Fractional Differential Calculus
Volume 7, Number 1 (2017), 135–149
doi:10.7153/fdc-07-03

CONTINUOUS DEPENDENCE OF THE SOLUTION OF RANDOM FRACTIONAL–ORDER DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

A. M. A. EL-SAYED, F. GAAFAR AND M. EL-GENDY

(*Communicated by N. N. Leonenko*)

Abstract. In this paper we are concerned with an initial value problem of random fractional-order differential equation with nonlocal condition. Continuous dependence and some other properties concerning the existence and uniqueness of the solution will be proved.

1. Introduction

The existence of solutions of nonlocal problems of differential equation of integer orders and fractional orders have been studied by some authors (see [4], [6], [7] and ([14]–[17]) for example).

Also stochastic problems are discussed in many papers, the reader is referred to ([1]–[3]), [5], ([8]–[13]) and [18].

Consider the random differential equation of fractional order

$$D^{\alpha}X(t) = c(t)f(X(t)) + b(t), \;\; t \in (0,T] \tag{1}$$

with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad a_k > 0, \quad \tau_k \in (0,T), \tag{2}$$

where X_0 is a second order random variable and a_k are positive real numbers.

Here we study the existence of a unique mean square continuous solution of the problem (1)–(2). The continuous dependence on the random variable X_0 and the deterministic coefficient a_k will be proved. The problem (1) with the integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0. \tag{3}$$

will be considered. It must be noted that, in [4] the authors proved the existence of unique solution of the deterministic case of the problem (1)–(2).

Mathematics subject classification (2010): 26A33, 34K50.
Keywords and phrases: Stochastic fractional calculus, Banach contraction fixed point theorem, nonlocal condition, continuous dependence, mean square solution.

2. Preliminaries

Let $I = [0,T]$ and $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\| X \|_C = \sup_{t\in[0,T]} \| X(t) \|_2 = \sup_{t\in[0,T]} \sqrt{E\,(X(t))^2}.$$

Now we have the following definitions.

DEFINITION 2.1. [4] Let $X \in C(I, L_2(\Omega))$ and $\beta \in R^+$ The mean square fractional-order integral I_a^β of $X(t)$ is defined by

$$I_a^\beta X(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds$$

where $\Gamma(.)$ denotes the gamma function.

DEFINITION 2.2. [4] The mean square Caputo fractional-order derivative of order $\alpha \in (0,1]$ of the absolutely mean square continuous process $X(t)$ is defined by

$$D_a^\alpha X(t) = I_a^{1-\alpha} \frac{d}{dt} X(t), \;\; 0 < \alpha \leqslant 1.$$

where D denotes the mean square differentiation and $X(t)$ is assumed to be mean square differentiable.

3. Integral equation representation

Throughout the paper we assume that the following assumptions hold

(H1) the function f satisfies the mean square Lipschitz condition

$$\| f(X_1(t)) - f(X_2(t)) \|_2 \leqslant k \| X_1(t) - X_2(t) \|_2 .$$

(H2) There exists a positive real number m such that

$$\sup_{t\in[0,T]} | f(0) | \leqslant m.$$

(H3) $c(t)$ and $b(t)$ are absolutely continuous functions where

$$c = \sup_t | c(t) |, \quad b = \sup_t | b(t) |$$

Now we have the following lemma.

LEMMA 3.1. *The solution of the nonlocal random problem* (1) *and* (2) *can be expressed by the integral equation*

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k}\right) + I^{\alpha} g(t,X(t)) \tag{4}$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$.

Proof. For simplicity let

$$c(t)f(x(t)) + b(t) = g(t,x(t)).$$

If $X(t)$ satisfies (1)–(2), then by using the properties of the stochastic fractional calculus, equation (1) can be written as

$$I^{1-\alpha}\dot{X}(t) = g(t,X(t)).$$

Operating by I^{α} on both sides in the last equation, we obtain

$$X(t) - X(0) = I^{\alpha} g(t,X(t))$$

substituting for the value of $X(0)$ from equation (2), we obtain

$$X(t) = X_0 - \sum_{k=1}^{n} a_k X(\tau_k) + I^{\alpha} g(t,X(t)).$$

For $t = \tau_k$, we have

$$X(\tau_k) = X_0 - \sum_{k=1}^{n} a_k X(\tau_k) + I^{\alpha}, g(t,X(t))|_{t=\tau_k},$$

