{\alpha,\nu-\sigma+1}(\lambda(t-s)^\alpha), & \sigma\leqslant\nu,\\ \lambda(t-s)^{\alpha-\sigma+\nu}\mathbf{E}_{\alpha,\alpha-\sigma+\nu+1}(\lambda(t-s)^\alpha), & \sigma>\nu.\end{cases} \end{aligned}$$

Suppose that x is a solution of BVP(1.4.7). By Theorem 3.2.1, there exist constants $c_{\nu j}\in\mathbb{R}$ $(\nu\in\mathbb{N}_0^1,\ j\in\mathbb{N}_0^m)$ such that

$$x(t) = \sum_{j=0}^i\sum_{\nu=0}^1 c_{\nu j}e_{\alpha,\nu+1}^{\nu,\alpha}(t,t_j)+\int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)g_x(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{4.3.2}$$

It follows that

$$\begin{aligned} x'(t) =& \left[\sum_{j=0}^i c_{0j}\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^\alpha)\right]'+\left[\sum_{j=0}^i c_{1j}(t-t_j)\mathbf{E}_{\alpha,2}(\lambda(t-t_j)^\alpha)\right]'\\ &+\left[\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)f_x(s)ds\right]'\\ =& \sum_{j=0}^i\left[c_{0j}e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,t_j)+c_{1j}e_{\alpha,1}^{0,\alpha}(t,t_j)\right]+\int_0^t e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,s)g_x(s)ds,\\ &t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned}$$

(i) From $x^{(\sigma)}(0)=x^{(\sigma)}(1)$, $\sigma\in\mathbb{N}_0^1$, we have

$$c_{\sigma 0} = \sum_{j=0}^m\left[\sum_{\nu=0}^{\sigma-1}c_{\nu j}e_{\alpha,\alpha+\nu-\sigma+1}^{\alpha+\nu-\sigma,\alpha}(1,t_j)+\sum_{\nu=\sigma}^{n-1}c_{\nu j}e_{\alpha,\nu-\sigma+1}^{\nu-\sigma,\alpha}(1,t_j)\right]$$

$$+\int_0^1 e_{\alpha,\alpha-\sigma}^{\alpha-\sigma-1,\alpha}(1,s)g_x(s)ds, \quad \sigma\in\mathbb{N}_0^{n-1}. \tag{4.3.3}$$

(ii) From $\Delta u(t_s)=\overline{I}_x(t_s)$, $\Delta u'(t_s)=\overline{J}_x(t_s)$, $s\in\mathbb{N}_1^m$, we have $c_{1s}=\overline{J}_x(t_s)$, $c_{2s}=\overline{I}_x(t_s)$ for $s\in\mathbb{N}_1^m$. Substituting $c_{\sigma s}$ into (4.3.2), we get

$$c_{00}=\sum_{j=0}^{m}\left[\overline{J}_x(t_j)e_{\alpha,1}^{0,\alpha}(1,t_j)+\overline{I}_x(t_j)e_{\alpha,2}^{1,\alpha}(1,t_j)\right]+\int_0^1 e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,s)g_x(s)ds,$$
$$c_{10}=\sum_{j=0}^{m}\left[\overline{J}_x(t_j)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_j)+\overline{I}_x(t_j)e_{\alpha,1}^{0,\alpha}(1,t_j)\right]+\int_0^1 e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(1,s)g_x(s)ds.$$

Substituting all $c_{\sigma s}$ into (4.3.2), we get (4.3.1). On the other hand, we can prove x is a solution of BVP(1.4.7) by direct computation if x satisfies (4.3.1). □

One can define nonlinear operator on X and establish existence result for BVP(1.4.7) by using Lemma 2.1. The details are omitted.

4.4. Solvability of BVP(1.4.8)

In this subsection, we establish existence result for solutions of BVP(1.4.8) by using Lemma 2.1. Denote

$$g_x(t)=g(t,x(t),{}^cD_{t_i^+}^p x(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,$$
$$\overline{J}_x(t_s)=\overline{J}(t_s,u(t_s),{}^cD_{t_{s-1}^+}^p x(t_s)), \quad s\in\mathbb{N}_1^m,$$
$$\overline{I}_x(t_s)=\overline{I}(t_s,u(t_s),{}^cD_{t_{s-1}^+}^p x(t_s)), \quad s\in\mathbb{N}_1^m.$$

LEMMA 4.4.1. *x is a solution of BVP(1.4.8) if and only if*

$$x(t)=\begin{cases}\left[\overline{J}_x(t_i)e_{\alpha,1}^{0,\alpha}(1,t_m)+\overline{I}_x(t_i)e_{\alpha,2}^{1,\alpha}(1,t_m)+\int_{t_m}^1 e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,s)g_x(s)ds\right]e_{\alpha,1}^{0,\alpha}(t,0)\\ \left[\overline{J}_{x,0}(t_i)e_{\alpha,\alpha}^{\alpha-1,\alpha}(1,t_m)+\overline{J}_{x,1}(t_i)e_{\alpha,1}^{0,\alpha}(1,t_m)+\int_{t_m}^1 e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(1,s)g_x(s)ds\right]e_{\alpha,2}^{1,\alpha}(t,0)\\ \quad +\int_0^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)g_x(s)ds, \quad t\in(0,t_1],\\ \overline{J}_x(t_i)e_{\alpha,1}^{0,\alpha}(t,t_i)+\overline{I}_x(t_i)e_{\alpha,2}^{1,\alpha}(t,t_i)+\int_{t_i}^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)g_x(s)ds,\\ \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.\end{cases} \tag{4.4.1}$$

Proof. Suppose that x is a solution of BVP(1.4.8). By Theorem 3.2.1, there exist constants $c_{vj}\in\mathbb{R}$ $(v\in\mathbb{N}_0^{n-1},\ j\in\mathbb{N}_0^m)$ such that

$$x(t)=c_{0i}e_{\alpha,1}^{0,\alpha}(t,t_i)+c_{1i}e_{\alpha,2}^{1,\alpha}(t,t_i)+\int_{t_i}^t e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,s)g_x(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{4.4.2}$$

Then

$$x'(t)=c_{0i}e_{\alpha,\alpha}^{\alpha-1,\alpha}(t,t_i)+c_{1i}e_{\alpha,1}^{0,\alpha}(t,t_i)+\int_{t_i}^t e_{\alpha,\alpha-1}^{\alpha-2,\alpha}(t,s)g_x(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{4.4.3}$$

(i) From $u^{(\sigma)}(0)=u^{(\sigma)}(1),\ j\in \mathbb{N}_0^1$ and (4.4.3), we have

$$c_{00}=c_{0i}e^{0,\alpha}_{\alpha,1}(1,t_m)+c_{1i}e^{1,\alpha}_{\alpha,2}(1,t_m)+\int_{t_m}^1 e^{\alpha-1,\alpha}_{\alpha,\alpha}(1,s)g_x(s)ds,$$
$$c_{10}=c_{0i}e^{\alpha-1,\alpha}_{\alpha,\alpha}(1,t_m)+c_{1i}e^{0,\alpha}_{\alpha,1}(1,t_m)+\int_{t_m}^1 e^{\alpha-2,\alpha}_{\alpha,\alpha-1}(1,s)g_x(s)ds. \tag{4.4.4}$$

(ii) From $u(t_s)=\overline{J}_x(t_s),\ u'(t_s)=\overline{I}_x(t_s),\ s\in\mathbb{N}_1^m$ and (4.4.3), we have $c_{0s}=\overline{J}_x(t_s), c_{1s}=\overline{I}_x(t_s)$.

Then

$$c_{\sigma 0}=\sum_{v=0}^{\sigma-1}\overline{J}_{x,v}(t_i)e^{\alpha+v-\sigma,\alpha}_{\alpha,\alpha+v-\sigma+1}(1,t_m)+\sum_{v=\sigma}^{n-1}\overline{J}_{x,v}(t_i)e^{v-\sigma,\alpha}_{\alpha,v-\sigma+1}(1,t_m)$$
$$+\int_{t_m}^1 e^{\alpha-\sigma-1,\alpha}_{\alpha,\alpha-\sigma}(1,s)g_x(s)ds,\quad \sigma\in\mathbb{N}_0^1.$$

Substituting all $c_{\sigma s}$ into (4.4.2), we get (4.4.2). On the other hand, we can prove x is a solution of BVP(1.4.8) by direct computation if x satisfies (4.4.1). □

One can define nonlinear operator on X and establish existence result for BVP(1.4.8) by using Lemma 2.1. The details are omitted.

4.5. Solvability of BVP(1.4.9)

In this subsection, we establish existence result for solutions of BVP(1.4.9) by using Lemma 2.1. Denote

$$f_{1x}(t)=f_1(t,x(t),D^q_{0^+}x(t)),\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,$$
$$I_{1x}(t_s)=I_1(t_s,x(t_s),D^q_{0^+}x(t_s)),\quad s\in\mathbb{N}_1^m,$$
$$J_{1x}(t_s)=J_1(t_s,x(t_s),D^q_{0^+}x(t_s)),\quad s\in\mathbb{N}_1^m.$$

LEMMA 4.5.1. *x is a solution of BVP(1.4.9) if and only if*

$$x(t)=\frac{e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,0)}{1-\mathbf{E}_{\alpha+\beta,1}(\lambda)}\sum_{j=1}^m e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_j)I_{1x}(t_j)+\sum_{j=1}^i e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,t_j)I_{1x}(t_j)$$
$$+\frac{e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,0)}{1-\mathbf{E}_{\alpha+\beta,1}(\lambda)}\sum_{j=0}^m e^{\alpha,\alpha+\beta}_{\alpha+\beta,\alpha+1}(1,t_j)J_{1x}(t_j)+\frac{e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,0)}{1-\mathbf{E}_{\alpha+\beta,1}(\lambda)}\sum_{j=1}^m e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_j)J_{1x}(t_j)$$
$$+\sum_{j=1}^i e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,t_j)J_{1x}(t_j)+\frac{e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,0)}{1-\mathbf{E}_{\alpha+\beta,1}(\lambda)}\int_0^1 e^{\alpha,\alpha+\beta}_{\alpha+\beta,\alpha+1}(1,u)f_{1x}(u)du$$
$$+\frac{e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,0)}{1-\mathbf{E}_{\alpha+\beta,1}(\lambda)}\int_0^1 e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,u)f_{1x}(u)du+\int_0^t e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,u)f_{1x}(u)du,$$
$$t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m. \tag{4.5.1}$$

Proof. Suppose that x is a solution of BVP(1.4.9). By Theorem 3.3.1, we know that there exist constants $c_j, d_j \in \mathbb{R}$ $(j \in \mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t) = & \sum_{j=0}^{i} d_j (t-t_j)^{\beta-1} \mathbf{E}_{\alpha+\beta,\beta}(\lambda(t-t_j)^{\alpha+\beta}) \\ & + \sum_{j=0}^{i} c_j (t-t_j)^{\alpha+\beta-1} \mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-t_j)^{\alpha+\beta}) \\ & + \int_0^t (t-u)^{\alpha+\beta-1} \mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta}) f_{1x}(u)du, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \end{aligned}$$

By Corollary 3.5.1, we have

$$\begin{aligned} I_{0^+}^{1-\beta} x(t) = & \sum_{j=0}^{i} d_j \mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_j)^{\alpha+\beta}) + \sum_{j=0}^{i} c_j (t-t_j)^{\alpha} \mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(t-t_j)^{\alpha+\beta}) \\ & + \int_0^t (t-u)^{\alpha} \mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(t-u)^{\alpha+\beta}) f_{1x}(u)du, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \end{aligned} \tag{4.5.2}$$

$$\begin{aligned} D_{0^+}^{\beta} x(t) = & \sum_{j=0}^{i} c_j (t-t_j)^{\alpha-1} \mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-t_j)^{\alpha+\beta}) \\ & + \int_0^t (t-u)^{\alpha-1} \mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta}) f_{1x}(u)du, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \end{aligned} \tag{4.5.3}$$

$$\begin{aligned} I_{0^+}^{1-\alpha} D_{0^+}^{\beta} x(t) = & \sum_{j=0}^{i} c_j \mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_j)^{\alpha+\beta}) \\ & + \int_0^t \mathbf{E}_{\alpha+\beta,1}(\lambda(t-u)^{\alpha+\beta}) f_{1x}(u)du, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \end{aligned} \tag{4.5.4}$$

(i) From $I_{0^+}^{1-\beta} x(0) = I_{0^+}^{1-\beta} x(1)$ and (4.5.2), we have

$$\begin{aligned} d_0 = & \sum_{j=0}^{m} d_j \mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta}) + \sum_{j=0}^{m} c_j (1-t_j)^{\alpha} \mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-t_j)^{\alpha+\beta}) \\ & + \int_0^1 (1-u)^{\alpha} \mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-u)^{\alpha+\beta}) f_{1x}(u)du. \end{aligned} \tag{4.5.5}$$

(ii) From $I_{0^+}^{1-\alpha} D_{0^+}^{\beta} x(0) = I_{0^+}^{1-\alpha} D_{0^+}^{\beta} x(1)$ and (4.5.3),

$$c_0 = \sum_{j=0}^{m} c_j \mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta}) + \int_0^1 \mathbf{E}_{\alpha+\beta,1}(\lambda(1-u)^{\alpha+\beta}) f_{1x}(u)du. \tag{4.5.6}$$

(iii) From $\lim_{t \to t_s^+} (t-t_s)^{1-\beta} x(t) = \frac{I_1(t_s, x(t_s), D_{0^+}^{q} x(t_s))}{\Gamma(\beta)}$, $s \in \mathbb{N}_1^m$ and (4.5.3), we get $d_s = I_{1x}(t_s)$ $(s \in \mathbb{N}_1^m)$.

(iv) From $\lim_{t \to t_s^+} (t-t_s)^{1-\alpha} D_{0^+}^{\beta} x(t) = \frac{J_1(t_s, x(t_s), D_{0^+}^{q} x(t_s))}{\Gamma(\alpha)}$, $s \in \mathbb{N}_1^m$ and (4.5.2), we get $c_s = J_{1x}(t_s)$ $(s \in \mathbb{N}_1^m)$.

It follows that

$$c_0 = \frac{1}{1-\mathbf{E}_{\alpha+\beta,1}(\lambda)}\left[\sum_{j=1}^{m} J_{1x}(t_j)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})+\int_0^1 \mathbf{E}_{\alpha+\beta,1}(\lambda(1-u)^{\alpha+\beta})f_{1x}(u)du\right],$$

$$\begin{aligned} d_0 = \frac{1}{1-\mathbf{E}_{\alpha+\beta,1}(\lambda)}&\left[\sum_{j=1}^{m} I_{1x}(t_j)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})\right.\\ &+\sum_{j=0}^{m} J_{1x}(t_j)(1-t_j)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-t_j)^{\alpha+\beta})\\ &\left.+\int_0^1 (1-u)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-u)^{\alpha+\beta})f_{1x}(u)du\right].\end{aligned}$$

Substituting c_s, d_s into (4.5.2), we get (4.5.1). On the other hand, we can prove x is a solution of BVP(1.4.9) by direct computation if x satisfies (4.5.1). □

One can define nonlinear operator on X and establish existence result for BVP(1.4.9) by using Lemma 2.1. The details are omitted.

4.6. Solvability of BVP(1.4.10)

In this subsection, we establish existence result for solutions of BVP(1.4.10) by using Lemma 2.1. Denote

$$\begin{aligned} f_{1x}(t) &= f_1(t,x(t),D^q_{t_i^+}x(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\ I_{1x}(t_s) &= I_1(t_s,x(t_s),D^q_{t_{s-1}^+}x(t_s)), \quad s\in\mathbb{N}_1^m,\\ J_{1x}(t_s) &= J_1(t_s,x(t_s),D^q_{t_{s-1}^+}x(t_s)), \quad s\in\mathbb{N}_1^m.\end{aligned}$$

LEMMA 4.6.1. *x is a solution of BVP(1.4.10) if and only if*

$$x(t)=\begin{cases} e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,0)e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_m)I_{1x}(t_m)+e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,0)e^{\alpha,\alpha+\beta}_{\alpha+\beta,\alpha}(1,t_m)J_{1x}(t_m)\\ \quad +e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,0)e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_m)J_{1x}(t_m)\\ \quad +e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,0)\int_{t_m}^1 e^{\alpha,\alpha+\beta}_{\alpha+\beta,\alpha+1}(1,u)f_{1x}(u)du\\ \quad +e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,0)\int_{t_m}^1 e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,u)f_{1x}(u)du\\ \quad +\int_0^t e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,u)f_{1x}(u)du, \quad t\in(t_0,t_1],\\ I_{1x}(t_i)e^{\beta-1,\alpha+\beta}_{\alpha+\beta,\beta}(t,t_i)+J_{1x}(t_i)e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,t_i)\\ \quad +\int_{t_i}^t e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,u)f_{1x}(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{cases} \tag{4.6.1}$$

Proof. Suppose that x is a solution of BVP(1.4.10). By Theorem 3.3.1, we know that there exist constants $c_j, d_j\in\mathbb{R}$ $(j\in\mathbb{N}_0^m)$ such that

$$x(t) = d_i(t-t_i)^{\beta-1}\mathbf{E}_{\alpha+\beta,\beta}(\lambda(t-t_i)^{\alpha+\beta})+c_i(t-t_i)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-t_i)^{\alpha+\beta})$$

$$+\int_{t_i}^t (t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})f_{1x}(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{4.6.2}$$

By Corollary 3.3.1, we have

$$I_{t_i^+}^{1-\beta}x(t) = d_i\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_i)^{\alpha+\beta}) + c_i(t-t_i)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(t-t_i)^{\alpha+\beta}) + \int_{t_i}^t (t-u)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(t-u)^{\alpha+\beta})f_{1x}(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{4.6.3}$$

$$D_{0^+}^{\beta}x(t) = c_i(t-t_i)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-t_i)^{\alpha+\beta}) + \int_{t_i}^t (t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})f_{1x}(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{4.6.4}$$

$$I_{0^+}^{1-\alpha}D_{0^+}^{\beta}x(t) = c_i\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_i)^{\alpha+\beta}) + \int_{t_i}^t \mathbf{E}_{\alpha+\beta,1}(\lambda(t-u)^{\alpha+\beta})f_{1x}(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{4.6.5}$$

(i) From $I_{0^+}^{1-\beta}x(0) = I_{t_m^+}^{1-\beta}x(1)$ and (4.6.2), we have

$$d_0 = d_m\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta}) + c_m(1-t_m)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-t_m)^{\alpha+\beta}) + \int_{t_m}^1 (1-u)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-u)^{\alpha+\beta})f_{1x}(u)du. \tag{4.6.6}$$

(ii) From $I_{0^+}^{1-\alpha}D_{0^+}^{\beta}x(0) = I_{t_m^+}^{1-\alpha}D_{t_m^+}^{\beta}x(1)$ and (4.6.4),

$$c_0 = c_m\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta}) + \int_{t_m}^1 \mathbf{E}_{\alpha+\beta,1}(\lambda(1-u)^{\alpha+\beta})f_{1x}(u)du. \tag{4.6.7}$$

(iii) From $\lim_{t\to t_s^+}(t-t_s)^{1-\beta}x(t) = \frac{I_1(t_s,x(t_s),D_{0^+}^q x(t_s))}{\Gamma(\beta)}$, $s\in\mathbb{N}_1^m$ and (4.6.5), we get $d_s = I_{1x}(t_s)$ $(s\in\mathbb{N}_1^m)$.

(iv) From $\lim_{t\to t_s^+}(t-t_s)^{1-\alpha}D_{0^+}^{\beta}x(t) = \frac{J_1(t_s,x(t_s),D_{0^+}^q x(t_s))}{\Gamma(\alpha)}$, $s\in\mathbb{N}_1^m$ and (4.6.5), we get $c_s = J_{1x}(t_s)$ $(s\in\mathbb{N}_1^m)$.

It follows that

$$c_0 = J_{1x}(t_m)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta}) + \int_{t_m}^1 \mathbf{E}_{\alpha+\beta,1}(\lambda(1-u)^{\alpha+\beta})f_{1x}(u)du,$$
$$d_0 = I_{1x}(t_m)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta}) + J_{1x}(t_m)(1-t_m)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-t_m)^{\alpha+\beta}) + \int_{t_m}^1 (1-u)^{\alpha}\mathbf{E}_{\alpha+\beta,\alpha+1}(\lambda(1-u)^{\alpha+\beta})f_{1x}(u)du.$$

Substituting c_s, d_s into (4.6.2), we get (4.6.1). On the other hand, we can prove x is a solution of BVP(1.4.10) by direct computation if x satisfies (4.36). □

One can define nonlinear operator on X and establish existence result for BVP(1.4.10) by using Lemma 2.1. The details are omitted.

4.7. Solvability of BVP(1.4.11)

In this subsection, we establish existence result for solutions of BVP(1.4.11) by using Lemma 2.1. Denote

$$\begin{aligned} g_{1x}(t) &= g_1(t,x(t),{}^cD^q_{0^+}x(t)), \quad t\in(t_i,t_{i+1}], \quad i\in \mathbb{N}_0^m,\\ \overline{I}_{1x}(t_s) &= \overline{I}_1(t_s,x(t_s),{}^cD^q_{0^+}x(t_s)), \quad s\in\mathbb{N}_1^m,\\ \overline{J}_{1x}(t_s) &= \overline{J}_1(t_s,x(t_s),{}^cD^q_{0^+}x(t_s)), \quad s\in\mathbb{N}_1^m. \end{aligned}$$

LEMMA 4.7.1. *x is a solution of BVP(1.4.11) if and only if*

$$\begin{aligned} x(t) =\ & e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)\sum_{j=0}^m e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_j)\overline{I}_{1x}(t_j)+\sum_{j=1}^i e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,t_j)\overline{I}_{1x}(t_j)\\ & -e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)\sum_{j=0}^m \frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_j)}{\Gamma(\beta+1)}\overline{J}_{1x}(t_j)+e^{0,\alpha+\beta}_{\alpha+\beta,\beta+1}(t,0)\sum_{j=0}^m e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_j)\overline{J}_{1x}(t_j)\\ & -\sum_{j=1}^i \frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,t_j)}{\Gamma(\beta+1)}\overline{J}_{1x}(t_j)+\sum_{j=1}^i e^{0,\alpha+\beta}_{\alpha+\beta,\beta+1}(t,t_j)\overline{J}_{1x}(t_j)\\ & +e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)\sum_{j=0}^m e^{0,\alpha}_{\alpha+\beta,\beta+1}(1,t_j)\overline{J}_{1x}(t_j)-\frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)}{\Gamma(\beta+1)}\sum_{j=0}^m e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_j)\overline{J}_{1x}(t_j)\\ & -\frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)}{\Gamma(\beta+1)}\int_0^1 e^{\alpha-1,\alpha+\beta}_{\alpha+\beta,\alpha}(1,u)g_{1x}(u)du+e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)\int_0^1 e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(1,u)g_{1x}(u)du\\ & +e^{0,\alpha+\beta}_{\alpha+\beta,\beta+1}(t,0)\int_0^1 e^{\alpha-1,\alpha+\beta}_{\alpha+\beta,\alpha}(1,u)g_{1x}(u)du+\int_0^t e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,u)g_{1x}(u)du,\\ & t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned} \tag{4.7.1}$$

Proof. Suppose that x is a solution of BVP(1.4.11). By Theorem 3.4.1, we know that there exist constants c_v, d_u $(v\in\mathbb{N}_0^m,\ u\in\mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t) =\ & \sum_{j=0}^i d_j\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_j)^{\alpha+\beta})+\sum_{j=0}^i c_j\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(t-t_j)^{\alpha+\beta})\\ & +\int_0^t(t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})g_{1x}(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned} \tag{4.7.2}$$

Then

$$\begin{aligned} {}^cD^\beta_{0^+}x(t) =\ & \sum_{j=0}^i c_j\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_j)^{\alpha+\beta})\\ & +\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})g_{1x}(u)du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned} \tag{4.7.3}$$

(i) From $x(0)=x(1)$ and ${}^cD^\beta_{0^+}x(0)={}^cD^\beta_{0^+}x(1)$ and (4.7.3), we get

$$d_0+\frac{c_0}{\Gamma(\beta+1)} = \sum_{j=0}^m d_j\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})+\sum_{j=0}^m c_j\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(1-t_j)^{\alpha+\beta})$$

$$+\int_0^1(1-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du,$$

$$c_0=\sum_{j=0}^{m}c_j\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})+\int_0^1(1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du.$$

(ii) From $\Delta u(t_s)=\overline{I}_{1x}(t_s)$ and $\Delta {}^cD_{0^+}^{\beta}x(t_s)=\overline{J}_{1x}(t_s)$ and (4.7.3), we get $c_s=\overline{J}_{1x}(t_s)$ and $d_s+\frac{c_s}{\Gamma(\beta+1)}=\overline{I}_{1x}(t_s)$. Thus

$$c_s=\overline{J}_{1x}(t_s),\quad d_s=\overline{I}_{1x}(t_s)-\frac{\overline{J}_{1x}(t_s)}{\Gamma(\beta+1)},\quad s\in\mathbb{N}_1^m.$$

Hence

$$\begin{aligned}
d_0=&\sum_{j=0}^{m}\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})\overline{I}_{1x}(t_j)-\sum_{j=0}^{m}\frac{\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})}{\Gamma(\beta+1)}\overline{J}_{1x}(t_j)\\
&+\sum_{j=0}^{m}\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(1-t_j)^{\alpha+\beta})\overline{J}_{1x}(t_j)-\frac{1}{\Gamma(\beta+1)}\sum_{j=0}^{m}\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})\overline{J}_{1x}(t_j)\\
&-\frac{1}{\Gamma(\beta+1)}\int_0^1(1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du\\
&+\int_0^1(1-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du,\\
c_0=&\sum_{j=0}^{m}J_{1x}(t_j)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})+\int_0^1(1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du.
\end{aligned}$$

Substituting c_s,d_s into (4.7.2), we get (4.7.1). On the other hand, we can prove x is a solution of BVP(1.4.11) by direct computation if x satisfies (4.7.1). □

One can define nonlinear operator on X and establish existence result for BVP(1.4.11) by using Lemma 2.1. The details are omitted.

4.8. Solvability of BVP(1.4.12)

In this subsection, we establish existence result for solutions of BVP(1.4.12) by using Lemma 2.1. Denote

$$\begin{aligned}
g_{1x}(t)&=g_1(t,x(t),{}^cD_{t_i^+}^{q}x(t)),\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\
\overline{I}_{1x}(t_s)&=\overline{I}_1(t_s,x(t_s),{}^cD_{t_{s-1}^+}^{q}x(t_s)),\quad s\in\mathbb{N}_1^m,\\
\overline{J}_{1x}(t_s)&=\overline{J}_1(t_s,x(t_s),{}^cD_{t_{s-1}^+}^{q}x(t_s)),\quad s\in\mathbb{N}_1^m.
\end{aligned}$$

LEMMA 4.8.1. *x is a solution of BVP(1.4.12) if and only if*

$$x(t)=\begin{cases} e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_m)\overline{I}_{1x}(t_m)-\frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_m)}{\Gamma(\beta+1)}\overline{J}_{1x}(t_m)\\ +e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)e^{0,\alpha+\beta}_{\alpha+\beta,\beta+1}(1,t_m)\overline{J}_{1x}(t_m)-\frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_m)}{\Gamma(\beta+1)}\overline{J}_{1x}(t_m)\\ +e^{0,\alpha+\beta}_{\alpha+\beta,\beta+1}(t,0)e^{0,\alpha+\beta}_{\alpha+\beta,1}(1,t_m)\overline{J}_{1x}(t_m)-\frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)}{\Gamma(\beta+1)}\int_{t_m}^1 e^{\alpha-1,\alpha+\beta}_{\alpha+\beta,\alpha}(1,u)g_{1x}(u)du\\ +e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,0)\int_{t_m}^1 e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(1,u)g_{1x}(u)du\\ +e^{0,\alpha+\beta}_{\alpha+\beta,\beta+1}(t,0)\int_{t_m}^1 e^{\alpha-1,\alpha+\beta}_{\alpha+\beta,\alpha}(1,u)g_{1x}(u)du+\int_{t_i}^t e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,u)g_{1x}(u)du,\\ \quad t\in(t_0,t_1],\\ e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,t_i)\overline{I}_{1x}(t_i)-\frac{e^{0,\alpha+\beta}_{\alpha+\beta,1}(t,t_i)}{\Gamma(\beta+1)}\overline{J}_{1x}(t_i)+e^{0,\alpha+\beta}_{\alpha+\beta,\beta+1}(t,t_i)\overline{J}_{1x}(t_i)\\ +\int_{t_i}^t e^{\alpha+\beta-1,\alpha+\beta}_{\alpha+\beta,\alpha+\beta}(t,u)g_{1x}(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_1^m. \end{cases}\tag{4.8.1}$$

Proof. Suppose that x is a solution of BVP(1.4.12). By Theorem 3.4.1, we know that there exist constants c_v,d_u $(v\in\mathbb{N}_0^m,\ u\in\mathbb{N}_0^m)$ such that

$$\begin{aligned}x(t)&=d_i\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_i)^{\alpha+\beta})+c_i\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(t-t_i)^{\alpha+\beta})\\&\quad+\int_{t_i}^t(t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})g_{1x}(u)du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}\tag{4.8.2}$$

Then

$$\begin{aligned}{}^cD^{\beta}_{t_i^+}x(t)&=c_i\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_i)^{\alpha+\beta})+\int_{t_i}^t(t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})g_{1x}(u)du,\\&\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}\tag{4.8.3}$$

(i) From $x(0)=x(1)$ and ${}^cD^{\beta}_{0^+}x(0)={}^cD^{\beta}_{t_m^+}x(1)$ and (4.8.2)–(4.8.3), we get

$$\begin{aligned}d_0+\frac{c_0}{\Gamma(\beta+1)}&=d_m\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta})+c_m\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(1-t_m)^{\alpha+\beta})\\&\quad+\int_{t_m}^1(1-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du,\end{aligned}$$

$$c_0=c_m\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta})+\int_{t_m}^1(1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)d.$$

(ii) From $x(t_s)=\overline{I}_{1x}(t_s)$ and ${}^cD^{\beta}_{0^+}x(t_s)=\overline{J}_{1x}(t_s)$ and (4.8.2)–(4.8.3), we get $c_s=\overline{J}_{1x}(t_s)$ and $d_s+\frac{c_s}{\Gamma(\beta+1)}=\overline{I}_{1x}(t_s)$. Thus

$$c_s=\overline{J}_{1x}(t_s),\quad d_s=\overline{I}_{1x}(t_s)-\frac{\overline{J}_{1x}(t_s)}{\Gamma(\beta+1)},\quad s\in\mathbb{N}_1^m.$$

Hence

$$d_0=\left(\overline{I}_{1x}(t_m)-\frac{\overline{J}_{1x}(t_m)}{\Gamma(\beta+1)}\right)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta})+\overline{J}_{1x}(t_m)\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(1-t_m)^{\alpha+\beta})$$

$$
\begin{aligned}
&-\tfrac{1}{\Gamma(\beta+1)}\overline{J}_{1x}(t_m)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta})\\
&-\tfrac{1}{\Gamma(\beta+1)}\int_{t_m}^1(1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du\\
&+\int_{t_m}^1(1-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du,\\
c_0 &= \overline{J}_{1x}(t_m)\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_m)^{\alpha+\beta})+\int_{t_m}^1(1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(1-u)^{\alpha+\beta})g_{1x}(u)du.
\end{aligned}
$$

Substituting c_s, d_s into (4.8.2), we get (4.8.1). On the other hand, we can prove x is a solution of BVP(1.4.12) by direct computation if x satisfies (4.8.1). □

One can define nonlinear operator on X and establish existence result for BVP(1.4.12) by using Lemma 2.1. The details are omitted.

5. Solvability of BVPs for higher order IFDEs

In this section, we establish existence results for BVP(1.4.13)–BVP(1.4.15). These boundary value problems are generalizations of BVPs for higher order ordinary differential equations which have been studied by many authors see [1].

5.1. Solvability of BVP(1.4.13)

Now we establish existence results for solutions of BVP(1.4.13). We suppose that f satisfies that $u \to f(t,u)$, $u \to I_i(t_s,u)$ are continuous functions, $t \to f(t,u)$ is measurable on $(t_i,t_{i+1}]$ $(i \in \mathbb{N}_0^m)$ for all $u \in \mathbb{R}$ and there exist constants $k > -1$, $l \in (-1,0]$ such that $|u| \leqslant r$ implies that there exists a constant $M_r, M_{r,I}$ such that $|f(t,u)| \leqslant M_r t^k (1-t)^l$ for almost all $t \in (0,1)$ and $|I_i(t_s,u)| \leqslant M_{r,I}$ for all $i \in \mathbb{N}_0^m$.

Denote

$$
M=(m_{i,j})_{n\times n}=\begin{pmatrix}
\frac{1}{\Gamma(2)} & 0 & 0 & 0 & \cdots & 0\\
\frac{1}{\Gamma(4)} & \frac{1}{\Gamma(2)} & 0 & 0 & \cdots & 0\\
\frac{1}{\Gamma(6)} & \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(2)} & 0 & \cdots & 0\\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot\\
\frac{1}{\Gamma(2n)} & \frac{1}{\Gamma(2n-2)} & \frac{1}{\Gamma(2n-4)} & \frac{1}{\Gamma(2n-6)} & \cdots & \frac{1}{\Gamma(2)}
\end{pmatrix},
$$

$$
N=(n_{i,j})_{n\times n}=\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0\\
\frac{1}{\Gamma(3)} & 1 & 0 & 0 & \cdots & 0\\
\frac{1}{\Gamma(5)} & \frac{1}{\Gamma(3)} & 1 & 0 & \cdots & 0\\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot\\
\frac{1}{\Gamma(2n-1)} & \frac{1}{\Gamma(2n-3)} & \frac{1}{\Gamma(2n-5)} & \frac{1}{\Gamma(2n-7)} & \cdots & 1
\end{pmatrix}.
$$

Then $|M| = |N| = 1$. One has for a determinant $|a_{i,j}|_{n\times n}$ that

$$
|a_{i,j}|_{n\times n} = \sum_{i=1}^{n} a_{i,n-j}A_{i,n-j}, \quad j \in \mathbb{N}_0^{n-1},
$$

where $A_{i,n-j}$ is the algebraic cofactor of $a_{i,n-j}$.

Suppose that $|a_{i,j}| \leqslant 1$. It is easy to show that

$$|A_{i,n-j}| \leqslant (n-1)! = \Gamma(n), \quad i \in \mathbb{N}_1^{n-k}, \quad j \in \mathbb{N}_0^{n-1}.$$

Then

$$M^{-1} = M^* = \begin{pmatrix} M_{11} & M_{21} & M_{31} & M_{41} & \cdots & M_{n1} \\ M_{12} & M_{22} & M_{32} & M_{42} & \cdots & M_{n2} \\ M_{13} & M_{23} & M_{33} & M_{43} & \cdots & M_{n3} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ M_{1n} & M_{2n} & M_{3n} & M_{4n} & \cdots & M_{nn} \end{pmatrix},$$

$$N^{-1} = N^* = \begin{pmatrix} N_{11} & N_{21} & N_{31} & N_{41} & \cdots & N_{n1} \\ N_{12} & N_{22} & N_{32} & N_{42} & \cdots & N_{n2} \\ N_{13} & N_{23} & N_{33} & N_{43} & \cdots & N_{n3} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ N_{1n} & N_{2n} & N_{3n} & N_{4n} & \cdots & N_{nn} \end{pmatrix},$$

where M_{ij} and N_{ij} are the algebraic cofactors of m_{ij} and n_{ij} respectively. M^* and N^* are the adjoint matrix of M and N respectively. We know that $|M_{ij}| \leqslant \Gamma(n)$ and $|N_{ij}| \leqslant \Gamma(n)$.

LEMMA 5.1.1. *Suppose that $u \in X$. Then $x \in X$ is a solution of*

$$\begin{cases} {}^cD_{0^+}^{\alpha} x(t) = f_u(t), \quad t \in (t_s, t_{s+1}], \quad s \in \mathbb{N}_0^m, \\ x^{(2i)}(0) = 0, \quad i \in \mathbb{N}_0^{n-1}, \\ x^{(2j)}(1) = 0, \quad j \in \mathbb{N}_0^{n-1}, \\ \Delta x^{(j)}(t_s) = I_{jx}(t_s), \quad j \in \mathbb{N}_0^{2n-1}, \quad s \in \mathbb{N}_1^m \end{cases} \tag{5.1.1}$$

if and only if

$$\begin{aligned} x(t) = & -\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{M_{j+1n-i}}{\Gamma(2i+2)} \sum_{w=1}^{m} \sum_{v=2(n-1-j)}^{2n-1} \frac{(1-t_w)^{v-2(n-1-j)} I_{vu}(t_w)}{\Gamma(v-2(n-1-j)+1)} t^{2i+1} \\ & -\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{M_{j+1n-i}}{\Gamma(2i+2)} \int_0^1 \frac{(1-s)^{\alpha-2(n-1-j)-1}}{\Gamma(\alpha-2(n-1-j))} f_u(s) ds t^{2i+1} \\ & + \sum_{w=1}^{s}\sum_{v=0}^{2n-1} \frac{(t-t_w)^v}{\Gamma(v+1)} I_{vu}(t_w) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_u(s) ds, \quad t \in (t_s, t_{s+1}], \quad s \in \mathbb{N}_1^m. \end{aligned} \tag{5.1.2}$$

Proof. Firstly we prove that x satisfies (5.1.2) if $x \in X$ and x is a solution of (5.1.1).

Since $u \in X$, there exists $r > 0$ such that

$$\|u\| = \max\left\{ \sup_{t\in(0,1]} |u(t)|, \sup_{t\in(0,1]} |{}^cD_{0^+}^{\beta} u(t)| \right\} \leqslant r. \tag{5.1.3}$$

Since f is a Carathéodory function, there exist constants $A_{r,f}\geqslant 0$ such that

$$|f(t,u(t))|\leqslant A_{r,f}t^p(1-t)^q,\quad t\in(t_s,t_{s+1}],\quad s\in\mathbb{N}_0^m. \tag{5.1.4}$$

Similarly, since I_j is a discrete Carathéodory function, there exist positive constants $A_{r,I_j,s}\geqslant 0$ $(s\in\mathbb{N}_1^m,\ j\in\mathbb{N}_0^{2n-1})$ such that

$$|I_j(t_s,u(t_s))|\leqslant A_{r,I_j,s}. \tag{5.1.5}$$

Suppose that $x\in X$ and x is a solution of (5.1.1). By Theorem 3.2.1, we know that there exist constants $c_{vj}\in\mathbb{R}$ such that

$$x(t)=\sum_{w=0}^{s}\sum_{v=0}^{2n-1}\frac{c_{vw}}{\Gamma(v+1)}(t-t_w)^v+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_u(s)ds,\quad t\in(t_s,t_{s+1}],\quad s\in\mathbb{N}_0^m. \tag{5.1.6}$$

By Definition 2, we have

$$\begin{aligned}{}^cD_{0^+}^{\beta}x(t)&=\sum_{w=0}^{s}\sum_{v=\beta}^{2n-1}\frac{c_{vw}}{\Gamma(v-\beta+1)}(t-t_w)^{v-\beta}+\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f_u(s)ds,\ t\in(t_s,t_{s+1}],\ s\in\mathbb{N}_0^m,\\ x^{(j)}(t)&=\sum_{w=0}^{s}\sum_{v=j}^{2n-1}\frac{c_{vw}}{\Gamma(v-j+1)}(t-t_w)^{v-j}+\int_0^t\frac{(t-s)^{\alpha-j-1}}{\Gamma(\alpha-j)}f_u(s)ds,\\ &\qquad t\in(t_s,t_{s+1}],\quad s\in\mathbb{N}_0^m,\quad j\in\mathbb{N}_0^{2n-1}.\end{aligned} \tag{5.1.7}$$

We have

$$\begin{aligned}\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_u(s)(s)ds\right|&\leqslant A_{r,f}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^p(1-s)^qds\\ &\leqslant A_{r,f}\int_0^t\frac{(t-s)^{\alpha+q-1}}{\Gamma(\alpha)}s^pds=A_{r,f}t^{\alpha+p+q}\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)},\\ \left|\int_0^t\frac{(t-s)^{\alpha-j-1}}{\Gamma(\alpha-j)}f_u(s)(s)ds\right|&\leqslant A_{r,f}\int_0^t\frac{(t-s)^{\alpha-j-1}}{\Gamma(\alpha-j)}s^p(1-s)^qds\\ &\leqslant A_{r,f}\int_0^t\frac{(t-s)^{\alpha+q-j-1}}{\Gamma(\alpha-j)}s^pds\\ &=A_{r,f}t^{\alpha-j+p+q}\frac{\mathbf{B}(\alpha-j+q,p+1)}{\Gamma(\alpha-j)},\quad j\in\mathbb{N}_0^{2n-1}.\end{aligned}$$

(i) It follows from $x^{(2j)}(0)=0$ that $c_{2j0}=0$ $(j\in\mathbb{N}_0^{n-1})$.
(ii) From $\Delta x^{(j)}(t_s)=I_{ju}(t_s)$ and (5.1.7), we get $c_{js}=I_{ju}(t_s)$ $(j\in\mathbb{N}_0^{2n-1},\ s\in\mathbb{N}_1^m)$.
(iii) From $x^{(2j)}(1)=0,\ j\in\mathbb{N}_0^{n-1}$, we get

$$\sum_{w=0}^{m}\sum_{v=2j}^{2n-1}\frac{c_{vw}}{\Gamma(v-2j+1)}(1-t_w)^{v-2j}+\int_0^1\frac{(1-s)^{\alpha-2j-1}}{\Gamma(\alpha-2j)}f_u(s)ds=0.$$

Use (i) and (ii), we get

$$\sum_{v=2j}^{2n-1}\frac{c_{v0}}{\Gamma(v-2j+1)}+\sum_{w=1}^{m}\sum_{v=2j}^{2n-1}\frac{(1-t_w)^{v-2j}}{\Gamma(v-2j+1)}I_{vu}(t_w)+\int_0^1\frac{(1-s)^{\alpha-2j-1}}{\Gamma(\alpha-2j)}f_u(s)ds=0,\quad j\in\mathbb{N}_0^{n-1}.$$

It follows that

$$\begin{pmatrix} \frac{1}{\Gamma(2)} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(2)} & 0 & 0 & \cdots & 0 \\ \frac{1}{\Gamma(6)} & \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(2)} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{\Gamma(2n)} & \frac{1}{\Gamma(2n-2)} & \frac{1}{\Gamma(2n-4)} & \frac{1}{\Gamma(2n-6)} & \cdots & \frac{1}{\Gamma(2)} \end{pmatrix} \begin{pmatrix} c_{2n-10} \\ c_{2n-30} \\ c_{2n-50} \\ \cdots \\ \\ c_{10} \end{pmatrix}$$

$$= - \begin{pmatrix} \sum_{w=1}^{m} \sum_{v=2(n-1)}^{2n-1} \frac{(1-t_w)^{v-2(n-1)}}{\Gamma(v-2(n-1)+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-2(n-1)-1}}{\Gamma(\alpha-2(n-1))} f_u(s)ds \\ \sum_{w=1}^{m} \sum_{v=2(n-2)}^{2n-1} \frac{(1-t_w)^{v-2(n-2)}}{\Gamma(v-2(n-2)+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-2(n-2)-1}}{\Gamma(\alpha-2(n-2))} f_u(s)ds \\ \sum_{w=1}^{m} \sum_{v=2(n-3)}^{2n-1} \frac{(1-t_w)^{v-2(n-3)}}{\Gamma(v-2(n-3)+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-2(n-3)-1}}{\Gamma(\alpha-2(n-3))} f_u(s)ds \\ \cdots \\ \sum_{w=1}^{m} \sum_{v=0}^{2n-1} \frac{(1-t_w)^{v}}{\Gamma(v+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f_u(s)ds \end{pmatrix}.$$

Hence

$$\begin{pmatrix} c_{2n-10} \\ c_{2n-30} \\ c_{2n-50} \\ \cdots \\ \\ c_{10} \end{pmatrix} = - \begin{pmatrix} \frac{1}{\Gamma(2)} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(2)} & 0 & 0 & \cdots & 0 \\ \frac{1}{\Gamma(6)} & \frac{1}{\Gamma(4)} & \frac{1}{\Gamma(2)} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{\Gamma(2n)} & \frac{1}{\Gamma(2n-2)} & \frac{1}{\Gamma(2n-4)} & \frac{1}{\Gamma(2n-6)} & \cdots & \frac{1}{\Gamma(2)} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} \sum_{w=1}^{m} \sum_{v=2(n-1)}^{2n-1} \frac{(1-t_w)^{v-2(n-1)}}{\Gamma(v-2(n-1)+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-2(n-1)-1}}{\Gamma(\alpha-2(n-1))} f_u(s)ds \\ \sum_{w=1}^{m} \sum_{v=2(n-2)}^{2n-1} \frac{(1-t_w)^{v-2(n-2)}}{\Gamma(v-2(n-2)+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-2(n-2)-1}}{\Gamma(\alpha-2(n-2))} f_u(s)ds \\ \sum_{w=1}^{m} \sum_{v=2(n-3)}^{2n-1} \frac{(1-t_w)^{v-2(n-3)}}{\Gamma(v-2(n-3)+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-2(n-3)-1}}{\Gamma(\alpha-2(n-3))} f_u(s)ds \cdots \\ \sum_{w=1}^{m} \sum_{v=0}^{2n-1} \frac{(1-t_w)^{v}}{\Gamma(v+1)} I_{vu}(t_w) + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f_u(s)ds \end{pmatrix}.$$

Hence

$$\begin{aligned} c_{2i+10} &= -\sum_{j=0}^{n-1} M_{j+1n-i} \left(\sum_{w=1}^{m} \sum_{v=2(n-1-j)}^{2n-1} \frac{(1-t_w)^{v-2(n-1-j)} I_{vu}(t_w)}{\Gamma(v-2(n-1-j)+1)} + \int_0^1 \frac{(1-s)^{\alpha-2(n-1-j)-1}}{\Gamma(\alpha-2(n-1-j))} f_u(s)ds \right) \\ &= -\sum_{j=0}^{n-1} M_{j+1n-i} \sum_{w=1}^{m} \sum_{v=2(n-1-j)}^{2n-1} \frac{(1-t_w)^{v-2(n-1-j)} I_{vu}(t_w)}{\Gamma(v-2(n-1-j)+1)} \\ &\quad -\sum_{j=0}^{n-1} M_{j+1n-i} \int_0^1 \frac{(1-s)^{\alpha-2(n-1-j)-1}}{\Gamma(\alpha-2(n-1-j))} f_u(s)ds. \end{aligned} \tag{5.1.8}$$

From (i), (ii) and (iii), we have (5.1.8) and

$$c_{2j0}=0,\ \ j\in\mathbb{N}_0^{n-1},\ \ c_{js}=I_{ju}(t_s)\ \ (s\in\mathbb{N}_1^m,\ j\in\mathbb{N}_0^{2n-1}). \tag{5.1.9}$$

Substituting (5.1.8) and (5.1.9) into (5.1.6), we get (5.1.2).

Secondly, we prove that $x\in X$ and x satisfies (5.1.1) if x satisfies (5.1.2). It is easy to see from (5.1.2) that $x\in X$ and

$$\lim_{t\to 0}x^{(2j)}(t)=0,\ \ j\in\mathbb{N}_0^{n-1},\ \ x^{(2j)}(1)=0,\ \ j\in\mathbb{N}_0^{n-1},$$
$$\Delta x^{(j)}(t_s)=I_{ju}(t_s),\ \ j\in\mathbb{N}_0^{2n-1},\ \ s\in\mathbb{N}_1^m.$$

Now, we prove that x satisfies ${}^cD_{0+}^{\alpha}x(t)=f_u(t)$ if (5.1.2) holds. We remember (5.1.8) and (5.1.9), then it suffices to prove ${}^cD_{0+}^{\alpha}x(t)=f_u(t)$ on $(0,1)$.

In fact, for $t\in(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m$, by Definition 2.2, we have

$$\begin{aligned}
D_{0+}^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}x^{(n)}(s)ds\\
&= \frac{1}{\Gamma(n-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}x^{(n)}(s)ds+\int_{t_i}^{t}(t-s)^{n-\alpha-1}x^{(n)}(s)ds\right]\\
&= \frac{\sum\limits_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{n-\alpha-1}\left(\sum\limits_{w=0}^{j}\sum\limits_{\theta=0}^{n-1}\frac{c_{\theta w}}{\Gamma(\theta+1)}(s-t_w)^{\theta}+\int_0^s\frac{(s-\sigma)^{\alpha-1}}{\Gamma(\alpha)}f_u(\sigma)d\sigma\right)^{(n)}ds}{\Gamma(n-\alpha)}\\
&\quad+\frac{\int_{t_i}^{t}(t-s)^{n-\alpha-1}\left(\sum\limits_{w=0}^{i}\sum\limits_{\theta=0}^{n-1}\frac{c_{\theta w}}{\Gamma(\theta+1)}(s-t_w)^{\theta}+\int_0^s\frac{(s-\sigma)^{\alpha-1}}{\Gamma(\alpha)}f_u(\sigma)d\sigma\right)^{(n)}ds}{\Gamma(n-\alpha)}\\
&= \frac{\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\frac{(s-\sigma)^{\alpha-1}}{\Gamma(\alpha)}f_u(\sigma)d\sigma\right)^{(n)}ds}{\Gamma(n-\alpha)}=\frac{\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\frac{(s-\sigma)^{\alpha-n}}{\Gamma(\alpha-n+1)}f_u(\sigma)d\sigma\right)'ds}{\Gamma(n-\alpha)}\\
&= \left[\frac{\int_0^t(t-s)^{n-\alpha}\left(\int_0^s\frac{(s-\sigma)^{\alpha-n}}{\Gamma(\alpha-n+1)}f_u(\sigma)d\sigma\right)'ds}{(n-\alpha)\Gamma(n-\alpha)}\right]'\\
&= \left[\frac{(t-s)^{n-\alpha}\left(\int_0^s\frac{(s-\sigma)^{\alpha-n}}{\Gamma(\alpha-n+1)}f_u(\sigma)d\sigma\right)\Big|_0^t+(n-\alpha)\int_0^t(t-s)^{n-\alpha-1}\left(\int_0^s\frac{(s-\sigma)^{\alpha-n}}{\Gamma(\alpha-n+1)}f_u(\sigma)d\sigma\right)ds}{(n-\alpha)\Gamma(n-\alpha)}\right]'\\
&= \left[\frac{\int_0^t\int_s^t(t-s)^{n-\alpha-1}\frac{(s-\sigma)^{\alpha-n}}{\Gamma(\alpha-n+1)}dsf_u(\sigma)d\sigma}{\Gamma(n-\alpha)}\right]'\\
&= \left[\frac{\int_0^t\int_0^1(1-w)^{n-\alpha-1}\frac{w^{\alpha-n}}{\Gamma(\alpha-n+1)}dwf_u(\sigma)d\sigma}{\Gamma(n-\alpha)}\right]'=f_u(t),\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0.
\end{aligned}$$

From above discussion, we know that $x\in X$ and x satisfies (5.1.1) if (5.1.2) holds. The proof is completed. □

Now, we define the operator T on X by

$$(Tx)(t) = -\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{M_{j+1n-i}}{\Gamma(2i+2)}\sum_{w=1}^{m}\sum_{v=2(n-1-j)}^{2n-1}\frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)}t^{2i+1}I_{vx}(t_w)$$
$$-\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{M_{j+1n-i}}{\Gamma(2i+2)}\int_0^1\frac{(1-s)^{\alpha-2(n-1-j)-1}}{\Gamma(\alpha-2(n-1-j))}f_x(s)dst^{2i+1}$$
$$+\sum_{w=1}^{s}\sum_{v=0}^{2n-1}\frac{(t-t_w)^v}{\Gamma(v+1)}I_{vx}(t_w)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_x(s)ds, \quad t\in(t_s,t_{s+1}], \quad s\in\mathbb{N}_1^m. \tag{5.1.10}$$

REMARK 5.1.1. By Lemmas 5.1.1, $T:X\to X$ is well defined and $x\in X$ is a solution of system (1.51) if and only if $x\in X$ is a fixed point of the operator T.