then

$$X(\tau_k) = X(t) - I^{\alpha} g(t,X(t)) + I^{\alpha} g(t,X(t))|_{t=\tau_k}.$$

So

$$\begin{aligned} X(t) &= X_0 - \sum_{k=1}^{n} a_k \left(X(t) - I^{\alpha} g(t,X(t)) + I^{\alpha} g(t,X(t))|_{t=\tau_k}\right) + I^{\alpha} g(t,X(t)) \\ &= X_0 - \sum_{k=1}^{n} a_k X(t) + \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t)) - \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k} + I^{\alpha} g(t,X(t)) \end{aligned}$$

and

$$\left(1 + \sum_{k=1}^{n} a_k\right) X(t) = X_0 - \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k} + \left(1 + \sum_{k=1}^{n} a_k\right) I^{\alpha} g(t,X(t)),$$

then

$$X(t) = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{n} a_k I^\alpha g(t, X(t))|_{t=\tau_k}\right) + I^\alpha g(t, X(t))$$

and finally

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k I^\alpha g(t, X(t))|_{t=\tau_k}\right) + I^\alpha g(t, X(t))$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$. □

4. Existence and uniqueness

Now define the operator F by

$$FX(t) = a\left(X_0 - \sum_{k=1}^{n} a_k I^\alpha g(t, X(t))|_{t=\tau_k}\right) + I^\alpha g(t, X(t)). \tag{5}$$

Then we can prove the following lemma.

LEMMA 4.2. $F : C \to C$.

Proof. Let $X \in C$, t_1, $t_2 \in [0, T]$ such that $| t_2 - t_1 |< \delta$, then

$$\begin{aligned}
&FX(t_2) - FX(t_1) \\
&= \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s)) + b(s))ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s)) + b(s))ds \\
&= \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s)) + b(s))ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s)) + b(s))ds \\
&\quad - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s)) + b(s))ds \\
&= \int_0^{t_1} \left[\frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)}\right](c(s)f(X(s)) + b(s))ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s)) + b(s))ds
\end{aligned}$$

and from assumption (H1) we have

$$\| f(X(t)) \|_2 - | f(0) | \leqslant \| f(X(t)) - f(0) \|_2 \leqslant k \| X(t) \|_2,$$

then we have

$$\| f(X(t)) \|_2 \leqslant k \| X(t) \|_2 + | f(0) | \leqslant k \| X \|_C + m.$$

Then

$$\begin{aligned}
&\| FX(t_2) - FX(t_1) \|_2 \\
\leqslant &\int_0^{t_1} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right] | c(s) | \| f(X(s)) \|_2 \, ds \\
&+ \int_0^{t_1} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right] | b(s) | \, ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} | c(s) | \| f(X(s)) \|_2 \, ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} | b(s) | \, ds \\
\leqslant &\, c[k \| X \|_C + m] + b \int_0^{t_1} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right] ds \\
&+ c[k \| X \|_C + m] + b \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
= &\, c[k \| X \|_C + m] + b) \left[-\frac{(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{t_2^{\alpha}}{\Gamma(\alpha+1)} - \frac{t_1^{\alpha}}{\Gamma(\alpha+1)} \right] \\
&+ c[k \| X \|_C + m] + b) \frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)} \\
= &\, c[k \| X \|_C + m] + b) \left[\frac{(t_2^{\alpha} - t_1^{\alpha})}{\Gamma(\alpha+1)} \right] \\
= &\, \frac{c[k \| X \|_C + m] + b)}{\Gamma(\alpha+1)} (t_2^{\alpha} - t_1^{\alpha})
\end{aligned}$$

which proves that $F : C \to C$. □

For the existence of a unique continuous solution $X \in C$ of the problem (1)–(2), we have the following theorem.

THEOREM 4.1. *Let the assumptions* $(H1)$–$(H3)$ *be satisfied. If* $A = \frac{kcT^{\alpha}}{\Gamma(\alpha+1)} < 1$, *then the problem* (1)–(2) *has a unique solution* $X \in C$.