LEMMA 5.1.2. *The operator $T:X\to X$ is completely continuous.*

Proof. The proof is standard and is omitted, one may see [81]. □

Now we are ready to present the main theorems. We need the following assumptions:

(H1) there exist nonnegative numbers σ_i, a_i, A_i $(i\in\mathbb{N}_0^n)$ such that

$$|f(t,x)| \leqslant [a_0+\sum_{i=1}^{\omega}a_i|x|^{\sigma_i}]t^p(1-t)^q, \quad t\in(0,1), \quad x\in\mathbb{R},$$
$$\left|I_j(t_s,x)\right| \leqslant A_0+\sum_{i=1}^{\omega}A_i|x|^{\sigma_i}, \quad s\in\mathbb{N}_1^m, \quad x\in\mathbb{R}.$$

Denote

$$\sigma = \max\{\sigma_i : i\in\mathbb{N}_1^n\},$$
$$M_0 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{\Gamma(n)}{\Gamma(2i+2)}\sum_{w=1}^{m}\sum_{v=2(n-1-j)}^{2n-1}\frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)}A_0+\sum_{w=1}^{m}\sum_{v=0}^{2n-1}\frac{1}{\Gamma(v+1)}A_0$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{\Gamma(n)}{\Gamma(2i+2)}\frac{\mathbf{B}(\alpha-2(n-1-j)+q,p+1)}{\Gamma(\alpha-2(n-1-j))}a_0+\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}a_0,$$
$$M_u = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{\Gamma(n)}{\Gamma(2i+2)}\sum_{w=1}^{m}\sum_{v=2(n-1-j)}^{2n-1}\frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)}A_u$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{\Gamma(n)}{\Gamma(2i+2)}\frac{\mathbf{B}(\alpha-2(n-1-j)+q,p+1)}{\Gamma(\alpha-2(n-1-j))}a_u+\sum_{w=1}^{m}\sum_{v=0}^{2n-1}\frac{1}{\Gamma(v+1)}A_u+\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}a_u, \; u\in\mathbb{N}_1^{\omega}.$$

THEOREM 5.1.1. *Suppose that (H1) holds. Then, the system (1.4.13) has at least one solution in X if*

(i) $\sigma<1$ or

(ii) $\sigma=1$ with $\sum_{u=1}^{\omega}M_u<1$ or

(iii) $\sigma > 1$ *with*

$$\sigma_{u_0} > 1, \quad M_0 + \sum_{u=1}^{\omega} M_u \left(\frac{M_0}{M_{u_0}(\sigma_{u_0}-1)}\right)^{\sigma_u/\sigma_{u_0}} \leqslant \left(\frac{M_0}{M_{u_0}(\sigma_{u_0}-1)}\right)^{1/\sigma_{u_0}}.$$

Proof. We shall apply the Schauder's fixed point theorem. From Lemma 5.2 and Remark 5.1 we note that T is completely continuous. If x is a fixed point of T, the system (1.4.13) has a solution x.

Let $\Omega_r = \{x \in X : ||x|| \leqslant r\}$. For $x \in \Omega_r$. Then $||x|| \leqslant r$, i.e., $|x(t)| \leqslant r$ for all $t \in (0,1]$. So (H1) implies

$$|f(t,x(t))| \leqslant [a_0 + \sum_{i=1}^{\omega} a_i |x(t)|^{\sigma_i}] t^p (1-t)^q \leqslant [a_0 + \sum_{i=1}^{\omega} a_i r^{\sigma_i}] t^p (1-t)^q,$$

$$\left|I_j(t_s, x(t_s))\right| \leqslant A_0 + \sum_{i=1}^{\omega} A_i |x(t_s)|^{\sigma_i} \leqslant A_0 + \sum_{i=1}^{\omega} A_i r^{\sigma_i}.$$

We know $|M_{ij}| \leqslant \Gamma(n)$. By (5.1.9), we have

$$\begin{aligned}
|(T_1x)(t)| \leqslant & \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{|M_{j+1n-i}|}{\Gamma(2i+2)} \sum_{w=1}^{m} \sum_{v=2(n-1-j)}^{2n-1} \frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)} t^{2i+1} |I_{vx}(t_w)| \\
& + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{|M_{j+1n-i}|}{\Gamma(2i+2)} \int_0^1 \frac{(1-s)^{\alpha-2(n-1-j)-1}}{\Gamma(\alpha-2(n-1-j))} |f_x(s)| ds t^{2i+1} \\
& + \sum_{w=1}^{s} \sum_{v=0}^{2n-1} \frac{(t-t_w)^v}{\Gamma(v+1)} |I_{vx}(t_w)| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_x(s)| ds \\
\leqslant & \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\Gamma(n)}{\Gamma(2i+2)} \sum_{w=1}^{m} \sum_{v=2(n-1-j)}^{2n-1} \frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)} \left[A_0 + \sum_{i=1}^{\omega} A_i r^{\sigma_i}\right] \\
& + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\Gamma(n)}{\Gamma(2i+2)} \int_0^1 \frac{(1-s)^{\alpha-2(n-1-j)-1}}{\Gamma(\alpha-2(n-1-j))} \left[a_0 + \sum_{i=1}^{\omega} a_i r^{\sigma_i}\right] s^p (1-s)^q ds \\
& + \sum_{w=1}^{m} \sum_{v=0}^{2n-1} \frac{1}{\Gamma(v+1)} \left[A_0 + \sum_{i=1}^{\omega} A_i r^{\sigma_i}\right] + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a_0 + \sum_{i=1}^{\omega} a_i r^{\sigma_i}\right] s^p (1-s)^q ds \\
\leqslant & \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\Gamma(n)}{\Gamma(2i+2)} \sum_{w=1}^{m} \sum_{v=2(n-1-j)}^{2n-1} \frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)} \left[A_0 + \sum_{i=1}^{\omega} A_i r^{\sigma_i}\right] \\
& + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\Gamma(n)}{\Gamma(2i+2)} \frac{\mathbf{B}(\alpha-2(n-1-j)+q, p+1)}{\Gamma(\alpha-2(n-1-j))} \left[a_0 + \sum_{i=1}^{\omega} a_i r^{\sigma_i}\right] \\
& + \sum_{w=1}^{m} \sum_{v=0}^{2n-1} \frac{1}{\Gamma(v+1)} \left[A_0 + \sum_{i=1}^{\omega} A_i r^{\sigma_i}\right] + \frac{\mathbf{B}(\alpha+q, p+1)}{\Gamma(\alpha)} \left[a_0 + \sum_{i=1}^{\omega} a_i r^{\sigma_i}\right] \\
= & \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\Gamma(n)}{\Gamma(2i+2)} \sum_{w=1}^{m} \sum_{v=2(n-1-j)}^{2n-1} \frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)} A_0 + \sum_{w=1}^{m} \sum_{v=0}^{2n-1} \frac{1}{\Gamma(v+1)} A_0 \\
& + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\Gamma(n)}{\Gamma(2i+2)} \frac{\mathbf{B}(\alpha-2(n-1-j)+q, p+1)}{\Gamma(\alpha-2(n-1-j))} a_0 + \frac{\mathbf{B}(\alpha+q, p+1)}{\Gamma(\alpha)} a_0
\end{aligned}$$

$$+\sum_{u=1}^{\omega}\left(\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{\Gamma(n)}{\Gamma(2i+2)}\sum_{w=1}^{m}\sum_{v=2(n-1-j)}^{2n-1}\frac{(1-t_w)^{v-2(n-1-j)}}{\Gamma(v-2(n-1-j)+1)}A_u\right.$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{\Gamma(n)}{\Gamma(2i+2)}\frac{\mathbf{B}(\alpha-2(n-1-j)+q,p+1)}{\Gamma(\alpha-2(n-1-j))}a_u$$
$$\left.+\sum_{w=1}^{m}\sum_{v=0}^{2n-1}\frac{1}{\Gamma(v+1)}A_u+\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}a_u\right)r^{\sigma_u}.$$

It follows that

$$||T_1x|| \leqslant M_0+\sum_{u=1}^{\omega}M_ur^{\sigma_u}. \tag{5.1.11}$$

In order to use Schauder's fixed point theorem, from (5.1.6), we should choose $r>0$ such that

$$M_0+\sum_{u=1}^{\omega}M_ur^{\sigma_u} \leqslant r. \tag{5.1.12}$$

Then $T\Omega_r \subseteq \Omega_r$. So T_1 has a fixed point in Ω_r. Then BVP(1.50) has a solution. We consider the following three cases:

Case 1. $\sigma<1$.

Since $\lim_{r\to\infty}\frac{M_0+\sum_{u=1}^{\omega}M_ur^{\sigma_u}}{r}=0$, we can choose $r>0$ sufficiently small such that (3.4) holds. Then $T\Omega_r\subseteq\Omega_r$. So T has a fixed point in Ω_r. Then BVP(1.4.13) has a solution.

Case 2. $\sigma=1$.

Since $\lim_{r\to\infty}\frac{M_0+\sum_{u=1}^{\omega}M_ur^{\sigma_u}}{r}<\sum_{u=1}^{n}M_u<1$, we can choose $r>0$ sufficiently small such that (5.1.12) holds. Then $T\Omega_r\subseteq\Omega_r$. So T has a fixed point in Ω_r. Then BVP(1.4.13) has a solution.

Case 3. $\sigma>1$.

Choose $r=\left(\frac{M_0}{M_{u_0}(\sigma_{u_0}-1)}\right)^{1/\sigma_{u_0}}$. Then we have by the inequality in (iii) that

$$||T_1x|| \leqslant M_0+\sum_{u=1}^{\omega}M_ur^{\sigma_u} \leqslant r.$$

Then $T\Omega_r\subseteq\Omega_r$. So T has a fixed point in Ω_r. Then BVP(1.4.13) has a solution.

The proof of Theorem 3.1 is completed. □

(H2) there exist constants M_f, $M_I \geqslant 0$ such that $|f(t,x)|\leqslant M_f$, $\left|I_j(t_s,x)\right|\leqslant M_I$ hold for all $t\in(0,1)$, $s\in\mathbb{N}_1^m$, $j\in\mathbb{N}_0^{n-1}$, $x\in\mathbb{R}$.

THEOREM 5.1.2. *Suppose that (H2) holds. Then BVP(1.4.13) has at least one solution.*

Proof. Choose $p=q=0$, $a_0=M_f, A_0=M_I$ and $a_i=0, A_i=0$, $\sigma_i=0$. One sees by (H2) that (H1) holds. By Theorem 5.1.1 (i), we get its proof. □

5.2. Existence of solutions of BVP(1.4.14)

The solvability of resonant boundary value problems for impulsive fractional differential equations were studied in [13, 14, 95]. Now we establish an existence theorem for BVP(1.4.14) by using Lemma 2.2. We consider the case $k<n-2$. The cases $k=n-2$ and $k=n-1$ are studied similarly.

We suppose that f satisfies that $u\to f(t,(t-t_i)^{n-\alpha}u)$, $u\to I_i(t_s,(t_s-t_{s-1})^{n-\alpha}u)$ are continuous functions, $t\to f(t,(t-t_i)^{n-\alpha}u)$ is measurable on $(t_i,t_{i+1}]$ $(i\in \mathbb{N}_0^m)$ for all $u\in\mathbb{R}$ and there exist constants $p>-1$, $q\in(-1,0]$ such that $|u|\leqslant r$ implies that there exists a constant M_r, $M_{r,I}$ such that $|f(t,(t-t_i)^{n-\alpha}u)|\leqslant M_r t^p(1-t)^q$ for almost all $t\in(0,1)$ and $|I_i(t_s,(t_s-t_{s-1})^{n-\alpha}u)|\leqslant M_{r,I}$ for all $i\in\mathbb{N}_0^m$.

Choose $Z=L^1(0,1)\times\mathbb{R}^{nm}$ with the norm $||(x,a_{ij}:i\in\mathbb{N}_0^{n-1},\ j\in\mathbb{N}_1^m)||=\max\{||x||_1,|a_{ij}|:i\in\mathbb{N}_1^n,\ j\in\mathbb{N}_1^m\}$.

Choose $E=PC_{n-\alpha}[0,1]=\{x\colon(0,1]\to\mathbb{R},\ x|_{(t_i,t_{i+1}]}\in C(t_i,t_{i+1}],\ \lim\limits_{t\to t_i^+}(t-t_i)^{n-\alpha}x(t)$ is finite with the norm $||x||=\max\left\{\sup\limits_{t\in(t_i,t_{i+1}]}(t-t_i)^{n-\alpha}|x(t)|:i\in\mathbb{N}_0^m\right\}$ and

$$D(L)=\{x\in X:D_{0^+}^{\alpha}x\in L^1(0,1),\ D_{0^+}^{\alpha-i}u(0)=0,\ i\in\mathbb{N}_1^k,\ D_{0^+}^{\alpha-j}u(1)=0,\ j\in\mathbb{N}_k^{n-1}\}.$$

Define the linear operator $L:X\bigcap D(L)\to Z$ and the nonlinear operator $N:X\to Z$ by

$$(Lx)(t)=\begin{pmatrix} D_{0^+}^{\alpha}x(t)\\ \Delta D_{0^+}^{\alpha-1}x(t_s):s\in\mathbb{N}_1^m\\ \cdots\\ \Delta D_{0^+}^{\alpha-(n-1)}x(t_s):s\in\mathbb{N}_1^m,\\ \Delta I_{0^+}^{n-\alpha}x(t_s):s\in\mathbb{N}_1^m\end{pmatrix},\quad (Nx)(t)=\begin{pmatrix} f_x(t)\\ I_{1x}(t_s):s\in\mathbb{N}_1^m\\ \cdots\\ I_{n-1x}(t_s):s\in\mathbb{N}_1^m,\\ I_{nx}(t_s):s\in\mathbb{N}_1^m\end{pmatrix}. \tag{5.2.1}$$

Denote M_{uv} be the algebraic cofactor of m_{uv} in

$$M=(m_{uv})_{(n-k-1)\times(n-k-1)}=\begin{pmatrix} 1 & 0 & \cdots & 0\\ \frac{1}{\Gamma(2)} & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ \frac{1}{\Gamma(n-k-1)} & \frac{1}{\Gamma(n-k-2)} & \cdots & 1\end{pmatrix}.$$

On sees that $|M_{uv}|\leqslant\Gamma(n-k-1)$ for all $u,v\in\mathbb{N}_1^k$.

LEMMA 5.2.1. *(i) L is a Fredholm operator of index zero.*
(ii) $N:\overline{\Omega}\to Z$ is called L-compact for bounded set $\overline{\Omega}\sqsubseteq X$.
(iii) x is a solution of BVP(1.51) if and only if $Lx=Nx$.

Proof. Firstly we prove that $L:D(L)\subset E\to Z$ is a Fredholm operator of index zero.

Claim 1.

$$\text{Ker}L=\{c_{n0}t^{\alpha-n}:c_{n0}\in\mathbb{R}\}. \tag{5.2.2}$$

In fact, $x\in$ Ker$L$ if and only if

$$\begin{pmatrix} D_{0^+}^{\alpha}x(t)\\ \Delta D_{0^+}^{\alpha-1}x(t_s):s\in \mathbb{N}_1^m\\ \cdots\\ \Delta D_{0^+}^{\alpha-(n-1)}x(t_s):s\in \mathbb{N}_1^m,\\ \Delta I_{0^+}^{n-\alpha}x(t_s):s\in \mathbb{N}_1^m\end{pmatrix}=\begin{pmatrix}0\\ 0:s\in \mathbb{N}_1^m\\ 0:s\in \mathbb{N}_1^m\\ \cdots\\ 0:s\in \mathbb{N}_1^m\end{pmatrix}.$$

Use Lemma Corollary 3.1.1 and Corollary 3.1.2, we have $x\in D(L)$ and

$$x(t)=\sum_{w=0}^{i}\sum_{v=1}^{n}\frac{c_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v},\quad t\in(t_i,t_{i+1}],\quad i\in \mathbb{N}_0^m,$$

$$D_{0^+}^{\alpha-j}x(t)=\sum_{w=0}^{i}\sum_{v=1}^{j}\frac{c_{vw}}{\Gamma(j-v+1)}(t-t_w)^{j-v},\quad t\in(t_i,t_{i+1}],\quad i\in \mathbb{N}_0^m,\quad j\in \mathbb{N}_1^{n-1},$$

$$I_{0^+}^{n-\alpha}x(t)=\sum_{w=0}^{i}\sum_{v=1}^{n}\frac{c_{vw}}{\Gamma(n-v+1)}(t-t_w)^{n-v},\quad t\in(t_i,t_{i+1}],\quad i\in \mathbb{N}_0^m.$$

(i) By $D_{0^+}^{\alpha-i}x(0)=0,\ i\in \mathbb{N}_1^k$, we get $c_{i0}=0,\ i\in \mathbb{N}_1^k$.
(ii) By $\Delta D_{0^+}^{\alpha-j}x(t_s)=0$, we get $c_{ji}=0,\ j\in \mathbb{N}_1^{n-1},\ i\in \mathbb{N}_1^m$.
(iii) By $\Delta I_{0^+}^{n-\alpha}x(t_s)$, we get $c_{ni}=0,\ i\in \mathbb{N}_1^m$.
(iv) By $D_{0^+}^{\alpha-j}x(1)=0,\ j\in \mathbb{N}_k^{n-1}$ together with (i)–(iii), we have

$$\sum_{v=k+1}^{j}\frac{c_{v0}}{\Gamma(j-v+1)}=0,\quad j\in \mathbb{N}_k^{n-1}.$$

Then $c_{i0}=0,\ i\in \mathbb{N}_k^{n-1}$. Hence $x(t)=c_{n0}t^{\alpha-n}$. On the other hand, we have $c_{n0}t^{\alpha-n}\in$ Ker$L$. Then (5.2.2) holds.

Claim 2.

$$\mathrm{Im}L=\left\{(u,a_{ij}):\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{a_{vw}}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}u(s)ds=0\right\}.\qquad(5.2.3)$$

In fact, $(u,a_{ij}:i\in \mathbb{N}_0^{n-1},\ j\in \mathbb{N}_1^m)\in \mathrm{Im}L$ if and only if there exists $x\in D(L)$ such that

$$\begin{pmatrix} D_{0^+}^{\alpha}x(t)\\ \Delta D_{0^+}^{\alpha-1}x(t_s):s\in \mathbb{N}_1^m\\ \cdots\\ \Delta D_{0^+}^{\alpha-(n-1)}x(t_s):s\in \mathbb{N}_1^m,\\ \Delta I_{0^+}^{n-\alpha}x(t_s):s\in \mathbb{N}_1^m\end{pmatrix}=\begin{pmatrix}u(t)\\ a_{1s}:s\in \mathbb{N}_1^m\\ a_{2s}:s\in \mathbb{N}_1^m\\ \cdots\\ a_{n-1s}:s\in \mathbb{N}_1^m\\ a_{ns}:s\in \mathbb{N}_1^m\end{pmatrix}.$$

By Corollary 3.1, we know that there exist constants $c_{vw}\in \mathbb{R}\ (v\in \mathbb{N}_1^m,\ w\in \mathbb{N}_0^m)$ such that $x\in D(L)\bigcap X$ and

$$x(t)=\sum_{w=0}^{s}\sum_{v=1}^{n}\frac{c_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds,\quad t\in(t_s,t_{s+1}],\quad s\in \mathbb{N}_0^m.$$

By Definition 2.2, we have for $j\in\mathbb{N}_0^{n-1}$ that

$$D_{0^+}^{\alpha-j}x(t)=\sum_{w=0}^{s}\sum_{v=1}^{j}\frac{c_{vw}}{\Gamma(j-v+1)}(t-t_w)^{j-v}+\int_0^t\frac{(t-s)^{j-1}}{\Gamma(j)}u(s)ds,\quad t\in(t_s,t_{s+1}],\quad s\in\mathbb{N}_0^m,$$
$$I_{0^+}^{n-\alpha}x(t)=\sum_{w=0}^{s}\sum_{v=1}^{n}\frac{c_{vw}}{\Gamma(n-v+1)}(t-t_w)^{n-v}+\int_0^t\frac{(t-s)^{n-1}}{\Gamma(n)}u(s)ds,\quad t\in(t_s,t_{s+1}],\quad s\in\mathbb{N}_0^m. \tag{5.2.4}$$

(i) From $\Delta D_{0^+}^{\alpha-j}x(t_s)=a_{js}$ and (5.2.4), we get $c_{js}=a_{js}$ $(j\in\mathbb{N}_1^{n-1},\ s\in\mathbb{N}_1^m)$.
(ii) From $D_{0^+}^{\alpha-i}x(0)=0,\ i\in\mathbb{N}_1^k$, we get $c_{i0}=0,\ i\in\mathbb{N}_{n-k}^{n-1}$.
(iii) From $\Delta I_{0^+}^{n-\alpha}x(t_s)=a_{ns}$ and (5.2.4), we get $c_{ns}=a_{ns}$ $(s\in\mathbb{N}_1^m)$.
(iv) By $D_{0^+}^{\alpha-j}x(1)=0,\ j\in\mathbb{N}_k^{n-1}$, we have from (5.2.4), (i)–(iii) that

$$\sum_{v=k+1}^{j}\frac{c_{v0}}{\Gamma(j-v+1)}+\sum_{w=1}^{m}\sum_{v=1}^{j}\frac{a_{vw}}{\Gamma(j-v+1)}(1-t_w)^{j-v}+\int_0^1\frac{(1-s)^{j-1}}{\Gamma(j)}u(s)ds=0,\quad j\in\mathbb{N}_k^{n-1}.$$

So

$$\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{a_{vw}}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}u(s)ds=0 \tag{5.2.5}$$

and

$$\begin{pmatrix}1&0&\cdots&0\\ \frac{1}{\Gamma(2)}&1&\cdots&0\\ \cdots&\cdots&\cdots&\cdots\\ \frac{1}{\Gamma(n-k-1)}&\frac{1}{\Gamma(n-k-2)}&\cdots&1\end{pmatrix}\begin{pmatrix}c_{k+10}\\ \cdots\\ c_{n-10}\end{pmatrix}$$
$$=-\begin{pmatrix}\sum_{w=1}^{m}\sum_{v=1}^{k+1}\frac{a_{vw}}{\Gamma(k+1-v+1)}(1-t_w)^{k+1-v}+\int_0^1\frac{(1-s)^{k+1-1}}{\Gamma(k+1)}u(s)ds\\ \sum_{w=1}^{m}\sum_{v=1}^{k+2}\frac{a_{vw}}{\Gamma(k+2-v+1)}(1-t_w)^{k+2-v}+\int_0^1\frac{(1-s)^{k+2-1}}{\Gamma(k+2)}u(s)ds\\ \cdots\\ \sum_{w=1}^{m}\sum_{v=1}^{n-1}\frac{a_{vw}}{\Gamma(n-1-v+1)}(1-t_w)^{n-1-v}+\int_0^1\frac{(1-s)^{n-1-1}}{\Gamma(n-1)}u(s)ds\end{pmatrix}.$$

Then

$$\begin{pmatrix}c_{k+10}\\ \cdots\\ c_{n-10}\end{pmatrix}=-\begin{pmatrix}1&0&\cdots&0\\ \frac{1}{\Gamma(2)}&1&\cdots&0\\ \cdots&\cdots&\cdots&\cdots\\ \frac{1}{\Gamma(n-k-1)}&\frac{1}{\Gamma(n-k-2)}&\cdots&1\end{pmatrix}^{-1}$$
$$\times\begin{pmatrix}\sum_{w=1}^{m}\sum_{v=1}^{k+1}\frac{a_{vw}}{\Gamma(k+1-v+1)}(1-t_w)^{k+1-v}+\int_0^1\frac{(1-s)^{k+1-1}}{\Gamma(k+1)}u(s)ds\\ \sum_{w=1}^{m}\sum_{v=1}^{k+2}\frac{a_{vw}}{\Gamma(k+2-v+1)}(1-t_w)^{k+2-v}+\int_0^1\frac{(1-s)^{k+2-1}}{\Gamma(k+2)}u(s)ds\\ \cdots\\ \sum_{w=1}^{m}\sum_{v=1}^{n-1}\frac{a_{vw}}{\Gamma(n-1-v+1)}(1-t_w)^{n-1-v}+\int_0^1\frac{(1-s)^{n-1-1}}{\Gamma(n-1)}u(s)ds\end{pmatrix}.$$

Then

$$\begin{pmatrix} c_{k+10} \\ \cdots \\ c_{n-10} \end{pmatrix} = -\begin{pmatrix} \sum_{\sigma=1}^{n-k-1} M_{\sigma 1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma} \frac{a_{vw}(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)} + \int_0^1 \frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)} u(s)ds\right) \\ \sum_{\sigma=1}^{n-k-1} M_{\sigma 2}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma} \frac{a_{vw}(1-t_w)^{k+\sigma-v}\Gamma(k+\sigma-v+1)}{\Gamma(k+\sigma-v+1)} + \int_0^1 \frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)} u(s)ds\right) \\ \cdots \\ \sum_{\sigma=1}^{n-k-1} M_{\sigma n-k-1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma} \frac{a_{vw}(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)} + \int_0^1 \frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)} u(s)ds\right) \end{pmatrix}. \tag{5.2.6}$$

On the other hand, if $(u, a_{ij} : i \in \mathbb{N}_0^{n-1},\ j \in \mathbb{N}_1^m)$ satisfies (5.2.3), we can prove that there exists $x \in D(L) \bigcap X$ such that $Lx = (u, a_{ij} : i \in \mathbb{N}_0^{n-1},\ j \in \mathbb{N}_1^m)$. In fact, from above discussion, we know that

$$\begin{aligned} x(t) = & \sum_{\sigma=1}^{n-k-1} M_{\sigma 1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma} \frac{a_{vw}(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)} + \int_0^1 \frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)} u(s)ds\right) t^{\alpha-(k+1)} + \cdots \\ & + \sum_{\sigma=1}^{n-k-1} M_{\sigma n-k-1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma} \frac{a_{vw}(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)} + \int_0^1 \frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)} u(s)ds\right) t^{\alpha-(n-1)} \\ & + c_{n0} t^{\alpha-n} + \sum_{w=1}^{i}\sum_{v=1}^{n} \frac{a_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds, t \in (t_i, t_{i+1}], \\ & i \in \mathbb{N}_1^m. \end{aligned} \tag{5.2.7}$$

Hence (5.2.3) is valid.

Claim 3. $L : D(L) \subset X \to Z$ is a Fredholm operator of index zero.

It follows from Claim 1 and Claim 2 that $\dim \mathrm{Ker} L = 1$ and $\mathrm{Im} L$ is closed in Z. Furthermore, define projectors $P : X \to \mathrm{Ker} L$ and $Q : Z \to \mathrm{Im} L$ by

$$Px(t) = \frac{I_{0^+}^{n-\alpha} x(0)}{\Gamma(\alpha-n+1)} t^{\alpha-n}, \quad x \in X,$$

$$Q(u, a_{ij} : i \in \mathbb{N}_1^n,\ j \in \mathbb{N}_1^m) = \left(\overline{Q}_{u,a_{ij}}, 0 : i \in \mathbb{N}_0^{n-1},\ j \in \mathbb{N}_1^m\right),$$

where

$$\overline{Q}_{u,a_{ij}} = \Gamma(k+1)\left[\sum_{w=1}^{m}\sum_{v=1}^{k} \frac{a_{vw}}{\Gamma(k-v+1)}(1-t_w)^{k-v} + \int_0^1 \frac{(1-s)^{k-1}}{\Gamma(k)} u(s)ds\right].$$

It is easy to see that $P : X \to \mathrm{Ker} L$ and $Q : Z \to Z/\mathrm{Im} L$ are well defined and

$$\mathrm{Im}\, P = \mathrm{Ker}\, L,\ \mathrm{Ker}\, Q = \mathrm{Im}\, L, \quad X = \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P, \quad Z = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q.$$

So $\dim \mathrm{Ker} L =$ co $\dim \mathrm{Im} L = 1 < +\infty$. Then $L : D(L) \subset X \to Z$ is a Fredholm operator of index zero.

The inverse of $L|_{D(L) \cap \mathrm{Ker}\, P} : D(L) \cap \mathrm{Ker}\, P \to \mathrm{Im}\, L$ is denoted by $K_P : \mathrm{Im} L \to D(L) \cap \mathrm{Ker}\, P$ with

$$K_p(u, a_{ij} : i \in \mathbb{N}_0^{n-1},\ j \in \mathbb{N}_1^m)(t)$$

$$= \sum_{\sigma=1}^{n-k-1} M_{\sigma 1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{a_{vw}(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}u(s)ds\right)t^{\alpha-(k+1)}+\cdots$$
$$+\sum_{\sigma=1}^{n-k-1} M_{\sigma n-k-1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{a_{vw}(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}u(s)ds\right)t^{\alpha-(n-1)}$$
$$+\sum_{w=1}^{i}\sum_{v=1}^{n}\frac{a_{vw}}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds, t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_1^m.$$

Claim 4. For each nonempty open bounded subset Ω of E satisfying $D(L)\cap\overline{\Omega}\neq\emptyset$, $N:\overline{\Omega}\to Z$ is L-compact.

We need to prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N(\overline{\Omega})$ is bounded and relatively compact.

One sees that

$$QNx(t)=Q\begin{pmatrix} f_x(t)\\ I_{1x}(t_s):s\in\mathbb{N}_1^m\\ I_{2x}(t_s):s\in\mathbb{N}_1^m\\ \cdots\\ I_{nx}(t_s):s\in\mathbb{N}_1^m\end{pmatrix}=\left(\overline{Q}_{f_x(t),I_{ix}(t_j)},0:i\in\mathbb{N}_0^{n-1},\ j\in\mathbb{N}_1^m\right),$$

where

$$\overline{Q}_{f_x(t),I_{ix}(t_j)}=\Gamma(k+1)\left[\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_{vx}(t_w)}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f_x(s)ds\right].$$

By direct computation, we have

$$K_p(I-Q)Nx(t)=K_PN_1x(t)-K_PQNx(t)=K_p\begin{pmatrix} f_x(t)-\overline{Q}_{f_x,I_{ix}(t_j)}\\ I_{1x}(t_s):s\in\mathbb{N}_1^m\\ I_{2x}(t_s):s\in\mathbb{N}_1^m\\ \cdots\\ I_{nx}(t_s):s\in\mathbb{N}_1^m\end{pmatrix}$$
$$=\sum_{\sigma=1}^{n-k-1}M_{\sigma 1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{I_{vx}(t_w)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}[f_x(s)-\overline{Q}_{f_x,I_{ix}(t_j)}]ds\right)t^{\alpha-(k+1)}+\cdots$$
$$+\sum_{\sigma=1}^{n-k-1}M_{\sigma n-k-1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{I_{vx}(t_w)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}[f_x(s)-\overline{Q}_{f_x,I_{ix}(t_j)}]ds\right)t^{\alpha-(n-1)}$$
$$+\sum_{w=1}^{i}\sum_{v=1}^{n}\frac{I_{vx}(t_w)}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[f_x(s)-\overline{Q}_{f_x,I_{ix}(t_j)}]ds,\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_1^m.$$

We can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N(\overline{\Omega})$ is bounded and relatively compact. Hence $N:\overline{\Omega}\to Z$ is called L-compact for bounded set $\overline{\Omega}\sqsubseteq X$.

Claim 5. x is a solution of BVP(1.51) if and only if $Lx=Nx$. □

We need the following assumptions:

(H1) there exist constants $p>-1$, $q\in(-1,0]$, nonnegative no-decreasing functions $\prod_f,\prod_j:[0,\infty)\to[0,\infty)$ such that

$$|f(t,x)|\leqslant\prod_f(|x|)t^p(1-t)^q,\quad t\in(t_s,t_{s+1}),\quad s\in\mathbb{N}_0^m,$$

$$\left|I_j(t_s,x)\right| \leqslant \prod_I(|x|), \quad s\in \mathbb{N}_1^m, \quad j\in \mathbb{N}_0^{n-1}.$$

(H2) there exists a constant $M>0$ such that for $x\in X$ with $|x(t)|>M$ for all $t\in(0,1]$ implies that

$$\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_{vx}(t_w)}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f_x(s)ds\neq 0.$$

(H3) there exists a constant $M_0>0$ such that

$$c\left[\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w,ct_w^{\alpha-n})}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f(s,cs^{\alpha-n})ds\right]>0$$

holds for all $|c|>M_0$ or

$$c\left[\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w,ct_w^{\alpha-n})}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f(s,cs^{\alpha-n})ds\right]<0.$$

holds for all $|c|>M_0$.

THEOREM 5.2.1. *Suppose that (H1)–(H3) hold. Then BVP(1.4.14) has at least one solution if*

$$\lim_{r\to\infty}\frac{r}{M+A\prod_I(r)+B\prod_f(r)}>1, \tag{5.2.8}$$

where

$$A=(4\Gamma(n-k)m+2m\Gamma(n-k+1))+\sum_{v=1}^{n}\frac{2m}{\Gamma(\alpha-v+1)},$$

$$B=4\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\frac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}+\frac{2\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}.$$

Proof. Let X, Z L and N be defined by (5.2.1). By (H1)–(H3), from Lemma 5.2.1, L be a Fredholm operator of index zero and N be L-compact on each closed nonempty set Ω centered at zero. We seek fixed point of the operator equation $Lx=Nx$. To apply Lemma 2.2, we should define an open bounded subset Ω of X centered at zero such that (i), (ii) and (iii) in Lemma 2.2 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps.

Step 1. Let $\Omega_1=\{x\in X\cap D(L)\setminus \mathrm{Ker}L,\ Lx=\lambda Nx \text{ for some } \lambda\in(0,1)\}$. We prove that Ω_1 is bounded.

In fact, for $x\in\Omega_1$, we have $Lx=\lambda Nx$ and $Nx\in \mathrm{Im}L$. Then

$$\begin{cases} D_{0^+}^{\alpha}x(t)=\lambda f(t,x(t)), \quad t\in(t_s,t_{s+1}], \quad s\in\mathbb{N}_0^m,\\ D_{0^+}^{\alpha-i}x(0)=0, \quad i\in\mathbb{N}_1^k,\\ D_{0^+}^{\alpha-j}x(1)=0, \quad j\in\mathbb{N}_k^{n-1},\\ \Delta I_{0^+}^{n-\alpha}x(t_s)=\lambda I_n(t_s,x(t_s)), \quad s\in\mathbb{N}_1^m,\\ \Delta D_{0^+}^{\alpha-j}x(t_s)=\lambda I_j(t_s,x(t_s)), \quad j\in\mathbb{N}_1^{n-1}, \quad s\in\mathbb{N}_1^m.\end{cases}$$

So

$$\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_{vx}(t_w)}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f_x(s)ds=0. \tag{5.2.9}$$

It follows from (H1) that

$$|f(t,x(t))|\leqslant\prod_f((t-t_s)^{n-\alpha}|x(t)|)t^p(1-t)^q\leqslant\prod_f(||x||)t^p(1-t)^q,\ t\in(t_s,t_{s+1}),\ s\in\mathbb{N}_0^m,$$
$$\left|I_j(t_s,x(t_s))\right|\leqslant\prod_I(||x||),\ s\in\mathbb{N}_1^m,\ j\in\mathbb{N}_1^n.$$

It follows from (H2) and (5.2.9) that there exists $\overline{t}\in(t_i,t_{i+1}]$ (for some $i\in\mathbb{N}_0^m$) such that

$$|x(\overline{t})|\leqslant M. \tag{5.2.10}$$

We have from Claim 2 that

$$\begin{aligned}x(\overline{t})=&\sum_{\sigma=1}^{n-k-1}M_{\sigma 1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{\lambda I_{vx}(t_w)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}\lambda f_x(s)ds\right)\overline{t}^{\alpha-(k+1)}+\cdots\\&+\sum_{\sigma=1}^{n-k-1}M_{\sigma n-k-1}\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{\lambda I_{vx}(t_w)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}\lambda f_x(s)ds\right)\overline{t}^{\alpha-(n-1)}\\&+c_{n0}\overline{t}^{\alpha-n}+\sum_{w=1}^{i}\sum_{v=1}^{n}\frac{\lambda I_{vx}(t_w)}{\Gamma(\alpha-v+1)}(\overline{t}-t_w)^{\alpha-v}+\int_0^{\overline{t}}\frac{(\overline{t}-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda f_x(s)ds.\end{aligned}$$

Then

$$\begin{aligned}&|c_{n0}|\\&=\overline{t}^{n-\alpha}\Bigg[|x(\overline{t})|+\sum_{\sigma=1}^{n-k-1}|M_{\sigma 1}|\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{|I_{vx}(t_w)|(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}|f_x(s)|ds\right)\overline{t}^{\alpha-(k+1)}\\&\quad+\cdots+\sum_{\sigma=1}^{n-k-1}|M_{\sigma n-k-1}|\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{|I_{vx}(t_w)|(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}|f_x(s)|ds\right)\overline{t}^{\alpha-(n-1)}\\&\quad+\sum_{w=1}^{i}\sum_{v=1}^{n}\frac{|I_{vx}(t_w)|}{\Gamma(\alpha-v+1)}(\overline{t}-t_w)^{\alpha-v}+\int_0^{\overline{t}}\frac{(\overline{t}-s)^{\alpha-1}}{\Gamma(\alpha)}|f_x(s)|ds\Bigg]\\&\leqslant M+\sum_{\sigma=1}^{n-k-1}\Gamma(n-k-1)\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{\prod_I(||x||)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}\prod_f(||x||)s^p(1-s)^q ds\right)\\&\quad+\cdots+\sum_{\sigma=1}^{n-k-1}\Gamma(n-k-1)\left(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{\prod_I(||x||)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\frac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}\prod_f(||x||)s^p(1-s)^q ds\right)\\&\quad+\sum_{w=1}^{i}\sum_{v=1}^{n}\frac{\prod_I(||x||)}{\Gamma(\alpha-v+1)}+\overline{t}^{n-\alpha}\int_0^{\overline{t}}\frac{(\overline{t}-s)^{\alpha-1}}{\Gamma(\alpha)}\prod_f(||x||)s^p(1-s)^q ds\\&\leqslant M+\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\left(\prod_I(||x||)\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{1}{\Gamma(k+\sigma-v+1)}+\prod_f(||x||)\frac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\right)\\&\quad+\cdots+\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\left(\prod_I(||x||)\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\frac{1}{\Gamma(k+\sigma-v+1)}+\frac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\prod_f(||x||)\right)\\&+\prod_I(||x||)\sum_{w=1}^{i}\sum_{v=1}^{n}\frac{1}{\Gamma(\alpha-v+1)}+\overline{t}^{n+p+q}\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\prod_f(||x||)\end{aligned}$$

$$\leqslant M+2\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\Big(m(k+\sigma)\prod_I(||x||)+\tfrac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\prod_f(||x||)\Big)$$
$$+\prod_I(||x||)\sum_{v=1}^{n}\tfrac{m}{\Gamma(\alpha-v+1)}+\tfrac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\prod_f(||x||).$$

Then for $t\in(t_s,t_{s+1}]$, we have

$$\begin{aligned}
&(t-t_s)^{n-\alpha}|x(t)|\\
&=(t-t_s)^{n-\alpha}\Bigg|\sum_{\sigma=1}^{n-k-1}M_{\sigma 1}\Big(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\tfrac{\lambda I_{vx}(t_w)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\tfrac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}\lambda f_x(s)ds\Big)t^{\alpha-(k+1)}\\
&\quad+\cdots+\sum_{\sigma=1}^{n-k-1}M_{\sigma n-k-1}\Big(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\tfrac{\lambda I_{vx}(t_w)(1-t_w)^{k+\sigma-v}}{\Gamma(k+\sigma-v+1)}+\int_0^1\tfrac{(1-s)^{k+\sigma-1}}{\Gamma(k+\sigma)}\lambda f_x(s)ds\Big)t^{\alpha-(n-1)}\\
&\quad+c_{n0}t^{\alpha-n}+\sum_{w=1}^{s}\sum_{v=1}^{n}\tfrac{\lambda I_{vx}(t_w)}{\Gamma(\alpha-v+1)}(t-t_w)^{\alpha-v}+\int_0^t\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda f_x(s)ds\Bigg|\\
&\leqslant\sum_{\sigma=1}^{n-k-1}|M_{\sigma 1}|\Big(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\tfrac{\prod_I(||x||)}{\Gamma(k+\sigma-v+1)}+\tfrac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\prod_f(||x||)\Big)\\
&\quad+\cdots+\sum_{\sigma=1}^{n-k-1}|M_{\sigma n-k-1}|\Big(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\tfrac{\prod_I(||x||)}{\Gamma(k+\sigma-v+1)}+\tfrac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\prod_f(||x||)\Big)\\
&\quad+|c_{n0}|+\sum_{w=1}^{s}\sum_{v=1}^{n}\tfrac{\prod_I(||x||)}{\Gamma(\alpha-v+1)}+\tfrac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\prod_f(||x||)\\
&\leqslant 2\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\Big(\sum_{w=1}^{m}\sum_{v=1}^{k+\sigma}\tfrac{1}{\Gamma(k+\sigma-v+1)}\prod_I(||x||)+\tfrac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\prod_f(||x||)\Big)\\
&\quad+|c_{n0}|+\sum_{v=1}^{n}\tfrac{m}{\Gamma(\alpha-v+1)}\prod_I(||x||)+\tfrac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\prod_f(||x||)\\
&\leqslant M+4\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\Big(m(k+\sigma)\prod_I(||x||)+\tfrac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\prod_f(||x||)\Big)\\
&\quad+\sum_{v=1}^{n}\tfrac{2m}{\Gamma(\alpha-v+1)}\prod_I(||x||)+\tfrac{2\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\prod_f(||x||)\\
&\leqslant M+(4\Gamma(n-k)m+2m\Gamma(n-k+1))\prod_I(||x||)\\
&\quad+4\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\tfrac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}\prod_f(||x||)\\
&\quad+\sum_{v=1}^{n}\tfrac{2m}{\Gamma(\alpha-v+1)}\prod_I(||x||)+\tfrac{2\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\prod_f(||x||).
\end{aligned}$$

It follows that

$$\begin{aligned}
||x||\leqslant M&+\Big[(4\Gamma(n-k)m+2m\Gamma(n-k+1))+\sum_{v=1}^{n}\tfrac{2m}{\Gamma(\alpha-v+1)}\Big]\prod_I(||x||)\\
&+\Big[4\Gamma(n-k-1)\sum_{\sigma=1}^{n-k-1}\tfrac{\mathbf{B}(k+q+\sigma,p+1)}{\Gamma(k+\sigma)}+\tfrac{2\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}\Big]\prod_f(||x||).
\end{aligned}\tag{5.2.11}$$

Since $\lim\limits_{r\to\infty}\frac{r}{M+A\prod_I(r)+B\prod_f(r)}>1$, we get from (6.21) that there exists a constant $M_4>0$ independent of λ such that $||x||\leqslant M_4$. It follows that Ω_1 is bounded.

Step 2. Let $\Omega_2=\{ct^{\alpha-n}\in \mathrm{Ker}L: N(ct^{\alpha-n})\in \mathrm{Im}L\}$. We prove that Ω_2 is bounded. For $ct^{\alpha-n}\in\Omega_2$, we have

$$N(ct^{\alpha-n})=\begin{pmatrix} f(t,ct^{\alpha-n})\\ I_1(t_s,ct_s^{\alpha-n}):s\in\mathbb{N}_1^m\\ I_2(t_s,ct_s^{\alpha-n}):s\in\mathbb{N}_1^m\\ \cdots\\ I_n(t_s.ct_s^{\alpha-n}):s\in\mathbb{N}_1^m\end{pmatrix}.$$

So

$$\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w,ct_w^{\alpha-n})}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f(s,cs^{\alpha-n})ds=0.$$

From (H3), we get that $|c|\leqslant M_0$. This shows Ω_2 is bounded.

Step 3. If the first inequality in (H3) holds, we prove that $\Omega_3=\{ct^{\alpha-n}\in \mathrm{Ker}\,L: \lambda\wedge(ct^{\alpha-n})+(1-\lambda)QN(ct^{\alpha-n})=0,\ \lambda\in[0,1]\}$ is bounded, where $\wedge:\mathrm{Ker}L\to Z/\mathrm{Im}L$ is the isomorphism given by $\wedge(ct^{\alpha-n})=(c,0:i\in\mathbb{N}_0^{n-1},j\in\mathbb{N}_1^m)$.

For $ct^{\alpha-n}\in \mathrm{Ker}\,L$, one sees that

$$\begin{aligned}-\lambda\wedge(ct^{\alpha-n}) &= -\lambda(c,0:i\in\mathbb{N}_0^{n-1},j\in\mathbb{N}_1^m)\\ &=(1-\lambda)\left(\overline{Q}_{ct^{\alpha-n},I_i(t_j,ct_j^{\alpha-n})},0:i\in\mathbb{N}_0^{n-1},j\in\mathbb{N}_1^m\right).\end{aligned}$$

where

$$\overline{Q}_{ct^{\alpha-n},I_i(t_j,ct_j^{\alpha-n})}=\Gamma(k+1)\left[\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w,ct_w^{\alpha-n})}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f(s,cs^{\alpha-n})ds\right].$$

Then

$$-\lambda c^2=(1-\lambda)\Gamma(k+1)\left[\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w,ct_w^{\alpha-n})}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f(s,cs^{\alpha-n})ds\right].$$

If $\lambda=1$, we get $c=0$. If $\lambda\in[0,1)$, and $|c|>M_0$, we get

$$\begin{aligned}&0\geqslant-\lambda c^2\\ &=(1-\lambda)\Gamma(k+1)c\left[\sum_{w=1}^{m}\sum_{v=1}^{k}\frac{I_v(t_w,ct_w^{\alpha-n})}{\Gamma(k-v+1)}(1-t_w)^{k-v}+\int_0^1\frac{(1-s)^{k-1}}{\Gamma(k)}f(s,cs^{\alpha-n})ds\right]>0,\end{aligned}$$

a contradiction. Then $|c|\leqslant M_0$. Then Ω_3 is bounded.

If the second inequality in (H3) holds we can prove that $\Omega_3=\{ct^{\alpha-n}\in \mathrm{Ker}\,L: \lambda\wedge(ct^{\alpha-n})-(1-\lambda)QN(ct^{\alpha-n})=0,\ \lambda\in[0,1]\}$ is bounded.

Step 4. We shall show that all conditions of Lemma 2.2 are satisfied.

Set Ω be a open bounded subset of X centered at zero such that $\Omega\supset\bigcup_{i=1}^{3}\overline{\Omega_i}$. By Lemma 4.4, L is a Fredholm operator of index zero and N_1 is L-compact on $\overline{\Omega}$. By the definition of Ω, we have

(a) $L(x) \neq \lambda N(x)$ for $x \in (D(L) \setminus \mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b) $N(x) \notin \mathrm{Im}L$ for $x \in \mathrm{Ker}L \cap \partial\Omega$.

(c) $\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$. In fact, let $H(x,\lambda) = \pm\lambda \wedge (x) + (1-\lambda)QN(x)$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}L$, thus by homotopy property of degree,

$$\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) = \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0)$$
$$= \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\wedge, \Omega \cap \mathrm{Ker}L, 0) \neq 0.$$

Thus by Lemma 2.2, $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$. Then x is a solution of BVP(1.4.14). The proof is complete. □

5.3. Positive solutions of BVP(1.4.15)

The motivation of this section is [90]. We establish existence results for solutions of BVP(1.4.15). We suppose that

(H1) f is nonnegative, $u \to f(t,u)$ is continuous on $[0,\infty)$, $t \to f(t,u)$ is measurable on $(t_i, t_{i+1}]$ $(i \in \mathbb{N}_0^m)$ for all $u \in [0,\infty)$ and there exist constants $p > -1$, $q \in (n-1-\alpha, 0]$ such that $u \in [0,r]$ implies that there exists a constant $M_r \geqslant 0$ such that $|f(t,u)| \leqslant M_r t^p (1-t)^q$ for almost all $t \in (0,1)$, $f(t,0) \not\equiv 0$ on each subinterval of $(0,1)$.

(H2) $u \to f(t,u)$ is nonnegative, $u \to I_i(t_s,u)$ is continuous, and $u \in [0,r]$ implies that there exists a constant $M_{r,I} \geqslant 0$ such that $|I_i(t_s,u)| \leqslant M_{r,I}$ for all $i \in \mathbb{N}_0^{n-1}$, $s \in \mathbb{N}_1^m$.

LEMMA 5.3.1. *Suppose that there exist constants $p > -1$, $q \in (n-1-\alpha, 0]$ such that $|h(t)| \leqslant t^p(1-t)^q$ for almost all $t \in (0,1)$. Then $x \in X$ is a solution of*

$$\begin{cases} {}^cD_{0^+}^{\alpha} x(t) = h(t), \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m, \\ \Delta x^{(j)}(t_i) = I_j(t_i, x(t_i)), \quad i \in \mathbb{N}_1^m, \quad j \in \mathbb{N}_0^{n-1}, \\ x^{(j)}(0) - k_j x^{(j)}(1) = 0, \quad j \in \mathbb{N}_0^{n-1} \end{cases} \tag{5.3.1}$$

if and only if

$$\begin{aligned} x(t) = & \sum_{u=0}^{n-1} \frac{t^u}{\Gamma(u+1)} \Bigg[\frac{k_u}{1-k_u} \left(\sum_{\tau=1}^{m} \sum_{\sigma=j}^{n-1} \frac{I_\sigma(t_\tau, x(t_\tau))}{\Gamma(\sigma-j+1)} (1-t_\tau)^{\sigma-j} + \int_0^1 \frac{(1-s)^{\alpha-j-1}}{\Gamma(\alpha-j)} h(s)ds \right) \\ & + \sum_{i=u+1}^{n-1} \prod_{v=i-1}^{n-2} \left(\frac{k_v}{\Gamma(n-v)(1-k_v)} \right) \left(\frac{k_i}{1-k_i} \left(\sum_{\tau=1}^{m} \sum_{\sigma=i}^{n-1} \frac{I_\sigma(t_\tau, x(t_\tau))}{\Gamma(\sigma-i+1)} (1-t_\tau)^{\sigma-i} \right.\right. \\ & \left.\left. + \int_0^1 \frac{(1-s)^{\alpha-i-1}}{\Gamma(\alpha-i)} h(s)ds \right) \right) \Bigg] + \sum_{\tau=1}^{i} \sum_{\sigma=0}^{n-1} \frac{I_\sigma(t_\tau, x(t_\tau))}{\Gamma(\sigma+1)} (t-t_\tau)^\sigma + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \\ & t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \end{aligned} \tag{5.3.2}$$

Proof. Suppose that $x\in X$ is a solution of (5.3.1). From Corollary 3.2.3, there exist constants $c_{vj}\in\mathbb{R}$ $(v\in\mathbb{N}_0^{n-1},\ j\in\mathbb{N}_0^m)$ such that

$$x(t)=\sum_{\tau=0}^{i}\sum_{\sigma=0}^{n-1}\frac{c_{\sigma\tau}}{\Gamma(\sigma+1)}(t-t_\tau)^\sigma+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m. \quad (5.3.3)$$

Then

$$x^{(j)}(t)=\sum_{\tau=0}^{i}\sum_{\sigma=j}^{n-1}\frac{c_{\sigma\tau}}{\Gamma(\sigma-j+1)}(t-t_\tau)^{\sigma-j}+\int_0^t\frac{(t-s)^{\alpha-j-1}}{\Gamma(\alpha-j)}h(s)ds,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.$$

From $\Delta x^{(j)}(t_i)=I_j(t_i,x(t_i))$, $i\in\mathbb{N}_1^m$, $j\in\mathbb{N}_0^{n-1}$, we have $c_{ji}=I_j(t_i,x(t_i))$, $i\in\mathbb{N}_1^m$, $j\in\mathbb{N}_0^{n-1}$. From $x^{(j)}(0)-k_jx^{(j)}(1)=0$, $j\in\mathbb{N}_0^{n-1}$, we have

$$c_{j0}=k_j\left[\sum_{\tau=0}^{m}\sum_{\sigma=j}^{n-1}\frac{c_{\sigma\tau}}{\Gamma(\sigma-j+1)}(1-t_\tau)^{\sigma-j}+\int_0^1\frac{(1-s)^{\alpha-j-1}}{\Gamma(\alpha-j)}h(s)ds\right].$$

It follows that

$$c_{j0}-k_j\sum_{\sigma=j}^{n-1}\frac{c_{\sigma0}}{\Gamma(\sigma-j+1)}=k_j\left[\sum_{\tau=1}^{m}\sum_{\sigma=j}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-j+1)}(1-t_\tau)^{\sigma-j}+\int_0^1\frac{(1-s)^{\alpha-j-1}}{\Gamma(\alpha-j)}h(s)ds\right].$$

That is

$$\begin{pmatrix}1&-\frac{k_0}{\Gamma(2)(1-k_0)}&-\frac{k_0}{\Gamma(3)(1-k_0)}&\cdots&-\frac{k_0}{\Gamma(n-1)(1-k_0)}\\0&1&-\frac{k_1}{\Gamma(2)(1-k_1)}&\cdots&-\frac{k_1}{\Gamma(n-2)(1-k_1)}\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&1\end{pmatrix}\begin{pmatrix}c_{00}\\c_{10}\\c_{20}\\\cdots\\c_{n-10}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{k_0}{1-k_0}\left[\sum_{\tau=1}^{m}\sum_{\sigma=0}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-0+1)}(1-t_\tau)^{\sigma-0}+\int_0^1\frac{(1-s)^{\alpha-0-1}}{\Gamma(\alpha-0)}h(s)ds\right]\\\frac{k_1}{1-k_1}\left[\sum_{\tau=1}^{m}\sum_{\sigma=1}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-1+1)}(1-t_\tau)^{\sigma-1}+\int_0^1\frac{(1-s)^{\alpha-1-1}}{\Gamma(\alpha-1)}h(s)ds\right]\\\frac{k_2}{1-k_2}\left[\sum_{\tau=1}^{m}\sum_{\sigma=2}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-2+1)}(1-t_\tau)^{\sigma-2}+\int_0^1\frac{(1-s)^{\alpha-2-1}}{\Gamma(\alpha-2)}h(s)ds\right]\\\cdots\\\frac{k_{n-1}}{1-k_{n-1}}\left[\sum_{\tau=1}^{m}\sum_{\sigma=n-1}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-(n-1)+1)}(1-t_\tau)^{\sigma-(n-1)}+\int_0^1\frac{(1-s)^{\alpha-(n-1)-1}}{\Gamma(\alpha-(n-1))}h(s)ds\right]\end{pmatrix}.$$

Then for $j\in\mathbb{N}_0^{n-1}$ we have

$$\begin{aligned}c_{j0}=&\frac{k_j}{1-k_j}\left[\sum_{\tau=1}^{m}\sum_{\sigma=j}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-j+1)}(1-t_\tau)^{\sigma-j}+\int_0^1\frac{(1-s)^{\alpha-j-1}}{\Gamma(\alpha-j)}h(s)ds\right]\\&+\sum_{i=j+1}^{n-1}\prod_{v=i-1}^{n-2}\left[\frac{k_v}{\Gamma(n-v)(1-k_v)}\right]\end{aligned}$$

$$\times\left[\frac{k_i}{1-k_i}\left(\sum_{\tau=1}^{m}\sum_{\sigma=i}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-i+1)}(1-t_\tau)^{\sigma-i}+\int_0^1\frac{(1-s)^{\alpha-i-1}}{\Gamma(\alpha-i)}h(s)ds\right)\right].$$

Substituting c_{ji} into (5.3.3), we get (5.3.2). On the other hand, if (5.3.2) holds, we can prove that $x\in X$ and x is a solution of (5.3.1). The proof is complete. □

Define the following operator T on X by

$$\begin{aligned}(Tx)(t) = &\sum_{u=0}^{n-1}\frac{t^u}{\Gamma(u+1)}\left[\frac{k_u}{1-k_u}\left(\sum_{\tau=1}^{m}\sum_{\sigma=u}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-u+1)}(1-t_\tau)^{\sigma-u}+\int_0^1\frac{(1-s)^{\alpha-u-1}}{\Gamma(\alpha-u)}f(s,x(s))ds\right)\right.\\ &+\sum_{i=u+1}^{n-1}\prod_{v=i-1}^{n-2}\left(\frac{k_v}{\Gamma(n-v)(1-k_v)}\right)\\ &\left.\times\left(\frac{k_i}{1-k_i}\left(\sum_{\tau=1}^{m}\sum_{\sigma=i}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma-i+1)}(1-t_\tau)^{\sigma-i}+\int_0^1\frac{(1-s)^{\alpha-i-1}}{\Gamma(\alpha-i)}f(s,x(s)ds\right)\right)\right]\\ &+\sum_{\tau=1}^{i}\sum_{\sigma=0}^{n-1}\frac{I_\sigma(t_\tau,x(t_\tau))}{\Gamma(\sigma+1)}(t-t_\tau)^{\sigma}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s)ds,\quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m.\end{aligned}$$

REMARK 5.3.1. By Lemmas 5.3.1, $T:X\to X$ is well defined and $x\in X$ is a solution of system (1.52) if and only if $x\in X$ is a fixed point of the operator T.