Proof. Let X and $Y \in C$, then

$$FX(t)-FY(t) = -a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} c(s)(f(X(s))-f(Y(s)))ds + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} c(s)(f(X(s))-f(Y(s)))ds$$

and

$$\begin{aligned}
&\| FX(t)-FY(t) \|_2 \\
&\leqslant a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} c(s) \parallel f(X(s))-f(Y(s)) \parallel_2 ds \\
&\quad + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid\parallel f(X(s))-f(Y(s)) \parallel_2 ds \\
&\leqslant ka\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid\parallel X(s)-Y(s) \parallel_2 ds \\
&\quad +k\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid\parallel X(s)-Y(s) \parallel_2 ds \\
&\leqslant ka \parallel X-Y \parallel_C \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid ds + k \parallel X-Y \parallel_C \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid ds \\
&\leqslant ka \parallel X-Y \parallel_C \sum_{k=1}^{n} \mid a_k \mid \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} \sup_s \mid c(s) \mid ds \\
&\quad +k \parallel X-Y \parallel_C \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sup_s \mid c(s) \mid ds \\
&\leqslant cka \parallel X-Y \parallel_C \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} ds + ck \parallel X-Y \parallel_C \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leqslant kc \parallel X-Y \parallel_C \left[a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&\leqslant kc \parallel X-Y \parallel_C \left[a\sum_{k=1}^{n} a_k \frac{\tau_k^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \right] \\
&\leqslant kc \parallel X-Y \parallel_C \left[a\sum_{k=1}^{n} a_k \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right]
\end{aligned}$$

$$\leqslant \frac{kcT^{\alpha}}{\Gamma(\alpha+1)} \parallel X-Y \parallel_C \left[a\sum_{k=1}^{n} a_k + 1\right] \leqslant \frac{2kcT^{\alpha}}{\Gamma(\alpha+1)} \parallel X-Y \parallel_C,$$

then

$$\parallel FX - FY \parallel_C \leqslant 2A \parallel X-Y \parallel_C$$

where

$$A = \frac{kcT^{\alpha}}{\Gamma(\alpha+1)}.$$

Which proves that the operator F given by equation (5) is contraction. Consequently, the integral equation (4) has a unique fixed point $X \in C$.

To obtain the equivalence of equation (4) with the random initial value problem (1)–(2), let X be the solution of problem (1)–(2). As in [4] we can obtain

$$\frac{d}{dt}X(t) = \frac{d}{dt}I^{\alpha}g(t,X(t)) = \frac{d}{dt}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,X(s))ds$$
$$= g(t,X(t))|_{t=0}\frac{t^{\alpha-1}}{\Gamma(\alpha)} + I^{\alpha}\frac{d}{dt}g(t,X(t)).$$

Operate by $I^{1-\alpha}$ in both sides to obtain

$$D^{\alpha}X(t) = g(t,X(t)) = c(t)f(X(t)) + b(t).$$

For $t=0$, we find the nonlocal condition (2) is satisfied as following, since we have

$$X(0) = a\left(X_0 - \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k}\right)$$

and

$$X(\tau_k) = a\left(X_0 - \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k}\right) + I^{\alpha}g(t,X(t))|_{t=\tau_k}$$

then

$$\sum_{k=1}^{n} a_k X(\tau_k) = \sum_{k=1}^{n} a_k \left[a\left(X_0 - \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k}\right) + I^{\alpha}g(t,X(t))|_{t=\tau_k}\right]$$
$$= a\sum_{k=1}^{n} a_k X_0 - a\sum_{k=1}^{n} a_k \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k} + \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k}$$

and

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k)$$
$$= a\left(X_0 - \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k}\right) + a\sum_{k=1}^{n} a_k X_0$$
$$-a\sum_{k=1}^{n} a_k \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k} + \sum_{k=1}^{n} a_k I^{\alpha} g(t,X(t))|_{t=\tau_k}$$

$$
\begin{aligned}
&= X_0\left(a+a\sum_{k=1}^{n} a_k\right)+\left(-a\sum_{k=1}^{n} a_k - a\sum_{k=1}^{n} a_k\sum_{k=1}^{n} a_k+\sum_{k=1}^{n} a_k\right) I^{\alpha} g(t,X(t))|_{t=\tau_k}\\
&= aX_0\left(1+\sum_{k=1}^{n} a_k\right)+\left(-a\sum_{k=1}^{n} a_k\left(1+\sum_{k=1}^{n} a_k\right)+\sum_{k=1}^{n} a_k\right) I^{\alpha} g(t,X(t))|_{t=\tau_k}\\
&= aa^{-1}X_0+\left(-aa^{-1}\sum_{k=1}^{n} a_k+\sum_{k=1}^{n} a_k\right) I^{\alpha} g(t,X(t))|_{t=\tau_k}\\
&= X_0
\end{aligned}
$$

which proves the equivalence. □

5. Continuous dependence

Consider the nonlocal random condition

$$
X(0)+\sum_{k=1}^{n} a_k X(\tau_k)=\tilde{X}_0, \qquad \tau_k\in(0,T) \tag{6}
$$

DEFINITION 5.3. The solution $X\in C$ of the nonlocal random problem (1)–(2) is continuously dependent (on the data X_0) if $\forall\varepsilon>0$, $\exists\delta>0$ such that $\| X_0-\tilde{X}_0 \|_2\leqslant\delta$ implies that $\| X-\tilde{X} \|_C\leqslant\varepsilon$.

Here, we study the continuous dependence (on the random data X_0) of the solution of the random fractional-order differential equation (1) and (2).

THEOREM 5.2. *Let the assumptions $(H1)$–$(H3)$ be satisfied. Then the solution of the nonlocal random problem (1)–(2) is continuously dependent on the random data X_0.*

Proof. Let

$$
\begin{aligned}
X(t) = a&\left(X_0-\sum_{k=1}^{n} a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s))+b(s))ds\right)\\
&+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s))+b(s))ds
\end{aligned}
$$

be the solution of the nonlocal random problem (1)–(2) and

$$
\begin{aligned}
\tilde{X}(t) = a&\left(\tilde{X}_0-\sum_{k=1}^{n} a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(\tilde{X}(s))+b(s))ds\right)\\
&+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(\tilde{X}(s))+b(s))ds
\end{aligned}
$$

be the solution of the nonlocal random problem (1) and (6). Then

$$
\begin{aligned}
&X(t)-\tilde{X}(t)\\
&= a(X_0-\tilde{X}_0) - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)(f(X(s))-f(\tilde{X}(s)))ds\\
&\quad + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} c(s)(f(X(s))-f(\tilde{X}(s)))ds
\end{aligned}
$$

Using our assumptions, we get

$$
\begin{aligned}
&\| X(t)-\tilde{X}(t) \|_2\\
&\leqslant a \| X_0-\tilde{X}_0 \|_2 + a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid \| f(X(s))-f(\tilde{X}(s)) \|_2 \, ds\\
&\quad + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid \| f(X(s))-f(\tilde{X}(s)) \|_2 \, ds\\
&\leqslant a \| X_0-\tilde{X}_0 \|_2 + ka\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid \| X(s)-\tilde{X}(s) \|_2 \, ds\\
&\quad + k\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid \| X(s)-\tilde{X}(s) \|_2 \, ds\\
&\leqslant a \| X_0-\tilde{X}_0 \|_2 + ka\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \sup_s \mid c(s) \mid \sup_s \| X(s)-\tilde{X}(s) \|_2 \, ds\\
&\quad + k\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sup_s \mid c(s) \mid \sup_s \| X(s)-\tilde{X}(s) \|_2 \, ds\\
&\leqslant a \| X_0-\tilde{X}_0 \|_2 + kc \| X-\tilde{X} \|_C \left[a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right]\\
&\leqslant a \| X_0-\tilde{X}_0 \|_2 + kc \| X-\tilde{X} \|_C \left[a\sum_{k=1}^{n} a_k \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right]\\
&\leqslant a \| X_0-\tilde{X}_0 \|_2 + kc \| X-\tilde{X} \|_C \frac{2T^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}
$$

then

$$
\| X-\tilde{X} \|_C \leqslant \frac{A\delta}{(1-2A)}.
$$

This complete the proof. □

Now consider the random fractional-order differential equation (1) with the nonlocal condition

$$X(0)+\sum_{k=1}^{n}\tilde{a}_k X(\tau_k)=X_0, \qquad \tau_k\in(0,T) \tag{7}$$

DEFINITION 5.4. The solution $X\in C$ of the nonlocal random problem (1)–(2) is continuously dependent (on the coefficient a_k of the nonlocal condition) if $\forall\varepsilon>0$, $\exists\delta>0$ such that $\mid a_k-\tilde{a}_k\mid\leqslant\delta$ implies that $\parallel X-\tilde{X}\parallel_C\leqslant\varepsilon$.