LEMMA 5.3.2. *The operator $T:X\to X$ is completely continuous.*

Proof. The proof is standard and is omitted, one may see [81]. □

Now, we present the main theorem and its proof in this subsection. Define the cone in Banach space X and N by

$$P=\{x\in X: x(t)\geqslant 0 \text{ for all } t\in(0,1)\},$$
$$K=\max\left\{1,\ \frac{k_i}{1-k_i}: i\in\mathbb{N}_0^{n-1}\right\},$$
$$N=mn^2IK+mn^3IK^{n+1}+mnI+\frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)}+\left(K+nK^{n+1}\right)\sum_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+q-i,p+1)}{\Gamma(\alpha-i)}.$$

THEOREM 5.3.1. *Suppose that there exists $a>0$ such that*

(H3) $f(t,x_1)\leqslant f(t,x_2)$, $I_\sigma(t,x_1)\leqslant I_\sigma(t,x_2)$ for all $t\in(0,1)$ and $0\leqslant x_1\leqslant x_2\leqslant a$, $\sigma\in\mathbb{N}_1^m$.

(H4) there exist nonnegative constant $p>-1$, $q\in(-1,0]$ and $I\geqslant 0$, $M\geqslant 0$ such that

$$f(t,u)\leqslant \tfrac{a}{n}t^p(1-t)^q,\quad t\in(0,1),\quad u\in[0,1],$$
$$I_\sigma(t_s,u)\leqslant \tfrac{a}{n}I,\quad \sigma\in\mathbb{N}_0^{n-1},\quad s\in\mathbb{N}_1^m,\quad u\in[0,a].$$

Then BVP(1.4.15) has two positive solutions w^* and v^* such that $0<||w^*||\leqslant a$ and $\lim_{\mu\to\infty}w_\mu=\lim_{\mu\to\infty}T^\mu w_0=w^*$ with $w_0(t)=a$, $t\in(0,1)$, and $0<||v^*||\leqslant a$ with $\lim_{\mu\to\infty}v_\mu=\lim_{\mu\to\infty}T^\mu v_0=w^*$ and $v_0(t)\equiv 0$ on $(0,1)$.

Proof. By Lemma 5.3.1, Remark 5.3.1 and the definition of T, we know that $T:P\to P$ is completely continuous. For $x_1,x_2\in\overline{P}_a$ with $x_1(t)\leqslant x_2(t)$ for all $t\in(0,1)$, from the definition of T and (H2), we can easily get that $(Tx_1)(t) \leqslant (Tx_2)(t)$ for all $t \in (0,1)$. We denote

$$\overline{P}_a = \{x \in P : ||x|| \leqslant a\}.$$

We first prove that $T : \overline{P}_a \to \overline{P}_a$. If $x \in \overline{P}_a$, then $||x|| \leqslant a$. Using (H3)–(H4), we get

$$\begin{aligned}
(Tx)(t) \leqslant & \sum_{u=0}^{n-1} \frac{1}{\Gamma(u+1)} \Bigg[\frac{k_u}{1-k_u} \left(\sum_{\tau=1}^{m} \sum_{\sigma=u}^{n-1} \frac{aI}{N\Gamma(\sigma-u+1)} + \int_0^1 \frac{(1-s)^{\alpha-u-1}}{\Gamma(\alpha-u)} \frac{a}{N} s^p (1-s)^q ds \right) \\
& + \sum_{i=u+1}^{n-1} \prod_{v=i-1}^{n-2} \left(\frac{k_v}{\Gamma(n-v)(1-k_v)} \right) \\
& \times \left(\frac{k_i}{1-k_i} \left(\sum_{\tau=1}^{m} \sum_{\sigma=i}^{n-1} \frac{aI}{N\Gamma(\sigma-i+1)} + \int_0^1 \frac{(1-s)^{\alpha-i-1}}{\Gamma(\alpha-i)} \frac{a}{N} s^p (1-s)^q ds \right) \right) \Bigg] \\
& + \sum_{\tau=1}^{i} \sum_{\sigma=0}^{n-1} \frac{aI}{N\Gamma(\sigma+1)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{a}{N} s^p (1-s)^q ds \\
\leqslant & \sum_{u=0}^{n-1} \frac{1}{\Gamma(u+1)} \Bigg[\frac{k_u}{1-k_u} \left(\sum_{\sigma=u}^{n-1} \frac{maI}{N\Gamma(\sigma-u+1)} + \frac{\mathbf{B}(\alpha+q-u,p+1)}{\Gamma(\alpha-u)} \frac{a}{N} \right) \\
& + \sum_{i=u+1}^{n-1} \prod_{v=i-1}^{n-2} \left(\frac{k_v}{\Gamma(n-v)(1-k_v)} \right) \left(\frac{k_i}{1-k_i} \left(\sum_{\sigma=i}^{n-1} \frac{maI}{N\Gamma(\sigma-i+1)} + \frac{\mathbf{B}(\alpha+q-i,p+1)}{\Gamma(\alpha-i)} \frac{a}{N} \right) \right) \Bigg] \\
& + \sum_{\tau=1}^{i} \sum_{\sigma=0}^{n-1} \frac{aI}{N\Gamma(\sigma+1)} + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)} \frac{a}{N} \\
\leqslant & \frac{mn^2 aIK}{N} + \frac{mn^3 aIK^{n+1}}{N} + \frac{mnaI}{N} + \frac{\mathbf{B}(\alpha+q,p+1)}{\Gamma(\alpha)} \frac{a}{N} + \left(\frac{aK}{N} + \frac{anK^{n+1}}{N} \right) \sum_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+q-i,p+1)}{\Gamma(\alpha-i)} \\
\leqslant & a.
\end{aligned}$$

Hence we have show that $T : \overline{P}_a \to \overline{P}_a$.

By w_0, one sees that $||w_0|| = a$. Then $w_0 \in \overline{P}_a$. Let $w_1 = Tw_0$ and $w_2 = Tw_1$. Then above discussion implies that $w_1, w_2 \in \overline{P}_a$. We denote $w_{u+1} = Tw_u = T^u w_0$ for $u = 1, 2, \cdots$. Since $T : \overline{P}_a \to \overline{P}_a$, we have $w_u \in \overline{P}_a$ for all $u = 1, 2, 3, \cdots$. It follows from the complete continuity of T that $\{w_u : u = 0, 1, 2, 3, \cdots\}$ is a sequentially compact set.

By (H3)–(H4), we get similarly to above discussion that

$$\begin{aligned}
w_1(t) = & \sum_{u=0}^{n-1} \frac{t^u}{\Gamma(u+1)} \Bigg[\frac{k_u}{1-k_u} \left(\sum_{\tau=1}^{m} \sum_{\sigma=u}^{n-1} \frac{I_\sigma(t_\tau, w_0(t_\tau))}{\Gamma(\sigma-u+1)} (1-t_\tau)^{\sigma-u} + \int_0^1 \frac{(1-s)^{\alpha-u-1}}{\Gamma(\alpha-u)} f(s, w_0(s)) ds \right) \\
& + \sum_{i=u+1}^{n-1} \prod_{v=i-1}^{n-2} \left(\frac{k_v}{\Gamma(n-v)(1-k_v)} \right) \\
& \times \left(\frac{k_i}{1-k_i} \left(\sum_{\tau=1}^{m} \sum_{\sigma=i}^{n-1} \frac{I_\sigma(t_\tau, w_0(t_\tau))(1-t_\tau)^{\sigma-i}}{\Gamma(\sigma-i+1)} + \int_0^1 \frac{(1-s)^{\alpha-i-1}}{\Gamma(\alpha-i)} f(s, w_0(s) ds \right) \right) \Bigg] \\
& + \sum_{\tau=1}^{i} \sum_{\sigma=0}^{n-1} \frac{I_\sigma(t_\tau, w_0(t_\tau))}{\Gamma(\sigma+1)} (t-t_\tau)^\sigma + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, w_0(s) ds \\
\leqslant & a = w_0(t), \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m.
\end{aligned}$$

So by the monotonic property of f, I_σ and the definition of T, we get

$$w_2(t) = (Tw_1)((t) \leqslant (Tw_0)(t) = w_1(t), \quad t \in (0,1).$$

By induction, we get

$$w_{u+1}(t) \leqslant w_u(t), \quad t \in (0,1), \quad u = 1,2,3,\cdots.$$

Thus there exists $w^* \in \overline{P}_a$ such that $w_u \to w^*$ as $u \to \infty$. Applying the continuity of T and $w_{u+1} = Tw_u$, we get that $w^* = Tw^*$.

Let $v_0(t) = 0$ for all $t \in (0,1)$. Then $v_0 \in \overline{P}_a$. Let $v_1 = Tv_0$ and $v_2 = Tv_1$. One can prove that $v_1 \in \overline{P}_a$ and $v_2 \in \overline{P}_a$. We denote $v_{u+1} = Tv_u = T^u v_0$ for $u = 1,2,3,\cdots$. Since $T : \overline{P}_a \to \overline{P}_a$, we have $v_u \in \overline{P}_a$. for all $u = 1,2,3,\cdots$. It follows from the complete continuity of T that $\{v_u\}$ is a sequentially compact set.

Since $v_1 = Tv_0 \in \overline{P}_a$, we have

$$v_1(t) = (Tv_0)(t) \geqslant 0 \equiv v_0, \quad t \in (0,1).$$

So by induction, we get

$$v_{u+1}(t) \geqslant v_u(t), \quad t \in (0,1), \quad u = 1,2,3,\cdots.$$

Thus there exists $v^* \in \overline{P}_a$ such that $v_u \to v^*$ as $u \to \infty$. Applying the continuity of T and $v_{u+1} = Tv_u$, we get that $v^* = Tv^*$.

Since $f(t,0) \not\equiv 0$, we see that the zero function is not the solution of BVP(1.4.15). Thus both w^* and v^* are positive solution of the operator equation $x = Tx$ in $\overline{P}_a$.

It is well known that each fixed point of T in P is a solution of BVP(1.4.15). Hence w^* and v^* are two positive solutions of BVP(1.52). The proof is complete. □

REMARK 5.3.1. The iterative schemes in Theorem 5.3.1 are $w_0(t) = a$ and $w_{u+1}(t) = (Tw_u)(t)$ for $u = 0,1,2,3,\cdots$ and $v_0(t) \equiv 0$, $v_{u+1}(t) = (Tv_u)(t)$ for $u = 0,1,2,3,\cdots$. They start off with a known simple function and the zero function respectively. This is convenient for application.

6. Some remarks

In this section, we point out some results in published paper are unsuitable.

REMARK 6.1. In [45], existence of positive solutions of (1.3.18) was studied. Suppose $\alpha \in (1,2)$. Lemma 3.1 [45] claimed that if $u \in PC[0,1]$ is fixed point of the operator $A : PC[0,1] \to PC[0,1]$ defined by

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds + t^{\alpha-1} \sum_{t<t_k<1} \frac{c_k}{1-c_k} u(t_k), \quad u \in PC[0,1],$$

where

$$G(t,s = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ (t(1-s))^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

then u is a solution of BVP(1.3.18). However we find that this lemma is in-correct. The reason is as follows:

Proof. In fact, if u is a fixed point of A, we have

$$u(t)=\int_0^1 G(t,s)f(s,u(s))ds+t^{\alpha-1}\sum_{t<t_k<1}\frac{c_k}{1-c_k}u(t_k),\quad u\in PC[0,1],$$

For $t\in(t_1,t_2)$, by Definition 2.2, we have by direct computation that

$$\begin{aligned}
D_{0^+}^{\alpha}u(t) &= \frac{1}{\Gamma(2-\alpha)}\left[\int_0^t(t-s)^{1-\alpha}\left(\int_0^1 G(s,v)f(v,u(v))dv+s^{\alpha-1}\sum_{s<t_k<1}\frac{c_k}{1-c_k}u(t_k)\right)ds\right]''\\
&=\frac{\left[\int_0^t(t-s)^{1-\alpha}\int_0^1 G(s,v)f(v,u(v))dvds\right]''+\left[\int_0^t(t-s)^{1-\alpha}s^{\alpha-1}\sum_{s<t_k<1}\frac{c_k}{1-c_k}u(t_k)ds\right]''}{\Gamma(2-\alpha)}\\
&=\frac{\left[\int_0^t(t-s)^{1-\alpha}\left(\int_0^s G(s,v)f(v,u(v))dv+\int_s^1 G(s,v)f(v,u(v))dv\right)ds\right]''}{\Gamma(2-\alpha)}\\
&\quad+\frac{\left[\int_0^{t_1}(t-s)^{1-\alpha}s^{\alpha-1}\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k)ds+\int_{t_1}^t(t-s)^{1-\alpha}s^{\alpha-1}\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)ds\right]''}{\Gamma(2-\alpha)}\\
&=\frac{\left[\int_0^t(t-s)^{1-\alpha}\int_0^s\left([s(1-v)]^{\alpha-1}-(s-v)^{\alpha-1}\right)f(v,u(v))dvds+\int_0^t(t-s)^{1-\alpha}\int_s^1[s(1-v)]^{\alpha-1}f(v,u(v))dvds\right]''}{\Gamma(2-\alpha)}\\
&\quad+\frac{\left[\int_0^{t_1}(t-s)^{1-\alpha}s^{\alpha-1}ds\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k)+\int_{t_1}^t(t-s)^{1-\alpha}s^{\alpha-1}ds\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)}.
\end{aligned}$$

By interchange integral order for the first term and $w=\frac{s}{t}$ for the second term, we get

$$\begin{aligned}
D_{0^+}^{\alpha}u(t) &= \frac{\left[\int_0^t\int_s^t(t-s)^{1-\alpha}\left([s(1-v)]^{\alpha-1}-(s-v)^{\alpha-1}\right)dsf(v,u(v))dv+\int_0^t\int_0^s(t-s)^{1-\alpha}s^{\alpha-1}ds(1-v)^{\alpha-1}f(v,u(v))dv\right]''}{\Gamma(2-\alpha)}\\
&\quad+\frac{\left[t\int_0^{\frac{t_1}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k)+w\int_{\frac{t_1}{t}}^1(1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)}\\
&=\frac{\left[\left(\int_0^t\int_0^t(t-s)^{1-\alpha}s^{\alpha-1}ds(1-v)^{\alpha-1}f(v,u(v))dv-\int_0^t\int_s^t(t-s)^{1-\alpha}(s-v)^{\alpha-1}ds\right)f(v,u(v))dv\right]''}{\Gamma(2-\alpha)}\\
&\quad+\frac{\left[t\int_0^{\frac{t_1}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=1}^{m}\frac{c_k}{1-c_k}u(t_k)+t\int_{\frac{t_1}{t}}^1(1-w)^{1-\alpha}w^{\alpha-1}dw\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)}\\
&=\frac{\left[\left(t\mathbf{B}(2-\alpha,\alpha)\int_0^t(1-v)^{\alpha-1}f(v,u(v))dv-\int_0^t\int_s^t(t-s)^{1-\alpha}(s-v)^{\alpha-1}ds\right)f(v,u(v))dv\right]''}{\Gamma(2-\alpha)}\\
&\quad+\frac{\left[t\int_0^{\frac{t_1}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\frac{c_1}{1-c_1}u(t_1)+t\mathbf{B}(2-\alpha,\alpha)\sum_{k=2}^{m}\frac{c_k}{1-c_k}u(t_k)\right]''}{\Gamma(2-\alpha)}\neq -f(t,u(t)).
\end{aligned}$$

Then Lemma 3.1[45] is in-correct. □

REMARK 6.2. The impulse conditions $u(t_k^+)=(1-c_k)u(t_k^-)$ in (1.3.18) is unsuitable for this kind of problem. The reason is as follows:

Proof. By Corollary 3.1.1, we have from $D_{0+}^{\alpha}u(t) = -f(t,u(t))$, $t \in (t_s,t_{s+1}]$, $s \in \mathbb{N}_0^m$ that

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \sum_{v=0}^{s} \frac{c_{1v}(t-t_v)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{v=0}^{s} \frac{c_{2v}(t-t_v)^{\alpha-2}}{\Gamma(\alpha-1)},\ t \in (t_s,t_{s+1}],\ s \in \mathbb{N}_0^m.$$

According to assumptions $u(0) = 0$ and u is right continuous at $t_k (k \in \mathbb{N}_1^m)$, we have $c_{2s} = 0$, $s \in \mathbb{N}_0^m$. Then

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \sum_{v=0}^{s} \frac{c_{1v}(t-t_v)^{\alpha-1}}{\Gamma(\alpha)}, \quad t \in (t_s,t_{s+1}], \quad s \in \mathbb{N}_0^m.$$

However, we find that $u(t_s^+) - u(t_s^-) \equiv 0$. So the impulse conditions $u(t_k^+) = (1-c_k)u(t_k^-)$ in (1.3.18) are unsuitable. □

REMARK 6.3. In [13, 14], the solvability of (1.3.20) and (1.3.21) were studied respectively. Results in [13, 14] are un-suitable.

Define $u_\alpha(t) = t^{2-\alpha}u(t)$ and

$$PC[0,1] = \{x : x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}], x(t_i^-), x(t_i^+) \text{ exist}, x(t_i^-) = x(t_i),\ i = 0,1,2,\cdots,k\}$$

with norm $||x|| = \sup_{t\in(0,1]} |x(t)|$ for $x \in PC[0,1]$,

$$X = \{u : u_\alpha, D_{0+}^{\alpha-1}u \in PC[0,1]\}$$

with norm $||u|| = \max\{||u_\alpha||, ||D_{0+}^{\alpha-1}||\}$ for $u \in X$,

$$Y = PC[0,1] \times R^{2k}$$

with norm $||(u,c)|| = \max\{||u||, |c|\}$ for $(u,c) \in Y$.

A linear operator L is defined in [13, 14] by

$$(Lx)(t) = (D_{0+}^{\alpha}u(t), \Delta u(t_1), \cdots, \Delta u(t_k), \Delta D_{0+}^{\alpha-1}u(t_1), \cdots, \Delta D_{0+}^{\alpha-1}u(t_k))$$

with its domain

$$D(L) = \begin{cases} \{u \in X : \lim_{t\to 0^+} t^{2-\alpha}u(t) = \sum_{i=1}^{n} a_i u(\xi_i),\ u(1) = \sum_{i=1}^{n} b_i u(\eta_i)\}, & \text{in [13]} \\ \{u \in X : D_{0+}^{\alpha-1}u(0) = \sum_{i=1}^{m} a_i D_{0+}^{\alpha-1}u(\xi_i),\ u(1) = \sum_{i=1}^{n} b_i \eta_i^{2-\alpha}u(\eta_i)\} & \text{in [14].} \end{cases}$$

(i) $D(L)$ is not well defined. In fact, $Lu \notin Y$ for $u \in X$. The definition of $D(L)$ in [13, 14] should be replaced by

$$D(L) = \begin{cases} \{u \in X : D_{0+}^{\alpha}u \in PC[0,1], \lim_{t\to 0^+} t^{2-\alpha}u(t) = \sum_{i=1}^{n} a_i u(\xi_i),\ u(1) = \sum_{i=1}^{n} b_i u(\eta_i)\}, \\ \qquad \text{for} \quad (1.3.20) \\ \{u \in X : D_{0+}^{\alpha}u \in PC[0,1], D_{0+}^{\alpha-1}u(0) = \sum_{i=1}^{m} a_i D_{0+}^{\alpha-1}u(\xi_i), u(1) = \sum_{i=1}^{n} b_i \eta_i^{2-\alpha}u(\eta_i)\} \\ \qquad \text{for} \quad (1.3.21). \end{cases}$$

(ii) In Lemma 2.3 in [13], it is claimed that L is a Fredholm mapping of index zero and $\mathrm{Ker}L = \{h_1 t^{\alpha-1} + h_2 t^{\alpha-2} : h_1, h_2 \in \mathbb{R}\}$. This claim is unsuitable. In fact $u \in \mathrm{Ker}L$ if and only if

$$\begin{cases} D_{0^+}^{\alpha} u(t) = 0, \quad t \in (0,1), \quad t \neq t_i, \quad i = 1,2,\cdots,k, \\ \lim_{t\to 0^+} t^{2-\alpha} u(t) = \sum_{i=1}^{n} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{n} b_i u(\eta_i), \\ \Delta u(t_i) = \Delta D_{0^+}^{\alpha-1} u(t_i) = 0, \quad i = 1,2,\cdots,k, \end{cases} \tag{6.1}$$

By Theorem 3.1.1 and $D_{0^+}^{\alpha} u(t) = 0$, we know that there exist constants c_{iv} $(i \in \mathbb{N}_0^m,$ $v \in \mathbb{N}_1^2)$ such that

$$u(t) = \sum_{j=0}^{i} c_{j1}(t-t_j)^{\alpha-1} + \sum_{j=0}^{i} c_{j2}(t-t_j)^{\alpha-2}, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{6.2}$$

So

$$D_{0^+}^{\alpha-1} u(t) = \Gamma(\alpha) \sum_{j=0}^{i} c_{j1}, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{6.3}$$

From $\Delta D_{0^+}^{\alpha-1} u(t_i) = 0$, we get $c_{i1} = 0$ $(i \in \mathbb{N}_1^m)$. By (6.2), we know $u(t_i^+)$ may be infinite. So this kind of impulse function ($\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i$ in (1.3.20), (1.3.21)) is unsuitable. Even $c_{i2} = 0 (i \in \mathbb{N}_1^m)$, this means $u(t_i^+)$ is finite, we have $u(t) = c_{01} t^{\alpha-1}$. So $\mathrm{Ker}L = \{h_1 t^{\alpha-1} : h_1 \in \mathbb{R}\}$. But some other boundary conditions in (1.3.20) and (1.3.21) are redundant. □

We propose the following impulsive boundary value problems and suggest readers investigate their solvability

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t), D_{0^+}^{\alpha-1} u(t)), \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m, \\ \lim_{t\to 0^+} t^{2-\alpha} u(t) = \sum_{i=1}^{n} a_i u(\xi_i), \ u(1) = \sum_{i=1}^{n} b_i u(\eta_i), \\ \lim_{t\to t_i^+} (t-t_i)^{2-\alpha} u(t) = I_i(u(t_i), D_{0^+}^{\alpha-1} u(t_i)), \Delta D_{0^+}^{\alpha-1} u(t_i) = J_i(u(t_i), D_{0^+}^{\alpha-1} u(t_i)), \ i \in \mathbb{N}_1^m \end{cases}$$

and

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t), D_{0^+}^{\alpha-1} u(t)), \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m, \\ D_{0^+}^{\alpha-1} u(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1} u(\xi_i), \ u(1) = \sum_{i=1}^{n} b_i \eta_i^{2-\alpha} u(\eta_i), \\ \lim_{t\to t_i^+} (t-t_i)^{2-\alpha} u(t) = I_i(u(t_i), D_{0^+}^{\alpha-1} u(t_i)), \Delta D_{0^+}^{\alpha-1} u(t_i) = J_i(u(t_i), D_{0^+}^{\alpha-1} u(t_i)), \ i \in \mathbb{N}_1^m. \end{cases}$$

REMARK 6.4. In [95], authors studied the existence of solutions of the following impulsive boundary value problem for fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}u(t)=f(t,v(t),D_{0^+}^{p}v(t)),\ D_{0^+}^{\beta}v(t)=g(t,u(t),D_{0^+}^{q}u(t)), & t\in(0,1),\\ \Delta u(t_i)=A_i(v(t_i),D_{0^+}^{p}v(t_i)),\ \Delta D_{0^+}^{q}u(t_i)=B_i(v(t_i),D_{0^+}^{p}v(t_i)), & i=1,2,\cdots,k,\\ \Delta v(t_i)=C_i(u(t_i),D_{0^+}^{q}u(t_i)),\ \Delta D_{0^+}^{p}v(t_i)=D_i(u(t_i),D_{0^+}^{q}u(t_i)), & i=1,2,\cdots,k,\\ D_{0^+}^{\alpha-1}u(0)-\sum\limits_{i=1}^{m}a_iD_{0^+}^{\alpha-1}u(\xi_i)=u(1)-\sum\limits_{i=1}^{m}b_i\eta_i^{2-\alpha}u(\eta_i)=0,\\ D_{0^+}^{\beta-1}v(0)-\sum\limits_{i=1}^{m}c_iD_{0^+}^{\beta-1}v(\zeta_i)=v(1)-\sum\limits_{i=1}^{m}d_i\theta_i^{2-\beta}v(\theta_i)=0, \end{cases} \tag{6.4}$$

where $1<\alpha,\beta<2$, $0<q\leqslant\alpha-1$, $0<p\leqslant\beta-1$, $0<\xi_1<\cdots<\xi_m<1$, $0<\eta_1<\cdots<\eta_m<1$, $0<\zeta_1<\cdots<\zeta_m<1$, $0<\theta_1<\cdots<\theta_m<1$, $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are Carathéodory functions, $A_i,B_i,C_i,D_i:\mathbb{R}^2\to\mathbb{R}$ are continuous functions, $\Delta w(t_i)=w(t_i^+)-w(t_i^-)$, $\Delta D_{0^+}^{\gamma}w(t_i)=D_{0^+}^{\gamma}w(t_i^+)-D_{0^+}^{\gamma}w(t_i^-)$, here $w\in\{u,v\}$, $\gamma\in\{p,q\}$, and $w(t_i^+),w(t_i^-)$ denote the right and left limits of $w(t)$ at $t=t_i$, respectively, and the fractional derivative is understood in the Riemann-Liouville sense. k, m, a_i, b_i, c_i, d_i $(i=a,2,\cdots,m)$ are fixed constant satisfying

$$\sum_{i=1}^{m}a_i=\sum_{i=1}^{m}b_i=\sum_{i=1}^{m}c_i=\sum_{i=1}^{m}d_i=\sum_{i=1}^{m}b_i\eta_i=\sum_{i=1}^{m}d_i\theta_i=1.$$

$$PC[0,1]=\{x:x|_{(t_i,t_{i+1}]}\in C(t_i,t_{i+1}],x(t_i^-),x(t_i^+)\text{ exist},x(t_i^-)=x(t_i),i=0,1,2,\cdots,k\}$$

with norm $||x||=\sup\limits_{t\in(0,1]}|x(t)|$ for $x\in PC[0,1]$,

$$\begin{aligned} Y_1&=\{u:u_\alpha,D_{0^+}^{\alpha-1}u\in PC[0,1]\}\text{ with norm }||u||=\max\{||u_\alpha||,||D_{0^+}^{\alpha-1}||\}\quad\text{for}\quad u\in Y_1,\\ Y_2&=\{u:u_\beta,D_{0^+}^{\beta-1}u\in PC[0,1]\}\text{ with norm }||u||=\max\{||u_\beta||,||D_{0^+}^{\beta-1}||\}\quad\text{for}\quad u\in Y_2,\\ Y&=Y_1\times Y_2\text{ is a Banach space with norm }||(y_1,y_2)||=\max\{||y_1||,||y_2||\}\\ &\qquad\text{for}\quad(y_1,y_2)\in Y_1\times Y_2,\\ Z_1&=Z_2=PC[0,1]\times R^{2k}\ \text{ with norm }\ ||(u,c)||=\max\{||u||,|c|\}\ \text{ for }\ (u,c)\in Z_1\ \text{ or }\ Z_2,\\ Z&=Z_1\times Z_2\ \text{ is a Banach space with norm }\ ||(z_1,z_2)||=\max\{||z_1||,||z_2||\}\\ &\qquad\text{for}\quad(z_1,z_2)\in Z_1\times Z_2. \end{aligned}$$

A linear operator $L:Y\cap D(L)\to Z$ and a nonlinear operator $N:Y\to Z$ are defined in [95]. The similar unsuitable things as in [13, 14] happened.