Here, we study the continuous dependence (on the coefficient a_k of the nonlocal condition) of the solution of the random fractional-order differential equation (1) and (2).

THEOREM 5.3. *Let the assumptions $(H1)$–$(H3)$ be satisfied. Then the solution of the nonlocal random problem (1)–(2) is continuously dependent on the coefficient a_k of the nonlocal condition.*

Proof. Let

$$X(t)=a\left(X_0-\sum_{k=1}^{n}a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s))+b(s))ds\right)$$
$$+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(X(s))+b(s))ds$$

be the solution of the nonlocal random problem (1)–(2) and

$$\tilde{X}(t)=\tilde{a}\left(X_0-\sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(\tilde{X}(s))+b(s))ds\right)$$
$$+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(c(s)f(\tilde{X}(s))+b(s))ds$$

be the solution of the nonlocal random problem (1) and (7).

Then

$$X(t)-\tilde{X}(t)$$
$$=(a-\tilde{a})X_0-a\sum_{k=1}^{n}a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)}c(s)f(X(s))ds+\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)}c(s)f(\tilde{X}(s))ds$$
$$-\left(a\sum_{k=1}^{n}a_k-\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\right)\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha-1}}{\Gamma(\alpha)}b(s)ds+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}c(s)[f(X(s))-f(\tilde{X}(s))]ds$$

Now we have

$$\begin{aligned} |a-\tilde{a}| &= \left| \left(1+\sum_{k=1}^{n} a_k\right)^{(-1)} - \left(1+\sum_{k=1}^{n} \tilde{a}_k\right)^{(-1)} \right| \\ &= \left| \sum_{k=1}^{n} (\tilde{a}_k - a_k) \left(1+\sum_{k=1}^{n} a_k\right)^{(-1)} \left(1+\sum_{k=1}^{n} \tilde{a}_k\right)^{(-1)} \right| \\ &\leqslant \left| \sum_{k=1}^{n} (\tilde{a}_k - a_k) \right| \leqslant n\delta \end{aligned}$$

and

$$\begin{aligned} &\tilde{a}\sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{X}(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s))ds \\ &= \tilde{a}\left(1+\sum_{k=1}^{n} \tilde{a}_k\right) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{X}(s))ds \\ &\quad - a\left(1+\sum_{k=1}^{n} a_k\right) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s))ds \\ &\quad - \tilde{a}\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{X}(s))ds + a\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s))ds \\ &= \tilde{a}(\tilde{a}^{-1}) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{X}(s))ds - a(a^{-1}) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s))ds \\ &\quad - \tilde{a}\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{X}(s))ds + a\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s))ds \end{aligned}$$

then

$$\begin{aligned} &\tilde{a}\sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{X}(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s))ds \\ &= -\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)[f(X(s)) - f(\tilde{X}(s))]ds + a\int_0^{\tau_k} f(s,X(s))ds - \tilde{a}\int_0^{\tau_k} f(\tilde{X}(s))ds \\ &\quad - \tilde{a}\int_0^{\tau_k} f(X(s))ds + \tilde{a}\int_0^{\tau_k} f(X(s))ds \end{aligned}$$

$$= -\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)[f(X(s)) - f(\tilde{X}(s))]ds + [a - \tilde{a}] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s)) ds$$
$$+\tilde{a} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)[f(X(s)) - f(\tilde{X}(s))]ds.$$

From assumption (H2) we have

$$\| f(X(t)) \|_2 \leqslant k \| X(t) \|_2 + | f(0) | \leqslant k \| X \|_C + m.$$

also we have

$$\left[a \sum_{k=1}^{n} a_k - \tilde{a} \sum_{k=1}^{n} \tilde{a}_k\right] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds$$
$$= \left[a \left(1 + \sum_{k=1}^{n} a_k\right) - \tilde{a} \left(1 + \sum_{k=1}^{n} \tilde{a}_k\right)\right] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds$$
$$-[a - \tilde{a}] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds$$
$$= \left[aa^{-1} - \tilde{a}\tilde{a}^{-1}\right] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds - [a - \tilde{a}] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds$$
$$= -[a - \tilde{a}] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds,$$

and

$$\| X(t) - \tilde{X}(t) \|_2$$
$$\leqslant | a - \tilde{a} | \| X_0 \|_2$$
$$+ \left\| -a \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s)) ds + \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(\tilde{X}(s)) ds \right\|_2$$
$$+ \left| a \sum_{k=1}^{n} a_k - \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \right| \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds$$
$$+ \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \| \| f(X(s)) - f(\tilde{X}(s)) \|_2 \, ds$$

$$
\begin{aligned}
&\leqslant |a-\tilde{a}| \| X_0 \|_2 + \left\| \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)[f(X(s)) - f(\tilde{X}(s))]ds \right\|_2 \\
&\quad + \left\| [a-\tilde{a}] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s) f(X(s))ds \right\|_2 \\
&\quad + \left\| \tilde{a} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} c(s)[f(X(s)) - f(\tilde{X}(s))]ds \right\|_2 \\
&\quad + \left| a \sum_{k=1}^{n} a_k - \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \right| \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} \mid b(s) \mid ds \\
&\quad + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mid c(s) \mid \| f(X(s)) - f(\tilde{X}(s)) \|_2 \, ds \\
&\leqslant n\delta \| X_0 \|_2 + kc \| X - \tilde{X} \|_C \int_0^{\tau_k} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + cn\delta [k \| X \|_C + m] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + c\tilde{a}k \| X - \tilde{X} \|_C \int_0^{\tau_k} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + nb\delta \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + kc \| X - \tilde{X} \|_C \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leqslant n\delta \left[\| X_0 \|_2 + c[k \| X \|_C + m] \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} + b \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&\quad + kc \| X - \tilde{X} \|_C \left[\int_0^{\tau_k} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \tilde{a} \int_0^{\tau_k} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&\leqslant n\delta \left[\| X_0 \|_2 + c[k \| X \|_C + m] \frac{T^\alpha}{\Gamma(\alpha+1)} + b \frac{T^\alpha}{\Gamma(\alpha+1)} \right] + \frac{3kcT^\alpha}{\Gamma(\alpha+1)} \| X - \tilde{X} \|_C
\end{aligned}
$$

then

$$
\| X - \tilde{X} \|_C \leqslant n\delta \left[\| X_0 \|_2 + c[k \| X \|_C + m] \frac{T^\alpha}{\Gamma(\alpha+1)} + b \frac{T^\alpha}{\Gamma(\alpha+1)} \right] + 3A \| X - \tilde{X} \|_C
$$

so

$$
\| X - \tilde{X} \|_C \leqslant \frac{n\delta \left[\| X_0 \|_2 + c[k \| X \|_C + m] \frac{T^\alpha}{\Gamma(\alpha+1)} + b \frac{T^\alpha}{\Gamma(\alpha+1)} \right]}{(1-3A)}
$$

This complete the proof. □

6. Nonlocal integral condition

Let $v(t)$ is a nondecreasing function such that

$$a_k = v(t_k) - v(t_{k-1}), \quad \tau_k \in (t_{k-1}, t_k)$$

where $0 < t_1 < t_2 < t_3 < \ldots < T$. Then, the nonlocal condition (2) will be in the form

$$X(0) + \sum_{k=1}^{n} X(\tau_k)(v(t_k) - v(t_{k-1})) = X_0.$$

From the mean square continuity of the solution of the nonlocal problem (1)–(2), we obtain from [18]

$$\underset{n\to\infty}{l.i.m} \sum_{k=1}^{n} X(\tau_k)(v(t_k) - v(t_{k-1})) = \int_0^T X(s)dv(s),$$

that is, the nonlocal conditions (2) is transformed to the mean square Riemann-Steltjes integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0.$$

Now, we have the following theorem.

THEOREM 6.4. *Let the assumptions* $(H1)-(H3)$ *be satisfied, then the stochastic differential equation* (1) *with the nonlocal integral condition* (3) *has a unique mean square continuous solution represented in the form*

$$X(t) = a\left(X_0 - \int_0^T \int_0^s \frac{(\tau_k - \theta)^{\alpha-1}}{\Gamma(\alpha)}(c(\theta)f(X(\theta)) + b(\theta))d\theta dv(s)\right) + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}(c(\theta)f(X(\theta)) + b(\theta))d\theta$$

where $a^\star = (1 + v(T) - v(0))^{-1}$.