Let ξ_i, η_i, ζ_i, $\theta_i\in(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m)$. We suggest readers study the following impulsive boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,v(t),D_{0+}^{p}v(t)), \quad D_{0+}^{\beta}v(t)=g(t,u(t),D_{0+}^{q}u(t)),\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\\ \lim\limits_{t\to t_i^+}(t-t_i)^{2-\alpha}u(t)=A_i(v(t_i),D_{0+}^{p}v(t_i)), \quad \Delta D_{0+}^{q}u(t_i)=B_i(v(t_i),D_{0+}^{p}v(t_i)),\ i\in\mathbb{N}_1^m,\\ \lim\limits_{t\to t_i^+}(t-t_i)^{2-\beta}v(t)=C_i(u(t_i),D_{0+}^{q}u(t_i)), \quad \Delta D_{0+}^{p}v(t_i)=D_i(u(t_i),D_{0+}^{q}u(t_i)),\ i\in\mathbb{N}_1^m,\\ D_{0+}^{\alpha-1}u(0)-\sum\limits_{i=0}^{m}a_iD_{0+}^{\alpha-1}u(\xi_i)=u(1)-\sum\limits_{i=0}^{m}b_i\eta_i^{2-\alpha}u(\eta_i)=0,\\ D_{0+}^{\beta-1}v(0)-\sum\limits_{i=0}^{m}c_iD_{0+}^{\beta-1}v(\zeta_i)=v(1)-\sum\limits_{i=0}^{m}d_i\theta_i^{2-\beta}v(\theta_i)=0. \quad \square \end{cases}$$

REMARK 6.5. In [12, 78], Bai studied the existence of solutions of BVP(1.3.19). The following lemma was proved:

LEMMA 2.1. ([12, 78]) *The linear impulsive boundary value problem*

$$\begin{cases} D_{0+}^{\alpha}u(t)-\lambda u(t)=\sigma(t), \quad t\in(0,1), \quad t\neq t_i, \quad i=1,2,\cdots,m,\\ \lim\limits_{t\to 0^+}t^{1-\alpha}u(t)-u(1)=k,\\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}[u(t)-u(t_i)]=a_i, \quad i=1,2,\cdots,m, \end{cases} \tag{6.5}$$

where $\alpha\in(0,1)$, $\lambda\neq 0$, k, $a_i\in\mathbb{R}$ *are constants and* $\sigma\in C[0,1]$, *has a unique solution* $u\in PC_{1-\alpha}[0,1]$ *given by*

$$u(t)=\frac{k\Gamma(\alpha)}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}t^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})+\int_0^1G_{\lambda,\alpha}(t,s)\sigma(s)ds+\Gamma(\alpha)\sum_{i=1}^{m}G_{\lambda,\alpha}(t,t_i)a_i, \tag{6.6}$$

where $G_{\lambda,\alpha}$ *is defined by*

$$G_{\lambda,\alpha}=\begin{cases} \frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})\mathbf{E}_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha}}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}+(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha}),\ 0\leqslant s\leqslant t\leqslant 1,\\ \frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})\mathbf{E}_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha}}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)},\ 0\leqslant t\leqslant s\leqslant 1. \end{cases} \tag{6.7}$$

Proof. Firstly it is proved that u is a solution of BVP(6.5). In [12] on page 215, it claimed that $\lim\limits_{t\to 0}t^{1-\alpha}w(t)-w(1)=0$ by $D_{0+}^{\alpha}w(t)-\lambda w(t)=0$, $t\in(0,1]\setminus\{t_1,\cdots,t_m\}$. This claim is unsuitable.

We present a direct proof for this lemma. In fact, by Theorem 3.1.1, if u is a solution of (6.5), we know that there exist constants $C_v\in\mathbb{R}$ such that

$$u(t)=\sum_{v=0}^{i}C_v(t-t_v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_v)^{\alpha})+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds,$$
$$t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{6.8}$$

One sees that $\lim_{t\to t_i^+}(t-t_i)^{1-\alpha}[u(t)-u(t_i)]=a_i$ implies $\lim_{t\to t_i^+}(t-t_i)^{1-\alpha}u(t)=a_i$. Then (6.5) implies $C_i=\Gamma(\alpha)a_i\ \ (i\in\mathbb{N}_1^m)$. One sees that $\lim_{t\to 0^+}t^{1-\alpha}u(t)-u(1)=k$ and (6.5) imply

$$\frac{C_0}{\Gamma(\alpha)}-\left[\sum_{v=0}^{m}C_v(1-t_v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_v)^{\alpha})+\int_0^1(1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds\right]=k.$$

It follows from $C_v=\Gamma(\alpha)a_v\ \ (v\in\mathbb{N}_1^m)$ that

$$\begin{aligned}C_0=&\frac{\Gamma(\alpha)^2}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}\sum_{v=1}^{m}a_v(1-t_v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_v)^{\alpha})\\&+\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}\int_0^1(1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds+\frac{k\Gamma(\alpha)}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}.\end{aligned}$$

Hence

$$\begin{aligned}u(t)=&\frac{k\Gamma(\alpha)}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}t^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})\\&+\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}\int_0^1(1-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)dst^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})\\&+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds\\&+\frac{\Gamma(\alpha)^2}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}\sum_{v=1}^{m}a_v(1-t_v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_v)^{\alpha})t^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})\\&+\Gamma(\alpha)\sum_{v=1}^{i}a_v(t-t_v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_v)^{\alpha})\\=&\frac{k\Gamma(\alpha)}{1-\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}t^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})+\int_0^1G_{\lambda,\alpha}(t,s)\sigma(s)ds+\Gamma(\alpha)\sum_{i=1}^{m}G_{\lambda,\alpha}(t,t_i)a_i.\end{aligned}$$

On the other hand, by direct computation, we can prove that u is a solution of (6.5) if u satisfies (6.6). □

We suggest readers to establish existence results for the following BVP

$$\begin{cases}D_{0^+}^{2\alpha}u(t)-\lambda u(t)=f(t,u(t),D_{0^+}^{\alpha}u(t)), & t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ \lim_{t\to 0}t^{1-\alpha}u(t)=u(1), \quad \lim_{t\to 0}t^{1-\alpha}D_{0^+}^{\alpha}u(t)=D_{0^+}^{\alpha}u(1),\\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha}u(t)=I(t_i,u(t_i),D_{0^+}^{\alpha}u(t_i)), & i\in\mathbb{N}_1^m,\\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha}D_{0^+}^{\alpha}u(t)=J(t_i,u(t_i),D_{0^+}^{\alpha}u(t_i)), & i\in\mathbb{N}_1^m,\end{cases}$$

where $\alpha\in(0,1)$, $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $*$, $D_{0^+}^{2\alpha}=D_{0^+}^{\alpha}D_{0^+}^{\alpha}$ is the sequential Riemann-Liouville fractional derivative, $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$, and $I,J:\{t_i:i\in\mathbb{N}_1^m\}\times\mathbb{R}^2\to\mathbb{R}$ are continuous, m is a fixed positive integer, $t_i\ \ (i\in\mathbb{N}_1^m)$ are fixed points with $0<t_1<t_2<\cdots<t_m<1$.

REMARK 6.6. From Corollary 3.2.1, we know that x is a solution of the impulsive problem

$${}^cD_{0^+}^{\alpha}x(t)=h(t),\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\ \Delta x(t_i)=y_i,\ i\in\mathbb{N}_1^m,\ x(0)=x_0,\alpha\in(0,1)$$

if and only if x satisfies

$$x(t)=x_0+\sum_{j=1}^{i} y_j+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.$$

Here $\alpha\in(0,1)$, x_0, y_i $(i\in\mathbb{N}_1^m)$ are constants, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, m a positive integer, $h\in L^1(0,1)$. This is just Lemma 2.7 in [32] and Lemma 3.3 in [80]. Furthermore, Lemma 2.7 in [33] is also derived from this result.

From Corollary 3.1.1, we know that x is a solution of the impulsive problem

$$D_{0^+}^{\alpha}x(t)=h(t),\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t)=y_i,\ i\in\mathbb{N}_0^m,\ \alpha\in(0,1)$$

if and only if x satisfies

$$x(t)=\sum_{j=0}^{i} y_j(t-t_j)^{\alpha-1}+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.$$

Here $\alpha\in(0,1)$, $y_i(i\in\mathbb{N}_0^m)$ are constants, $h\in L^1(0,1)$. We suggest readers study the existence and uniqueness of solutions of the following problem

$$\begin{aligned} lD_{0^+}^{\alpha}x(t)&=f(t,x(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t)&=I(t_i,x(t_i)), \quad i\in\mathbb{N}_0^m, ax(0)+bx(T)=c, \end{aligned}$$

where $\alpha\in(0,1)$, $a,b,c\in\mathbb{R}$, f,I satisfies some assumptions.

REMARK 6.7. In [70], BVP(1.3.15) was studied in which the fractional differential equation has one starting point. In [72, 89], authors studied the existence of solutions of the following boundary value problem of nonlinear fractional order impulsive differential equation which has multiple starting points

$$\begin{cases} {}^cD_{t_i^+}^{\alpha}x(t)+f(t,x(t))=0, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\ \Delta x(t_i)=I_i(x(t_i)), \quad \Delta x'(t_i)=J_i(x(t_i)), \quad i\in\mathbb{N}_1^m,\\ \alpha_1x(0)-\beta_1x'(0)=x_0, \quad \alpha_2x(1)+\beta_2x'(1)=x_1, \end{cases}$$

where $\alpha_1,\beta_2,\alpha_2,\beta_2\in\mathbb{R}$ with $\eta=:\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1\neq0$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, x_0, $x_1\in\mathbb{R}$.

In fact, suppose that x is a solution of above problem, by Corollary 3.2.1, there exists a_i, b_i $(i\in\mathbb{N}_0^m)$ such that

$$x(t)=a_i+b_i(t-t_i)+\int_{t_i}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.$$

From $\alpha_1x(0)-\beta_1x'(0)=0$, $\alpha_2x(1)+\beta_2x'(1)=0$, we get

$$\alpha_1a_0-\beta_1b_0=x_0,$$

$$\alpha_2 a_m + (\beta_2 + \alpha_2(1-t_m))\, b_m + \int_{t_m}^1 \left[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right] f(s,x(s))ds = x_1.$$

From $\Delta x(t_i) = I_i(x(t_i))$, $\Delta x'(t_i) = J_i(x(t_i))$, we have for $i \in \mathbb{N}_1^m$ that

$$a_i - \left(a_{i-1} + b_{i-1}(t_i - t_{i-1}) + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\right) = I_i(x(t_i)),$$
$$b_i - \left(b_{i-1} + \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds\right) = J_i(x(t_i))$$

Use $\sum_{j=2}^{i}\sum_{\sigma=1}^{j-1} a_j b_\sigma = \sum_{\sigma=1}^{i}\sum_{j=\sigma+1}^{i} a_j b_\sigma$, we have

$$\begin{aligned}
b_i &= b_0 + \sum_{j=1}^{i} J_j(x(t_j)) + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds, \quad i \in \mathbb{N}_1^m,\\
a_i &= a_0 + \sum_{j=1}^{i} b_{j-1}(t_j - t_{j-1}) + \sum_{j=1}^{i} I_j(x(t_j)) + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\\
&= a_0 + t_i b_0 + \sum_{j=1}^{i} I_j(x(t_j)) + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\\
&\quad + \sum_{j=2}^{i}(t_j - t_{j-1}) \sum_{\sigma=1}^{j-1} J_\sigma(x(t_\sigma)) + \sum_{j=2}^{i}(t_j - t_{j-1}) \sum_{\sigma=1}^{j-1} \int_{t_{\sigma-1}}^{t_\sigma} \frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds\\
&= a_0 + t_i b_0 + \sum_{j=1}^{i} I_j(x(t_j)) + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\\
&\quad + \sum_{\sigma=1}^{i}\sum_{j=\sigma+1}^{i} (t_j - t_{j-1}) J_\sigma(x(t_\sigma)) + \sum_{\sigma=1}^{i}\sum_{j=\sigma+1}^{i} \int_{t_{\sigma-1}}^{t_\sigma} (t_j - t_{j-1}) \frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds\\
&= a_0 + t_i b_0 + \sum_{j=1}^{i} I_j(x(t_j)) + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\\
&\quad + \sum_{\sigma=1}^{i} (t_i - t_\sigma) J_\sigma(x(t_\sigma)) + \sum_{\sigma=1}^{i} \int_{t_{\sigma-1}}^{t_\sigma} (t_i - t_\sigma) \frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds, \quad i \in \mathbb{N}_1^m.\\
a_m &= a_0 + t_m b_0 + \sum_{j=1}^{m} I_j(x(t_j)) + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\\
&\quad + \sum_{\sigma=1}^{m} (t_m - t_\sigma) J_\sigma(x(t_\sigma)) + \sum_{\sigma=1}^{m} \int_{t_{\sigma-1}}^{t_\sigma} (t_m - t_\sigma) \frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds\\
b_m &= b_0 + \sum_{j=1}^{m} J_j(x(t_j)) + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s))ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\alpha_1 a_0 - \beta_1 b_0 = x_0,\\
&\alpha_2 a_0 + (\alpha_2 + \beta_2)\, b_0 + \alpha_2 \sum_{j=1}^{m} I_j(x(t_j)) + \sum_{\sigma=1}^{m} (\beta_2 + \alpha_2(1-t_\sigma)) J_\sigma(x(t_\sigma))\\
&+ \sum_{\sigma=1}^{m} \int_{t_{\sigma-1}}^{t_\sigma} \left[\frac{\alpha_2(t_m-t_\sigma)(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\alpha_2(t_\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}\right] f(s,x(s))ds
\end{aligned}$$

$$+\int_{t_m}^{1}\left[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]f(s,x(s))ds=x_1.$$

It follows that

$$\begin{aligned}
a_0 &= \frac{(\alpha+\beta_2)x_0+\beta_1x_1}{\eta}-\frac{\alpha_2\beta_1}{\eta}\sum_{j=1}^{m}I_j(x(t_j))-\sum_{\sigma=1}^{m}\frac{(\beta_2+\alpha_2(1-t_\sigma))\beta_1}{\eta}J_\sigma(x(t_\sigma))\\
&\quad -\frac{\beta_1}{\eta}\sum_{\sigma=1}^{m}\int_{t_{\sigma-1}}^{t_\sigma}\left[\frac{\alpha_2(t_m-t_\sigma)(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{\alpha_2(t_\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}\right]f(s,x(s))ds\\
&\quad -\frac{\beta_1}{\eta}\int_{t_m}^{1}\left[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]f(s,x(s))ds,\\
b_0 &= \frac{\alpha_1x_1-\alpha_2x_0}{\eta}-\frac{\alpha_1\alpha_2}{\eta}\sum_{j=1}^{m}I_j(x(t_j))-\sum_{\sigma=1}^{m}\frac{(\beta_2+\alpha_2(1-t_\sigma))\alpha_1}{\eta}J_\sigma(x(t_\sigma))\\
&\quad -\frac{\alpha_1}{\eta}\sum_{\sigma=1}^{m}\int_{t_{\sigma-1}}^{t_\sigma}\left[\frac{\alpha_2(t_m-t_\sigma)(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{\alpha_2(t_\sigma-s)^{\alpha-1}}{\Gamma(\alpha)}\right]f(s,x(s))ds\\
&\quad -\frac{\alpha_1}{\eta}\int_{t_m}^{1}\left[\frac{\alpha_2(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]f(s,x(s))ds.
\end{aligned}$$

Hence

$$x(t)=\begin{cases} a_0+b_0t+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds, & t\in(t_0,t_1],\\ A_i+B_i(t-t_i)+\int_{t_i}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds, & t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_1^m,\end{cases}$$

where a_0, b_0 are defined above and A_i, B_i are defined by

$$\begin{aligned}
B_i &= b_0+\sum_{j=1}^{i}J_j(x(t_j))+\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x(s))ds,\quad i\in\mathbb{N}_1^m,\\
A_i &= a_0+t_ib_0+\sum_{j=1}^{i}I_j(x(t_j))+\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds\\
&\quad +\sum_{\sigma=1}^{i}(t_i-t_\sigma)J_\sigma(x(t_\sigma))+\sum_{\sigma=1}^{i}\int_{t_{\sigma-1}}^{t_\sigma}(t_i-t_\sigma)\frac{(t_\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x(s))ds,\quad i\in\mathbb{N}_1^m.
\end{aligned}$$

One sees that Lemma 2.2 in [72] and Theorem 2.1 in [89] are unsuitable. □

REMARK 6.8. Suppose that $a+b\neq 0$. In [88], author studied the solvability of (1.3.26) and (1.3.27) respectively. We find that Lemma 2.9 and Lemma 2.10 [88] are incorrect.

In fact, by Theorem 3.4.1, we know that x is a piecewise continuous solution of

$${}^cD_{0,t}^{\alpha}({}^cD_{0,t}^{\beta}u(t))+\lambda u(t)=f(t,u(t)),\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m$$

if and only if there exist constants C_j, $D_j\in\mathbb{R}$ $(j\in\mathbb{N}_0^m)$ such that

$$x(t)=\sum_{j=0}^{i}D_j\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_j)^{\alpha+\beta})+\sum_{j=0}^{i}C_j(t-t_j)^{\beta}\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(t-t_j)^{\alpha+\beta})$$

$$+\int_0^t (t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})f(u,x(u))du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \tag{6.9}$$

By Corollary 3.4.2, we know

$$\begin{aligned}{}^cD_{0^+}^{\beta}x(t) = &\sum_{j=0}^{i} C_j\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_j)^{\alpha+\beta})\\ &+\int_0^t (t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})f(u,x(u))du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.\end{aligned} \tag{6.10}$$

From $\Delta x(t_i)=y_i \ (i\in\mathbb{N}_1^m)$ and (6.9), we get $D_i=y_i \ (i\in\mathbb{N}_1^m)$. By ${}^cD_{0^+}^{\beta}x(t_i)=d_i$ $(i\in\mathbb{N}_0^m)$ and (6.10), we get $C_0=d_0$ and

$$C_i+\sum_{j=0}^{i-1}C_j\mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_j)^{\alpha+\beta})+\int_0^{t_i}(t_i-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_i-u)^{\alpha+\beta})f(u,x(u))du=d_i,\quad i\in\mathbb{N}_1^m.$$

This is

$$\begin{pmatrix} 1 & 0 & \cdots & 0\\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_2-t_1)^{\alpha+\beta}) & 1 & \cdots & 0\\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_3-t_1)^{\alpha+\beta}) & \mathbf{E}_{\alpha+\beta,1}(\lambda(t_3-t_2)^{\alpha+\beta}) & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_1)^{\alpha+\beta}) & \mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_2)^{\alpha+\beta}) & \cdots & 1\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\\ \cdots\\ C_i\end{pmatrix}$$

$$=\begin{pmatrix} d_1-d_0-\int_0^{t_1}(t_1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_1-u)^{\alpha+\beta})h(u)du\\ d_2-d_0-\int_0^{t_2}(t_2-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_2-u)^{\alpha+\beta})f(u,x(u))du\\ d_3-d_0-\int_0^{t_3}(t_3-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_3-u)^{\alpha+\beta})f(u,x(u))du\\ \cdots\cdots\\ d_i-d_0-\int_0^{t_i}(t_i-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_i-u)^{\alpha+\beta})f(u,x(u))du\end{pmatrix}.$$

Then

$$\begin{pmatrix}C_1\\C_2\\C_3\\ \cdots\\ C_i\end{pmatrix}=\begin{pmatrix} 1 & 0 & \cdots & 0\\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_2-t_1)^{\alpha+\beta}) & 1 & \cdots & 0\\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_3-t_1)^{\alpha+\beta}) & \mathbf{E}_{\alpha+\beta,1}(\lambda(t_3-t_2)^{\alpha+\beta}) & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_1)^{\alpha+\beta}) & \mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_2)^{\alpha+\beta}) & \cdots & 1\end{pmatrix}^{-1}$$

$$\times\begin{pmatrix} d_1-d_0-\int_0^{t_1}(t_1-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_1-u)^{\alpha+\beta})h(u)du\\ d_2-d_0-\int_0^{t_2}(t_2-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_2-u)^{\alpha+\beta})f(u,x(u))du\\ d_3-d_0-\int_0^{t_3}(t_3-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_3-u)^{\alpha+\beta})f(u,x(u))du\\ \cdots\cdots\\ d_i-d_0-\int_0^{t_i}(t_i-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_i-u)^{\alpha+\beta})f(u,x(u))du\end{pmatrix}.$$

It follows that

$$C_i = d_i - d_0 - \int_0^{t_i} (t_i-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_i-u)^{\alpha+\beta})f(u,x(u))du$$
$$+\sum_{\sigma=1}^{i-1} M_\sigma \left(d_\sigma - d_0 - \int_0^{t_\sigma}(t_\sigma-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_\sigma-u)^{\alpha+\beta})f(u,x(u))du\right),$$
$$t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,$$

where M_σ is the algebraic cofactor of $m_{\sigma i}$ in M, i.e,

$$M_\sigma = (-1)^{i+\sigma}\times$$

$$\begin{vmatrix} \mathbf{E}_{\alpha+\beta,1}(\lambda(t_{\sigma+1}-t_\sigma)^{\alpha+\beta}) & 1 & 0 & \cdots & 0 \\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_{\sigma+2}-t_\sigma)^{\alpha+\beta}) & \mathbf{E}_{\alpha+\beta,1}(\lambda(t_{\sigma+2}-t_{\sigma+1})^{\alpha+\beta}) & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_\sigma)^{\alpha+\beta}) & \mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_{\sigma+1})^{\alpha+\beta}) & \cdots & \cdots & \mathbf{E}_{\alpha+\beta,1}(\lambda(t_i-t_{i-1})^{\alpha+\beta}) \end{vmatrix}.$$

From $ax(0)+bx(1)=c$, we get

$$aD_0 + b\Big[D_0 + d_0\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda) + \sum_{j=1}^m y_j\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})$$
$$+\sum_{j=1}^m C_j(1-t_j)^\beta\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(1-t_j)^{\alpha+\beta})$$
$$+\int_0^1(1-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(1-u)^{\alpha+\beta})f(u,x(u))du\Big] = c.$$

Then

$$D_0 = \frac{c}{a+b} - \frac{bd_0}{a+b}\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda) + \frac{b}{a+b}\sum_{j=1}^m y_j\mathbf{E}_{\alpha+\beta,1}(\lambda(1-t_j)^{\alpha+\beta})$$
$$+\frac{b}{a+b}\sum_{j=1}^m C_j(1-t_j)^\beta\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(1-t_j)^{\alpha+\beta})$$
$$+\frac{b}{a+b}\int_0^1(1-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(1-u)^{\alpha+\beta})f(u,x(u))du.$$

Hence BVP(1.3.26) is converted to

$$x(t) = \sum_{j=1}^i y_j\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_j)^{\alpha+\beta}) + d_0t^\beta\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda t^{\alpha+\beta}) + D_0\mathbf{E}_{\alpha+\beta,1}(\lambda t^{\alpha+\beta})$$
$$+\sum_{j=1}^i\Big[d_j - d_0 - \int_0^{t_j}(t_j-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_j-u)^{\alpha+\beta})f(u,x(u))du + \sum_{\sigma=1}^{j-1}M_\sigma d_\sigma$$
$$-d_0\sum_{\sigma=1}^{j-1}M_\sigma - \sum_{\sigma=1}^{j-1}M_\sigma\int_0^{t_\sigma}(t_\sigma-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t_\sigma-u)^{\alpha+\beta})f(u,x(u))du\Big]$$
$$\times(t-t_j)^\beta\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(t-t_j)^{\alpha+\beta})$$
$$+\int_0^t(t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})f(u,x(u))du,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m. \tag{6.11}$$

Hence (2.5) in Lemma 2.9 in [88] is wrong. Similarly we can verify (2.14) in Lemma 2.10 in [88] is also wrong. We omit the details. □

REMARK 6.9. Consider the following problems:

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}({}^cD^{\beta}_{t_i^+}u(t))+\lambda u(t)=f(t,u(t)), & t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m, \\ \Delta u(t_i)=u(t_i^+)-u(t_i^-)=y_i, & i\in\mathbb{N}_1^m, \\ au(0)+bu(1)=c, \quad [{}^cD^{\beta}_{t_i^+}u]|_{t=t_i}=d_i, & i\in\mathbb{N}_0^m, \end{cases} \tag{6.12}$$

and

$$\begin{cases} {}^cD^{\alpha}_{t_i^+}({}^cD^{\beta}_{t_i^+}u(t))+\lambda u(t)=f(t,u(t)), & t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m, \\ \Delta u(t_i)=u(t_i^+)-u(t_i^-)=y_i, & i\in\mathbb{N}_1^m, \\ a[{}^cD^{\beta}_{0,t}u]|_{t=0}+b[{}^cD^{\beta}_{0,t}u]|_{t=1}=c, \quad u(t_i)=d_i, & i\in\mathbb{N}_0^m. \end{cases} \tag{6.13}$$

By Theorem 3.4.1, we know that x is a piecewise continuous solution of

$${}^cD^{\alpha}_{t_i^+}({}^cD^{\beta}_{t_i^+}u(t))+\lambda u(t)=f(t,u(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m$$

if and only if there exist constants $C_j,D_j\in\mathbb{R}$ $(j\in\mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t)&=D_i\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_i)^{\alpha+\beta})+C_i(t-t_i)^{\beta}\mathbf{E}_{\alpha+\beta,\beta+1}(\lambda(t-t_i)^{\alpha+\beta}) \\ &\quad+\int_{t_i}^t(t-u)^{\alpha+\beta-1}\mathbf{E}_{\alpha+\beta,\alpha+\beta}(\lambda(t-u)^{\alpha+\beta})f(u,x(u))du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned} \tag{6.14}$$

By Corollary 3.4.2, we know

$$\begin{aligned} {}^cD^{\beta}_{0^+}x(t)&=C_i\mathbf{E}_{\alpha+\beta,1}(\lambda(t-t_i)^{\alpha+\beta}) \\ &\quad+\int_{t_i}^t(t-u)^{\alpha-1}\mathbf{E}_{\alpha+\beta,\alpha}(\lambda(t-u)^{\alpha+\beta})f(u,x(u))du, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned} \tag{6.15}$$

Using the boundary conditions, impulse functions in (6.12) together with (6.14)–(6.15), we can convert BVP(6.12) to an integral equation. Similarly use (6.13) and (6.14)–(6.15), we can convert BVP(6.26) to an integral equation. We omit the details. □

REMARK 6.10. Let $\rho=1-\int_0^1 sg(s)ds$. Lemma 2.3 in [52] claimed that x is a solution of

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t)=y(t), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m, \quad \alpha\in(1,2), \\ \Delta x(t_i)=h_i, \quad \Delta{}^cD^{\beta}_{0^+}x(t_i)=h_i^c, \quad i\in\mathbb{N}_1^m, \\ x(0)=0, \quad x(1)=\int_0^1 x(s)g(s)ds \end{cases} \tag{6.16}$$

if and only if

$$x(t) = \int_0^1 G(t,s)y(s)ds + \sum_{i=1}^{m} H(t,t_i)h_i + \sum_{i=1}^{m} K(t,t_i)h_i^c, \quad t \in [0,1] \tag{6.17}$$

where

$$G(t,s) = \begin{cases} \frac{t}{\rho} I_{1^-}^{\alpha} g(s) - \frac{t}{\rho\Gamma(\alpha)}(1-s)^{\alpha-1} + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{t}{\rho} I_{1^-}^{\alpha} g(s) - \frac{t}{\rho\Gamma(\alpha)}(1-s)^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$H(t,t_i) = \begin{cases} \frac{t}{\rho}\left(\int_{t_i}^1 g(s)ds - 1\right) + 1, & 0 < t_i < t \leqslant 1, \\ \frac{t}{\rho}\left(\int_{t_i}^1 g(s)ds - 1\right), & 0 \leqslant t \leqslant t_i < 1, \end{cases}$$

$$K(t,t_i) = -\Gamma(2-\beta) \begin{cases} t_i^{\beta} + \frac{t_i^{\beta}}{\rho}\left(\int_{t_i}^1 g(s)ds - 1\right) + \frac{t}{\rho}\frac{1}{t_i^{1-\beta}}\int_0^{t_i} sg(s)ds, & 0 < t_i < t \leqslant 1, \\ \frac{t_i^{\beta}}{\rho}\left(\int_{t_i}^1 g(s)ds - 1\right) + \frac{t}{\rho}\frac{1}{t_i^{1-\beta}}\int_0^{t_i} sg(s)ds, & 0 \leqslant t \leqslant t_i < 1. \end{cases}$$

Suppose that x is a solution of BVP(6.16) and $\rho = 1 - \int_0^1 sg(s)ds \neq 0$. By Corollary 3.2.1, there exists constants $C_i, D_i \ (i \in \mathbb{N}_0^m)$ such that

$$x(t) = \sum_{j=0}^{i} (C_j + D_j(t-t_j)) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{6.18}$$

By direct computation, we have

$${}^cD_{0^+}^{\beta} x(t) = \sum_{j=0}^{i} \frac{D_j}{\Gamma(2-\beta)}(t-t_j)^{1-\beta} + \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)ds, \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{6.19}$$

From $\Delta x(t_i) = h_i$ and (6.18), we have $C_i = h_i$ for $i \in \mathbb{N}_1^m$. $x(0) = 0$ and (6.18) imply that $C_0 = 0$. $x(1) = \int_0^1 x(s)g(s)ds$ and (6.31) imply that

$$D_0 + \sum_{j=1}^{m} (h_j + h_j^c(1-t_j)) + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds$$
$$= \sum_{\sigma=0}^{m} \int_{t_\sigma}^{t_{\sigma+1}} g(s) \left[D_0 s + \sum_{j=1}^{\sigma} (h_j + h_j^c(s-t_j)) + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)du \right] ds.$$

It follows that

$$D_0 = \frac{1}{1-\int_0^1 sg(s)ds} \sum_{\sigma=0}^{m} \int_{t_\sigma}^{t_{\sigma+1}} g(s) \left(\sum_{j=1}^{\sigma} (h_j + h_j^c(s-t_j)) + \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)du \right) ds$$
$$- \frac{1}{1-\int_0^1 sg(s)ds} \sum_{j=1}^{m} (h_j + h_j^c(1-t_j)) - \frac{1}{1-\int_0^1 sg(s)ds} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.$$

We note $\Delta {}^cD_{0^+}^{\beta} x(t_i) = {}^cD_{0^+}^{\beta} x(t_i^+) - {}^cD_{0^+}^{\beta} x(t_i^-) = h_i^c$ and $0 < \beta < \alpha - 1 < 1$. Since

$${}^cD_{0^+}^{\beta} x(t_i^+) = \sum_{j=0}^{i} \frac{D_j}{\Gamma(2-\beta)}(t_i - t_j)^{1-\beta} + \int_0^{t_i} \frac{(t_i-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)ds$$

and

$$ {}^cD_{0^+}^{\beta}x(t_i^-)=\sum_{j=0}^{i-1}\frac{D_j}{\Gamma(2-\beta)}(t_i-t_j)^{1-\beta}+\int_0^{t_i}\frac{(t_i-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds, $$

we have $h_i^c=0$. So the impulsive function $\Delta^cD_{0^+}^{\beta}x(t_i)=h_i^c$ is unsuitable. Then Lemma 2.3 in [52] is wrong. □

7. Some examples

In this section, to illustrate the usefulness of our main result, we present some examples.

EXAMPLE 7.1. Consider the following impulsive boundary value problem

$$ \begin{cases} D_{0^+}^{\frac{8}{5}}u(t)-u(t)=t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}}\left[b_0+a_0|(t-i/2)^{2/5}u(t)|^{\sigma}+c_0|(t-i/2]^{3/5}D_{0^+}^{\frac{1}{5}}u(t)|^{\sigma}\right], \\ t\in\left(\frac{i}{2},\frac{i}{2}+\frac{1}{2}\right], \quad i=0,1, \\ I_{0^+}^{\frac{2}{5}}u(0)=I_{0^+}^{\frac{2}{5}}u(1), \quad D_{0^+}^{\frac{3}{5}}u(0)=D_{0^+}^{\frac{3}{5}}u(1), \\ \lim\limits_{t\to s^+}(t-1/2)^{\frac{2}{5}}u(t)=I, \quad \Delta D_{0^+}^{\frac{3}{5}}u(1/2)=J, \end{cases} \tag{7.1} $$

where b_0, a_0, c_0, I,J, $\sigma\geqslant 0$ are constants. Then BVP(7.1) has at least one solution if $\sigma<1$.

Proof. Corresponding to BVP(1.4.5), we have $0=t_0<t_1=1/2<t_2=1$, $\alpha=\frac{8}{5}$ with $n=2$ and $p=\frac{1}{5}$, $m=1$, $k=l=-\frac{1}{5}$,

$$ f(t,x,y)=t^{-\frac{1}{5}}(1-t)^{-\frac{1}{5}}\left[b_0+a_0[(t-i/2)^{2/5}x]^{\sigma}+c_0(t-i/2]^{3/5}|y|^{\sigma}\right], $$

$t\in\left(\frac{i}{2},\frac{i}{2}+\frac{1}{2}\right]$, $i=0,1$, $I_2(1/2,x,y)=I$, and $I_1(1/2,x,y)=J$, f is a II-Caratheódory function. By Theorem 4.1.1, BVP(7.1) has at least one solution. □

Claim 7.1. Let

$$ M=(m_{uv})_{i\times i}=\begin{pmatrix} b & 0 & 0 & \cdots & 0 & 0 \\ a_1 & b & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1} & a_{i-1} & a_{i-1} & \cdots & a_{i-1} & b \end{pmatrix}. $$

Then the algebraic co-factor $M_{ui}(u\in\mathbb{N}_1^i)$ of m_{ui} in M is given by

$$ M_{ui}=\begin{cases} (-1)^{u+i}a_{i-1}b^{u-1}\prod\limits_{j=u}^{i-2}(a_j-b), & 1\leqslant u\leqslant i-1, \\ b^{i-1}, \quad u=i. \end{cases} $$

EXAMPLE 7.2. Let $h \in L^1(0,1)\cap C(0,1)$, $\alpha \in (1,2)$, c_s, $d_s \in \mathbb{R}$ $(s \in \mathbb{N}_1^m)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and there exist constant $k > -1$, $l \in (-1,0]$ such that $|h(t)| \leqslant t^k(1-l)^l$ for all $t \in (0,1)$. Then Dirichlet boundary value problem for impulsive fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}x(t) = h(t), \text{ a.e., } t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m, \\ I_{0^+}^{2-\alpha}x(0) = I_{0^+}^{2-\alpha}x(1) = 0, \\ \Delta I_{0^+}^{2-\alpha}x(t_s) = c_s I_{0^+}^{2-\alpha}(t_s), \quad \Delta D_{0^+}^{\alpha-1}x(t_s) = d_s D_{0^+}^{\alpha-1}x(t_s), \quad s \in \mathbb{N}_1^m \end{cases} \tag{7.2}$$

has a unique solution

$$\begin{aligned} x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + A_0t^{\alpha-1} + \sum_{j=1}^{i} A_j(t-t_j)^{\alpha-1} + \sum_{j=1}^{i} B_j(t-t_j)^{\alpha-2}, \\ t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m, \end{aligned} \tag{7.3}$$

where

$$\begin{aligned} A_i = {} & \frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}d_\sigma \int_0^{t_\sigma} h(s)ds + \Gamma(\alpha-1)^{i-1}d_i\int_0^{t_i}h(s)ds \\ & + \left[\frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}\Gamma(\alpha)d_\sigma + \Gamma(\alpha-1)^{i-1}\Gamma(\alpha)d_i\right]A_0, \\ & \qquad i \in \mathbb{N}_1^m, \end{aligned}$$

$$\begin{aligned} B_i = {} & \frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}c_\sigma\int_0^{t_\sigma}(t_\sigma-s)h(s)ds \\ & + \Gamma(\alpha-1)^{i-1}c_i\int_0^{t_i}(t_i-s)h(s)ds \\ & + \frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}\Gamma(\alpha)c_\sigma\sum_{j=0}^{\sigma-1}A_j(t_\sigma-t_j) \\ & + \Gamma(\alpha-1)^{i-1}\Gamma(\alpha)c_i\sum_{j=0}^{i-1}A_j(t_i-t_j) \\ & + \frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}\Gamma(\alpha-1)c_\sigma B_0 \\ & + \Gamma(\alpha-1)^{i-1}\Gamma(\alpha-1)c_iB_0, \quad i \in \mathbb{N}_1^m, \end{aligned}$$

$$A_0 = -\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)h(s)ds - \sum_{j=1}^{m}A_j(t-t_j) + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}\sum_{j=1}^{m}B_i.$$

Proof. From Corollary 3.1, we know that there exist constants A_i, $B_i \in \mathbb{R}$ such that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=0}^{i}A_j(t-t_j)^{\alpha-1} + \sum_{j=0}^{i}B_j(t-t_j)^{\alpha-2}, \quad t \in (t_i,t_{i+1}], \quad i \in \mathbb{N}_0^m. \tag{7.4}$$

Then Corollary 3.2 implies that

$$I_{0^+}^{2-\alpha}x(t)=\int_0^t(t-s)h(s)ds+\Gamma(\alpha)\sum_{j=0}^{i}A_j(t-t_j)+\Gamma(\alpha-1)\sum_{j=0}^{i}B_j,\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m$$

and

$$D_{0^+}^{\alpha-1}x(t)=\int_0^t h(s)ds+\Gamma(\alpha)\sum_{j=0}^{i}A_j,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.$$

From $I_{0^+}^{2-\alpha}x(0)=I_{0^+}^{2-\alpha}x(1)=0$, we get

$$B_0=0,\quad \int_0^1(1-s)h(s)ds+\Gamma(\alpha)\sum_{j=0}^{m}A_j(t-t_j)+\Gamma(\alpha-1)\sum_{j=0}^{m}B_i=0.$$

So

$$A_0=-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)h(s)ds-\sum_{j=1}^{m}A_j(t-t_j)+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}\sum_{j=1}^{m}B_i.$$

From $\Delta I_{0^+}^{2-\alpha}x(t_s)=c_sI_{0^+}^{2-\alpha}(t_s)$, $\Delta D_{0^+}^{\alpha-1}x(t_s)=d_sD_{0^+}^{\alpha-1}x(t_s)$, we get

$$\Gamma(\alpha-1)B_i=c_i\left(\int_0^{t_i}(t_i-s)h(s)ds+\Gamma(\alpha)\sum_{j=0}^{i-1}A_j(t_i-t_j)+\Gamma(\alpha-1)\sum_{j=0}^{i-1}B_j\right),$$

$$\Gamma(\alpha-1)A_i=d_i\left(\int_0^{t_i}h(s)ds+\Gamma(\alpha)\sum_{j=0}^{i-1}A_j\right),\quad i\in\mathbb{N}_1^m.$$

It turns to

$$\begin{pmatrix}\Gamma(\alpha-1) & 0 & \cdots & 0\\ -\Gamma(\alpha)d_1 & \Gamma(\alpha-1) & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ -\Gamma(\alpha)d_i & -\Gamma(\alpha)d_i & \cdots & \Gamma(\alpha-1)\end{pmatrix}\begin{pmatrix}A_1\\ A_2\\ \cdots\\ A_i\end{pmatrix}=\begin{pmatrix}d_1\int_0^{t_1}h(s)ds+\Gamma(\alpha)d_1A_0\\ d_2\int_0^{t_2}h(s)ds+\Gamma(\alpha)d_2A_0\\ \cdots\\ d_i\int_0^{t_i}h(s)ds+\Gamma(\alpha)d_iA_0\end{pmatrix}.$$

So

$$\begin{pmatrix}A_1\\ A_2\\ \cdots\\ A_i\end{pmatrix}=\begin{pmatrix}\Gamma(\alpha-1) & 0 & \cdots & 0\\ -\Gamma(\alpha)d_1 & \Gamma(\alpha-1) & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ -\Gamma(\alpha)d_i & -\Gamma(\alpha)d_i & \cdots & \Gamma(\alpha-1)\end{pmatrix}^{-1}\begin{pmatrix}d_1\int_0^{t_1}h(s)ds+\Gamma(\alpha)d_1A_0\\ d_2\int_0^{t_2}h(s)ds+\Gamma(\alpha)d_2A_0\\ \cdots\\ d_i\int_0^{t_i}h(s)ds+\Gamma(\alpha)d_iA_0\end{pmatrix}.$$

By Claim 7.1 ($b=\Gamma(\alpha-1)$ and $a_i=-\Gamma(\alpha)d_i$), we have

$$\begin{aligned}A_i&=\frac{1}{\Gamma(\alpha-1)^i}\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)\\ &\quad\times\left(d_\sigma\int_0^{t_\sigma}h(s)ds+\Gamma(\alpha)d_\sigma A_0\right)+\Gamma(\alpha-1)^{i-1}\left[d_i\int_0^{t_i}h(s)ds+\Gamma(\alpha)d_iA_0\right]\\ &=\frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}d_\sigma\int_0^{t_\sigma}h(s)ds+\Gamma(\alpha-1)^{i-1}d_i\int_0^{t_i}h(s)ds\end{aligned}$$

$$+\left[\frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}\Gamma(\alpha)d_\sigma+\Gamma(\alpha-1)^{i-1}\Gamma(\alpha)d_i\right]A_0,$$

$$i\in\mathbb{N}_1^m.$$

Similarly

$$\Gamma(\alpha-1)B_i=c_i\int_0^{t_i}(t_i-s)h(s)ds+\Gamma(\alpha)c_i\sum_{j=0}^{i-1}A_j(t_i-t_j)+\Gamma(\alpha-1)c_i\sum_{j=0}^{i-1}B_j,\quad i\in\mathbb{N}_1^m$$

is turned to

$$\begin{pmatrix}B_1\\B_2\\\cdots\\B_i\end{pmatrix}=\begin{pmatrix}\Gamma(\alpha-1)&0&\cdots&0\\-\Gamma(\alpha-1)c_1&\Gamma(\alpha-1)&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\-\Gamma(\alpha-1)c_i&-\Gamma(\alpha-1)c_i&\cdots&\Gamma(\alpha-1)\end{pmatrix}^{-1}$$

$$\times\begin{pmatrix}c_1\int_0^{t_1}(t_1-s)h(s)ds+\Gamma(\alpha)c_1\sum_{j=0}^{1-1}A_j(t_1-t_j)+\Gamma(\alpha-1)c_1B_0\\c_2\int_0^{t_2}(t_2-s)h(s)ds+\Gamma(\alpha)c_2\sum_{j=0}^{2-1}A_j(t_2-t_j)+\Gamma(\alpha-1)c_2B_0\\\cdots\\c_i\int_0^{t_i}(t_i-s)h(s)ds+\Gamma(\alpha)c_i\sum_{j=0}^{i-1}A_j(t_i-t_j)+\Gamma(\alpha-1)c_iB_0\end{pmatrix}.$$

So

$$\begin{aligned}B_i=&\frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}\\&\times\left(c_\sigma\int_0^{t_\sigma}(t_\sigma-s)h(s)ds+\Gamma(\alpha)c_\sigma\sum_{j=0}^{\sigma-1}A_j(t_\sigma-t_j)+\Gamma(\alpha-1)c_\sigma B_0\right)\\&+\Gamma(\alpha-1)^{i-1}\left[c_i\int_0^{t_i}(t_i-s)h(s)ds+\Gamma(\alpha)c_i\sum_{j=0}^{i-1}A_j(t_i-t_j)+\Gamma(\alpha-1)c_iB_0\right]\\=&\frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}c_\sigma\int_0^{t_\sigma}(t_\sigma-s)h(s)ds\\&+\Gamma(\alpha-1)^{i-1}c_i\int_0^{t_i}(t_i-s)h(s)ds\\&+\frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}\Gamma(\alpha)c_\sigma\sum_{j=0}^{\sigma-1}A_j(t_\sigma-t_j)\\&+\Gamma(\alpha-1)^{i-1}\Gamma(\alpha)c_i\sum_{j=0}^{i-1}A_j(t_i-t_j)\end{aligned}$$

$$+\frac{\sum_{\sigma=1}^{i-1}\left(\Gamma(\alpha-1)^{\sigma-1}\Gamma(\alpha)d_{i-1}\prod_{j=\sigma}^{i-2}(\Gamma(\alpha)d_j+\Gamma(\alpha-1))\right)}{\Gamma(\alpha-1)^i}\Gamma(\alpha-1)c_\sigma B_0$$
$$+\Gamma(\alpha-1)^{i-1}\Gamma(\alpha-1)c_iB_0, \quad i\in\mathbb{N}_1^m.$$

Substituting A_i,B_i into (7.4), we get (7.3). On the other hand, if x satisfies (7.3), we can prove that x satisfies (7.2) by direct computation. □

REMARK 7.1. Let $h\in L^1(0,\infty)\cap C(0,\infty)$, $\alpha\in(0,1)$, b_0, $c_s(s\in\mathbb{R})\in\mathbb{R}$, $0=t_0<t_1\ll t_2<\cdots$ and there exist a constant $k>-1$ such that $|h(t)|\leqslant t^k$ for all $t\in(0,\infty)$. We suggest readers to establish the solution of the generalized periodic boundary value problem for impulsive fractional differential equation

$$\begin{cases} D_{0^+}^\alpha x(t)=h(t), \text{ a.e.}, & t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^\infty,\\ I_{0^+}^{1-\alpha}x(0)=b_0\lim_{t\to\infty}I_{0^+}^{1-\alpha}x(t), \\ \Delta I_{0^+}^{1-\alpha}x(t_s)=c_sI_{0^+}^{1-\alpha}(t_s), \quad s\in\mathbb{N}_1^\infty.\end{cases}$$

REMARK 7.2. Let $h\in L^1(0,\infty)\cap C(0,\infty)$, $\alpha\in(1,2)$, c_s, $d_s\in\mathbb{R}$ $(s\in\mathbb{N}_1^\infty)$, $0=t_0<t_1<t_2<\cdots$ and there exist a constant $k>-1$ such that $|h(t)|\leqslant t^k$ for all $t\in(0,\infty)$. We suggest readers to establish the solution of the boundary value problem for impulsive fractional differential equation

$$\begin{cases} D_{0^+}^\alpha x(t)=h(t), \text{ a.e.}, & t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^\infty,\\ I_{0^+}^{2-\alpha}x(0)=\lim_{t\to\infty}D_{0^+}^{\alpha-1}x(t)=0, \\ \Delta I_{2-\alpha}x(t_s)=c_sI_{0^+}^{2-\alpha}(t_s), \quad \Delta D_{0^+}^{\alpha-1}x(t_s)=d_sD_{0^+}^{\alpha-1}x(t_s), \quad s\in\mathbb{N}_1^\infty.\end{cases}$$

In [76], authors studied the existence of at least one solution of the following anti-periodic boundary value problem

$$\begin{cases} {}^cD_{t_i^+}^\alpha u(t)=f(t,u(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\ \Delta u(t_i)=I_i(u(t_i)), \quad \Delta u'(t_i)=J_i(u(t_i)), \quad \Delta u''(t_i)=Q_i(u(t_i)), \quad i\in\mathbb{N}_1^m,\\ u(0)=-u(1), \quad u'(0)=-u'(1), \quad u''(0)=-u''(1),\end{cases}$$

where $\alpha\in(2,3)$, ${}^cD_{0^+}$ is the Caputo fractional derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous, I_i, J_i, $Q_i:\mathbb{R}\to\mathbb{R}$ are continuous, $0=t_0<t_s<\cdots<t_m<t_{m+1}=1$.

Motivated by [76], we can investigate the following anti-periodic boundary value problems

$$\begin{cases} {}^cD_{t_i^+}^\alpha u(t)-\lambda u(t)=f(t,u(t)), \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\ u(t_i^+)=I_i(u(t_i)), \quad u'(t_i^+)=J_i(u(t_i)), \quad u''(t_i^+)=Q_i(u(t_i)), \quad i\in\mathbb{N}_1^m,\\ u(0)=-u(1), \quad u'(0)=-u'(1), \quad u''(0)=-u''(1),\end{cases} \tag{7.5}$$

and

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t)-\lambda u(t)=f(t,u(t)), \quad t\in(t_i,t_{i+1}], \quad i\in \mathbb{N}_0^m, \\ \Delta u(t_i)=I_i(u(t_i)), \quad \Delta u'(t_i)=J_i(u(t_i)), \quad \Delta u''(t_i)=Q_i(u(t_i)), \quad i\in\mathbb{N}_1^m, \\ u(0)=-u(1), \quad u'(0)=-u'(1), \quad u''(0)=-u''(1), \end{cases} \tag{7.6}$$

where $\alpha\in(2,3)$, $\lambda\in\mathbb{R}$, f,I_i,J_i are continuous functions. Choose the Banach space

$$PC[0,1]=\left\{x:(0,1]\to\mathbb{R}: x|_{(t_i,t_{i+1}]}\in C(t_i,t_{i+1}],\ \lim_{t\to t_i^+}x(t)\text{ is finite, } i\in\mathbb{N}_0^m\right\}$$

with the norm $||x||=\sup_{t\in(0,1]}|x(t)|$. We will seek solutions of BVP(7.5) and BVP(7.6) in X.