Proof. Taking the limit of equation (3) we get the proof. □

REFERENCES

[1] L. ARNOLD, *Stochastic differential equations: theory and applications*, A Wiley-Interscience Publication Copyright by J. Wiley and Sons, New York, 1974.

[2] A. T. BHARUCHA-TEID, *Fixed point theorems in probabilistic analysis*, Bulletin of the American Mathematical Society **82** (5), 1976.

[3] A. M. A. EL-SAYED, *On the stochastic fractional calculus operators*, Journal of Fractional Calculus and Applications **6** (1) 2015, pp. 101–109.

[4] A. M. A. EL-SAYED, R. O. ABD EL-SALAM SH. A., *On the stability of a fractional-order differential equation with nonlocal initial condition*, Electronic Journal of Qualitative Theory of Differential Equations **29** (2008) pp. 1–8.

[5] A. M. A. EL-SAYED, R. O. ABDELRAHMAN AND M. EL-GENDY, *On the maximal and minimal solutions of the nonlocal problem of Itô Stochastic Differential Equation*, Nonlinear Analysis and Differential Equations **4** (6) 2016, pp. 269–281.

[6] A. M. A. EL-SAYED AND E. O. BIN-TAHER, *A nonlocal Problem for a multi-term fractional order differential equation*, Journal of Math. Analysis **5** (29) 2011, pp. 1445–1451.

[7] A. M. A. EL-SAYED AND E. O. BIN-TAHER, *An arbitraty fractional order differential equation with internal nonlocal and integral conditions*, Advances in pure mathematics **1** (3) 2011, pp. 59–62.

[8] A. M. A. EL-SAYED, E. E. EL-ADDAD AND H. F. A. MADKOUR, *On some equivalent problems of stochastic differential equations of fractional order*, Journal of Fractional Calculus and Applications **6** (2) 2015, pp. 115–122.

[9] A. M. A. EL-SAYED, M. A. EL-TAWIL, M. S. M. SAIF AND F. M. HAFIZ, *The mean square Riemann-Liouville stochastic fractional derivative and stochastic fractional order differential equation*, Math. Sci. Res. J. **9** (6) 2005, pp. 142–150.

[10] A. M. A. EL-SAYED, F. GAAFAR AND M. EL-GENDY, *Continuous dependence of the solution of Ito stochastic differential equation with nonlocal conditions*, Applied Mathematical Sciences **10** (40) 2016, pp. 1971–1982.

[11] D. ISAACSON, *Stochastic integrals and derivatives*, The Annals of Mathematical Statistics **40** (5) 1969, pp. 1610–1616.

[12] K. ITO, *On stochastic differential equations*, Mem. A. M. S. **4**, 1951, pp. 1–51.

[13] S. ITOH, *Random fixed point theorems with an application to random differential equations in Banach spaces*, Journal Of Mathematical Analysis And Applications **67**, 1979, pp. 261–273.

[14] K. MILLER, AND B. ROSS, *An introduction to the fractional calculus and fractional differential equations*, John Wiley, New York, 1993.

[15] I. PODLUBNY, *Fractional differential equations*, Acad. press, San Diego-New York-London, 1999.

[16] I. PODLUBNY, AND A. M. A. EL-SAYED, *On two definitions of fractional calculus*, Slovak Academy of science-Institute of Experimental phys, preprint UEF 03-96 (ISBN 80-7099-252-2), 1999.

[17] S. SAMKO, A. KILBAS, AND O. L. MARICHEV, *Fractional integrals and derivatives*, Gordon and Breach Science Publisher, 1993.

[18] T. T. SOONG, *Random differential equations in science and engineering*, Mathematics in Science and Engineering **103**, 1973.

(Received March 14, 2016)

A. M. A. El-Sayed
Faculty of Science
Alexandria University
Egypt
e-mail: amasyed@alexu.edu.eg

F. Gaafar
Faculty of Science
Damanhour University
Egypt
e-mail: fatmagaafar2@yahoo.com

M. El-Gendy
Faculty of Science
Damanhour University
Egypt
e-mail: maysa_elgendy@yahoo.com