EXAMPLE 7.3. $x\in PC(0,1]$ is a solution of BVP(7.5) if and only if

$$x(t)=\begin{cases} -A_0(t)I_m(x(t_m))-B_0(t)J_m(x(t_m))-C_0(t)Q_m(x(t_m))-\int_{t_m}^1G(t,s)f(s,x(s))ds \\ +\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds, \quad t\in(t_0,t_1], \\ \mathbf{E}_{\alpha,1}(\lambda(t-t_i)^{\alpha})I_i(x(t_i))+(t-t_i)\mathbf{E}_{\alpha,2}(\lambda(t-t_i)^{\alpha})J_i(x(t_i)) \\ +(t-t_i)^2\mathbf{E}_{\alpha,3}(\lambda(t-t_i)^{\alpha})Q_i(u(t_i)) \\ +\int_{t_i}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_1^m, \end{cases} \tag{7.6}$$

where

$$\begin{aligned} A_0(t)&=\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^{\alpha})+t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})\lambda(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_m)^{\alpha}) \\ &\quad +t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\lambda(1-t_m)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_m)^{\alpha}), \\ B_0(t)&=\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-t_m)\mathbf{E}_{\alpha,2}(\lambda(1-t_m)^{\alpha})+t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^{\alpha}) \\ &\quad +t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\lambda(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_m)^{\alpha}), \\ C_0(t)&=\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-t_m)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_m)^{\alpha})+t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-t_m)\mathbf{E}_{\alpha,2}(\lambda(1-t_m)^{\alpha}) \\ &\quad +t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^{\alpha}), \\ G(t,s)&=\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})+t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha}) \\ &\quad +t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha}). \end{aligned}$$

Proof. Suppose that x is a solution of BVP(7.5). We know from Theorem 3.2.1 that there exist constants a_i, b_i, c_i $(i\in\mathbb{N}_0^m)$ such that

$$\begin{aligned} x(t)&=a_i\mathbf{E}_{\alpha,1}(\lambda(t-t_i)^{\alpha})+b_i(t-t_i)\mathbf{E}_{\alpha,2}(\lambda(t-t_i)^{\alpha})+c_i(t-t_i)^2\mathbf{E}_{\alpha,3}(\lambda(t-t_i)^{\alpha}) \\ &\quad +\int_{t_i}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m. \end{aligned} \tag{7.7}$$

Then

$$x'(t)=a_i\lambda(t-t_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_i)^{\alpha})+b_i\mathbf{E}_{\alpha,1}(\lambda(t-t_i)^{\alpha})+c_i(t-t_i)\mathbf{E}_{\alpha,2}(\lambda(t-t_i)^{\alpha})$$

$$
\begin{aligned}
&+\int_{t_i}^t (t-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(t-s)^\alpha)f(s,x(s))ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m,\\
x''(t) &= a_i\lambda(t-t_i)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(t-t_i)^\alpha)+b_i\lambda(t-t_i)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_i)^\alpha)\\
&+c_i\mathbf{E}_{\alpha,1}(\lambda(t-t_i)^\alpha)+\int_{t_i}^t (t-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(t-s)^\alpha)f(s,x(s))ds,\\
&\qquad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.
\end{aligned}
$$

Then $x(0)=-x(1)$, $x'(0)=-x'(1)$, $x''(0)=-x''(1)$ imply

$$
\begin{aligned}
a_0 &= -a_m\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^\alpha)-b_m(1-t_m)\mathbf{E}_{\alpha,21}(\lambda(1-t_m)^\alpha)\\
&\quad -c_m(1-t_m)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_m)^\alpha)-\int_{t_m}^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha)f(s,x(s))ds,\\
b_0 &= -a_m\lambda(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_m)^\alpha)-b_m\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^\alpha)\\
&\quad -c_m(1-t_m)\mathbf{E}_{\alpha,2}(\lambda(1-t_m)^\alpha)-\int_{t_m}^1(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^\alpha)f(s,x(s))ds,\\
c_0 &= -a_m\lambda(1-t_m)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_m)^\alpha)-b_m\lambda(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_m)^\alpha)\\
&\quad -c_m\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^\alpha)-\int_{t_m}^1(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^\alpha)f(s,x(s))ds.
\end{aligned}
$$

By $x(t_i^+)=I_i(x(t_i))$, $x'(t_i^+)=J_i(x(t_i))$, $x''(t_i^+)=Q_i(x(t_i))$, we have

$$
a_i=I_i(x(t_i)), \quad b_i=J_i(x(t_i)), \quad c_i=Q_i(x(t_i)), \quad i\in\mathbb{N}_1^m.
$$

Then

$$
\begin{aligned}
a_0 &= -\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^\alpha)I_m(x(t_m))-(1-t_m)\mathbf{E}_{\alpha,21}(\lambda(1-t_m)^\alpha)J_m(x(t_m))\\
&\quad -(1-t_m)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_m)^\alpha)Q_m(x(t_m))-\int_{t_m}^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha)f(s,x(s))ds,\\
b_0 &= -\lambda(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_m)^\alpha)I_m(x(t_m))-\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^\alpha)J_m(x(t_m))\\
&\quad -(1-t_m)\mathbf{E}_{\alpha,2}(\lambda(1-t_m)^\alpha)Q_m(x(t_m))-\int_{t_m}^1(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^\alpha)f(s,x(s))ds,\\
c_0 &= -\lambda(1-t_m)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_m)^\alpha)I_m(x(t_m))\\
&\quad -\lambda(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_m)^\alpha)J_m(x(t_m))-\mathbf{E}_{\alpha,1}(\lambda(1-t_m)^\alpha)Q_m(x(t_m))\\
&\quad -\int_{t_m}^1(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^\alpha)f(s,x(s))ds.
\end{aligned}
$$

Substituting all a_s, b_s, c_s into (7.7), we get (7.6). On the other hand, if x satisfies (7.6), we can get (7.5) by direct computation. □

EXAMPLE 7.4. Let

$$
M=(m_{ij})=\begin{pmatrix} 1+\mathbf{E}_{\alpha,1}(\lambda) & \mathbf{E}_{\alpha,2}(\lambda) & \mathbf{E}_{\alpha,3}(\lambda)\\ \mathbf{E}_{\alpha,\alpha}(\lambda) & 1+\mathbf{E}_{\alpha,1}(\lambda) & \mathbf{E}_{\alpha,2}(\lambda)\\ \mathbf{E}_{\alpha,\alpha-1}(\lambda) & \mathbf{E}_{\alpha,\alpha}(\lambda) & 1+\mathbf{E}_{\alpha,1}(\lambda)\end{pmatrix}
$$

and M_{ij} be the algebraic cofactor of m_{ij} in M. Suppose that $|M|\neq 0$. Then $x\in PC(0,1]$ is a solution of BVP(7.6) if and only if

$$
x(t)=-\sum_{j=1}^m A_j(t)I_j(x(t_j))-\sum_{j=1}^m B_j(t)J_j(x(t_j))-\sum_{j=1}^m C_j(t)Q_j(x(t_j))
$$

$$
\begin{aligned}
&-\int_0^1 G(t,s)f(s,x(s))ds+\sum_{j=1}^{i}\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^{\alpha})I_j(x(t_j))\\
&+\sum_{j=1}^{i}(t-t_j)\mathbf{E}_{\alpha,2}(\lambda(t-t_j)^{\alpha})J_j(x(t_j))+\sum_{j=1}^{i}(t-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(t-t_j)^{\alpha})Q_j(x(t_j))\\
&+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m, \qquad (7.9)
\end{aligned}
$$

where

$$
\begin{aligned}
A_j(t) =\ & \tfrac{M_{11}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{21}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})\lambda(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{31}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})\lambda(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{12}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{22}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})\lambda(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{32}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})\lambda(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{13}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{23}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\lambda(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{33}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\lambda(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha}),\\
B_j(t) =\ & \tfrac{M_{11}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{21}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})\\
&+\lambda\tfrac{M_{31}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{12}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{22}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})\\
&+\lambda\tfrac{M_{32}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{13}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{23}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})\\
&+\lambda\tfrac{M_{33}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha}),\\
C_j(t) =\ & \tfrac{M_{11}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{21}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{31}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{12}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{22}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{32}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{13}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})\\
&+\tfrac{M_{23}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})+\tfrac{M_{33}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha}),\\
G(t,s) =\ & \tfrac{M_{11}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\\
&+\tfrac{M_{21}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})\\
&+\tfrac{M_{31}}{|M|}\mathbf{E}_{\alpha,1}(\lambda t^{\alpha})(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha})\\
&+\tfrac{M_{12}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\\
&+\tfrac{M_{22}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})
\end{aligned}
$$

$$+\frac{M_{32}}{|M|}t\mathbf{E}_{\alpha,2}(\lambda t^{\alpha})(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha})$$
$$+\frac{M_{13}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})$$
$$+\frac{M_{23}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})$$
$$+\frac{M_{33}}{|M|}t^2\mathbf{E}_{\alpha,3}(\lambda t^{\alpha})(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha}).$$

Proof. Suppose x is a solution of BVP(7.6). By Theorem 3.2.1, we know that there exist constants a_{iv} $(i\in\mathbb{N}_0^m,\ v=0,1,2))$ such that

$$\begin{aligned}x(t) =& \sum_{j=0}^{i} a_{j0}\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^{\alpha}) + \sum_{j=0}^{i} a_{j1}(t-t_j)\mathbf{E}_{\alpha,2}(\lambda(t-t_j)^{\alpha})\\ &+\sum_{j=0}^{i} a_{j2}(t-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(t-t_j)^{\alpha}) + \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds,\\ & t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m. \qquad (7.10)\end{aligned}$$

Then

$$\begin{aligned}x'(t) =& \lambda\sum_{j=0}^{i} a_{j0}(t-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_j)^{\alpha}) + \sum_{j=0}^{i} a_{j1}\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^{\alpha})\\ &+\sum_{j=0}^{i} a_{j2}(t-t_j)\mathbf{E}_{\alpha,2}(\lambda(t-t_j)^{\alpha})+\int_0^t(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(t-s)^{\alpha})f(s,x(s))ds,\\ & t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m,\\ x''(t) =& \lambda\sum_{j=0}^{i} a_{j0}(t-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(t-t_j)^{\alpha}) + \lambda\sum_{j=0}^{i} a_{j1}(t-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_j)^{\alpha})\\ &+\sum_{j=0}^{i} a_{j2}\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^{\alpha})+\int_0^t(t-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(t-s)^{\alpha})f(s,x(s))ds,\\ & t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_0^m.\end{aligned}$$

Then $x(0)=-x(1)$, $x'(0)=-x'(1)$, $x''(0)=-x''(1)$ imply

$$\begin{aligned}a_{00} =& -\sum_{j=0}^{m} a_{j0}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha}) - \sum_{j=0}^{m} a_{j1}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})\\ &-\sum_{j=0}^{m} a_{j2}(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})\\ &-\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})f(s,x(s))ds,\\ a_{01} =& -\lambda\sum_{j=0}^{m} a_{j0}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha}) - \sum_{j=0}^{m} a_{j1}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})\\ &-\sum_{j=0}^{m} a_{j2}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha}) - \int_0^1(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})f(s,x(s))ds,\\ a_{02} =& -\lambda\sum_{j=0}^{m} a_{j0}(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha})-\lambda\sum_{j=0}^{m} a_{j1}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})\end{aligned}$$

$$-\sum_{j=0}^{m} a_{j2}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})-\int_0^1(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha})f(s,x(s))ds.$$

By $\Delta x(t_i^+)=I_i(x(t_i))$, $\Delta x'(t_i^+)=J_i(x(t_i))$, $\Delta x''(t_i^+)=Q_i(x(t_i))$, we have

$$a_{i0}=I_i(x(t_i)),\quad a_{i1}=J_i(x(t_i)),\quad a_{i2}=Q_i(x(t_i)),\quad i\in\mathbb{N}_1^m.$$

Then

$$\begin{aligned}
&(1+\mathbf{E}_{\alpha,1}(\lambda))a_{00}+\mathbf{E}_{\alpha,2}(\lambda)a_{01}+\mathbf{E}_{\alpha,3}(\lambda)a_{02}\\
=&-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))-\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})f(s,x(s))ds,\\
&\mathbf{E}_{\alpha,\alpha}(\lambda)a_{00}+(1+\mathbf{E}_{\alpha,1}(\lambda))a_{01}+\mathbf{E}_{\alpha,2}(\lambda)a_{02}\\
=&=-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))-\int_0^1(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})f(s,x(s))ds,\\
&\mathbf{E}_{\alpha,\alpha-1}(\lambda)a_{00}+\mathbf{E}_{\alpha,\alpha}(\lambda)a_{01}+(1+\mathbf{E}_{\alpha,1}(\lambda))a_{02}\\
=&=-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))\\
&-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))\\
&-\int_0^1(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha})f(s,x(s))ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
a_{00}=&\frac{M_{11}}{|M|}\Big[-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))-\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big]\\
&+\frac{M_{21}}{|M|}\Big[-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))-\int_0^1(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big]\\
&+\frac{M_{31}}{|M|}\Big[-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))\\
&-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))\\
&-\int_0^1(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big],
\end{aligned}$$

$$
\begin{aligned}
a_{01} = \tfrac{M_{12}}{|M|}\Big[&-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))-\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big]\\
&+\tfrac{M_{22}}{|M|}\Big[-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))-\int_0^1(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big]\\
&+\tfrac{M_{32}}{|M|}\Big[-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))\\
&-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))\\
&-\int_0^1(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big],\\
a_{02} = \tfrac{M_{13}}{|M|}\Big[&-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))-\int_0^1(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big]\\
&+\tfrac{M_{23}}{|M|}\Big[-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))\\
&-\sum_{j=1}^{m}(1-t_j)\mathbf{E}_{\alpha,2}(\lambda(1-t_j)^{\alpha})Q_j(xt_j))-\int_0^1(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big]\\
&+\tfrac{M_{33}}{|M|}\Big[-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-2}\mathbf{E}_{\alpha,\alpha-1}(\lambda(1-t_j)^{\alpha})I_j(x(t_j))\\
&-\lambda\sum_{j=1}^{m}(1-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(1-t_j)^{\alpha})J_j(x(t_j))-\sum_{j=1}^{m}\mathbf{E}_{\alpha,1}(\lambda(1-t_j)^{\alpha})Q_j(x(t_j))\\
&-\int_0^1(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(\lambda(1-s)^{\alpha})f(s,x(s))ds\Big]
\end{aligned}
$$

Here M_{ij} is the is the algebraic cofactor of m_{ij} in $M=(m_{ij})$ defined in Theorem. Substituting all a_{s0}, a_{s1}, a_{s2} into (7.10), we get (7.9). On the other hand, if x satisfies (7.9), we can get (7.6) by direct computation. □

THEOREM 7.1. *Suppose that there exist constants $M_f \geqslant 0,\ M_I \geqslant 0,\ M_J \geqslant 0,\ M_Q \geqslant 0$ such that*

$$|f(t,x)| \leqslant M_f,\ t\in(0,1],\ x\in\mathbb{R},\ |I_i(x)|\leqslant M_i,\ |J_i(x)|\leqslant M_J,\ |Q_i(x)|\leqslant M_Q,\ i\in\mathbb{N}_1^m$$

Then BVP(7.5) has at least one solution.

Proof. Define the operator T on X by

$$(Tx)(t)=\begin{cases}-A_0(t)I_m(x(t_m))-B_0(t)J_m(x(t_m))-C_0(t)Q_m(x(t_m))-\int_{t_m}^1 G(t,s)f(s,x(s))ds\\+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds,\quad t\in(t_0,t_1],\\ \mathbf{E}_{\alpha,1}(\lambda(t-t_i)^{\alpha})I_i(x(t_i))+(t-t_i)\mathbf{E}_{\alpha,2}(\lambda(t-t_i)^{\alpha})J_i(x(t_i))\\+(t-t_i)^2\mathbf{E}_{\alpha,3}(\lambda(t-t_i)^{\alpha})Q_i(u(t_i))\\+\int_{t_i}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds,\quad t\in(t_i,t_{i+1}],\quad i\in\mathbb{N}_1^m.\end{cases}$$

Then $T:X\to X$ is well defined and T is completely continuous, x is a solution of BVP(7.5) if and only if x is a fixed point of T in X. By the definition of T, we have

$$\begin{aligned}|(Tx)(t)|&\leqslant \sup_{t\in(t_0,t_1]}|(Tx)(t)+\sup_{t\in(t_i,t_{i+1}]}|(Tx)(t)|\\
&\leqslant \big[\mathbf{E}_{\alpha,1}(|\lambda|)^2+|\lambda|\mathbf{E}_{\alpha,2}(|\lambda|)\mathbf{E}_{\alpha,\alpha}(|\lambda|)\ +|\lambda|\mathbf{E}_{\alpha,3}(|\lambda|)\mathbf{E}_{\alpha,\alpha-1}(|\lambda|)]M_I\\
&\quad+[\mathbf{E}_{\alpha,1}(|\lambda|)\mathbf{E}_{\alpha,1}(|\lambda|)+\mathbf{E}_{\alpha,2}(|\lambda|)\mathbf{E}_{\alpha,1}(|\lambda|)\ +|\lambda|\mathbf{E}_{\alpha,3}(|\lambda|)\mathbf{E}_{\alpha,\alpha}(|\lambda|)]M_J\\
&\quad+[\mathbf{E}_{\alpha,1}(|\lambda|)\mathbf{E}_{\alpha,3}(|\lambda|)+\mathbf{E}_{\alpha,2}(|\lambda|)\mathbf{E}_{\alpha,2}(|\lambda|)\ +\mathbf{E}_{\alpha,3}(|\lambda|)\mathbf{E}_{\alpha,1}(|\lambda|)]M_Q\\
&\quad+\int_{t_m}^1\big[\mathbf{E}_{\alpha,1}(|\lambda|)(1-s)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)+\mathbf{E}_{\alpha,2}(|\lambda|)(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(|\lambda|)\\
&\quad+\mathbf{E}_{\alpha,3}(|\lambda|)(1-s)^{\alpha-3}E_{\alpha,\alpha-2}(|\lambda|)\big]M_fds+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)M_fds\\
&\quad+\mathbf{E}_{\alpha,1}(|\lambda|)M_I+\mathbf{E}_{\alpha,2}(|\lambda|)M_J+\mathbf{E}_{\alpha,3}(|\lambda|)M_Q+\int_{t_i}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)M_fds\\
&\leqslant \big[\mathbf{E}_{\alpha,1}(|\lambda|)^2+|\lambda|\mathbf{E}_{\alpha,2}(|\lambda|)\mathbf{E}_{\alpha,\alpha}(|\lambda|)\ +|\lambda|\mathbf{E}_{\alpha,3}(|\lambda|)\mathbf{E}_{\alpha,\alpha-1}(|\lambda|)]M_I\\
&\quad+[\mathbf{E}_{\alpha,1}(|\lambda|)\mathbf{E}_{\alpha,1}(|\lambda|)+\mathbf{E}_{\alpha,2}(|\lambda|)\mathbf{E}_{\alpha,1}(|\lambda|)\ +|\lambda|\mathbf{E}_{\alpha,3}(|\lambda|)\mathbf{E}_{\alpha,\alpha}(|\lambda|)]M_J\\
&\quad+[\mathbf{E}_{\alpha,1}(|\lambda|)\mathbf{E}_{\alpha,3}(|\lambda|)+\mathbf{E}_{\alpha,2}(|\lambda|)\mathbf{E}_{\alpha,2}(|\lambda|)\ +\mathbf{E}_{\alpha,3}(|\lambda|)\mathbf{E}_{\alpha,1}(|\lambda|)]M_Q\\
&\quad+\Big[\frac{\mathbf{E}_{\alpha,1}(|\lambda|)(1-s)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{|alpha}+\frac{\mathbf{E}_{\alpha,2}(|\lambda|)E_{\alpha,\alpha-1}(|\lambda|)}{\alpha-1}\\
&\quad+\frac{\mathbf{E}_{\alpha,3}(|\lambda|)E_{\alpha,\alpha-2}(|\lambda|)}{\alpha-2}\Big]M_f+\frac{E_{\alpha,\alpha}(|\lambda|)M_f}{\alpha}+\mathbf{E}_{\alpha,1}(|\lambda|)M_I+\mathbf{E}_{\alpha,2}(|\lambda|)M_J\\
&\quad+\mathbf{E}_{\alpha,3}(|\lambda|)M_Q+\frac{E_{\alpha,\alpha}(|\lambda|)M_f}{\alpha}=:M_0\quad\text{a constant independent of } x.\end{aligned}$$

Set $\Omega=\{x\in X:||x||\leqslant M_0+1\}$. Then $T\Omega\subseteq\Omega$. Lemma 2.1 implies that T has fixed point $x\in X$ which is a solution of BVP(7.5). The proof is complete. □

THEOREM 7.2. *Suppose that $M\neq 0$ and there exist constants $M_f\geqslant 0$, $M_I\geqslant 0$, $M_J\geqslant 0$, $M_Q\geqslant 0$ such that*

$$|f(t,x)|\leqslant M_f,\ t\in(0,1],\ x\in\mathbb{R},\ |I_i(x)|\leqslant M_i,\ |J_i(x)|\leqslant M_J,\ |Q_i(x)|\leqslant M_Q,\ i\in\mathbb{N}_1^m$$

Then BVP(7.6) has at least one solution.

Proof. Set

$$\overline{M}=\max\{1+\mathbf{E}_{\alpha,1}(\lambda),\mathbf{E}_{\alpha,2}(\lambda),\mathbf{E}_{\alpha,3}(\lambda),\mathbf{E}_{\alpha,\alpha}(\lambda),\mathbf{E}_{\alpha,2}(\lambda)\ \mathbf{E}_{\alpha,\alpha-1}(\lambda),\mathbf{E}_{\alpha,\alpha}(\lambda)\}.$$

Then we know M_{ij} (the algebraic cofactor of m_{ij} in M) satisfies $|M_{ij}| \leqslant 2\overline{M}^2$. Define the operator T on X by

$$\begin{aligned}(Tx)(t) = &-\sum_{j=1}^{m} A_j(t)I_j(x(t_j)) - \sum_{j=1}^{m} B_j(t)J_j(x(t_j)) - \sum_{j=1}^{m} C_j(t)Q_j(x(t_j))\\ &-\int_0^1 G(t,s)f(s,x(s))ds + \sum_{j=1}^{i} \mathbf{E}_{\alpha,1}(\lambda(t-t_j)^{\alpha})I_j(x(t_j))\\ &+\sum_{j=1}^{i}(t-t_j)\mathbf{E}_{\alpha,2}(\lambda(t-t_j)^{\alpha})J_j(x(t_j)) + \sum_{j=1}^{i}(t-t_j)^2\mathbf{E}_{\alpha,3}(\lambda(t-t_j)^{\alpha})Q_j(x(t_j))\\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds, \quad t\in(t_i,t_{i+1}], \quad i\in\mathbb{N}_0^m.\end{aligned}$$

Then $T : X \to X$ is well defined, T is completely continuous and we will seek fixed points of T in X which is solutions of BVP(7.6). One sees that there exists $M_0 > 0$ such that $||Tx|| \leqslant M_0$. Set $\Omega = \{x \in X : ||x|| \leqslant M_0 + 1\}$. Then $T\Omega \subseteq \Omega$. Lemma 2.1 implies that T has fixed point $x \in X$ which is a solution of BVP(7.6). The proof is complete. □

8. Possible trends of researches

The advantages of fractional calculus have been described and pointed out in the last few decades by many authors. It has been shown that the fractional order models of real systems are regularly more adequate than usually used integer order models. Applications of these fractional order models are in many fields, as for example, rheology, mechanics, chemistry, physics, bioengineering, robotics and many others [21].

Kilbas and Trujillo in [42, 43], Agarwal, Benchohra and Hamani [3] gave a survey of methods and some results in the theory of fractional differential equations respectively. Some problems and research directions were also presented in [42, 43]. In recent years, there has been much interest by many authors in developing theoretical analysis like asymptotic periodicity, asymptotic behavior in general, numerical methods and computational simulation.

In [21], authors presented applications and implementations of fractional order systems. It provides a brief theoretical introduction to fractional order system dedicating almost all the space to the modelling issue, fractional chaotic system control and fractional order controller theory and realization.

In [34], Hilfer reviewed phenomenological and physical arguments for the general importance of fractional derivatives. Many physical applications of fractional calculus such as the Polymer science applications, the decimation transformation of random walk models, the rouse model, the rheological constitutive modelling, the relaxation and diffusion models and the unorthodox applications were reviewed in [34].

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such characteristics arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the reader to [49]. It is natural and interesting to make deep studies on impulsive fractional differential equations (systems).

As we have seen above, only the first steps were done in constructing the theory of impulsive fractional differential equations. The first paper concerned with the solvability of impulsive fractional differential equations with the Caputo type fractional derivatives may be [4]. When $\alpha \to 2$, $\delta \to 0$, we know that $D_{0^+}^{\alpha} u \to u''$, $D_{0^+}^{\delta} u \to u$, $D_{0^+}^{\alpha-1} u \to u'$ (see (2.105) and (2.90) in [65]). So (1) becomes

$$\begin{cases} u''(t) = p(t) f(t, u(t), u(t), u'(t)), \quad t \in (t_i, t_{i+1}), \quad i \in \mathbb{N}_0^m, \\ u'(0) = 0, \quad u(1) = 0, \\ u(t_i^+) = I(t_i, u(t_i), u(t_i), u'(t_i)), \quad i \in \mathbb{N}_1^m, \\ \Delta u'(t_i) = J(t_i, u(t_i), u(t_i), u'(t_i)), \quad i \in \mathbb{N}_1^m. \end{cases} \tag{8.1}$$

The boundary conditions $u'(0) = u(1) = 0$ is called mixed boundary conditions. BVP(8.1) was studied in [56, 94].

In this paper, by considering some boundary value problems for impulsive fractional differential equations with the Caputo fractional derivatives or the Riemann-Liouville fractional derivatives, we have presented methods (the continuation theorem in coincidence degree theory and the Schauder's fixed point theorem) and results based on a reduction to integral equations.

One knows that some other kinds of fractional derivatives have also been proposed such as the Hadamard type fractional derivative, fractional q-difference and so on, see text books [34, 43, 63, 65]. So there have been many questions on impulsive fractional differential equations for investigating in this field.

From research methods, many methods have been applied successfully in studying boundary value problems of ordinary differential equations or impulsive ordinary differential equations or functional differential (difference) equations such as the continuation theorem in coincidence degree theory, the Schauder's fixed point theorem, upper and lower solution method, monotone iterative technique, fixed point theorems in cone in Banach space, critical point theorems and so on. However, these methods seem to be difficult to be applied in impulsive fractional differential equations. There has been few published papers.

We hope that the specialists in the theory of ordinary differential, fractional differential, partial and fractional partial differential equations shall pay their attention to this field.

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