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SOLVABILITY FOR A SYSTEM OF NONLINEAR FRACTIONAL HIGHER–ORDER THREE–POINT BOUNDARY VALUE PROBLEM

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Abstract. Existence of eigenvalues yielding single and multiple positive solutions for a system of higher order fractional differential equations along with boundary conditions is established. The results are obtained by the use of a Guo-Krasnosel'skii fixed point theorem in cones.

1. Introduction

In recent years, boundary value problems of nonlinear fractional differential equations have been studied by many researchers. Fractional differential equations appear naturally in various fields of science and engineering, and constitute an important field of research. As a matter of fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [17, 29, 30]. For more details of some recent theoretical results on fractional differential equations and their applications, we refer the reader to [4, 5, 7, 8, 21, 31]. Some recent work on boundary value problems of fractional order can be found in [12, 14, 15, 22, 24, 25, 27, 32] and the references therein.

In this paper, we consider the system of nonlinear fractional differential equations

$$\begin{aligned} D^{\alpha}_{a^+}u(t)+\lambda f(t,u(t),v(t))=0,\ t\in(a,b),\ n-1<\alpha\leqslant n,\\ D^{\beta}_{a^+}v(t)+\mu g(t,u(t),v(t))=0,\ t\in(a,b),\ m-1<\beta\leqslant m, \end{aligned} \tag{1}$$

with the boundary conditions

$$\begin{aligned} u^{(i)}(a)=0,\ 0\leqslant i\leqslant n-2,\ u^{(\alpha_1)}(b)=\xi u^{(\alpha_1)}(\eta),\ 1\leqslant\alpha_1\leqslant\alpha-2,\ \text{(but fixed)}\\ v^{(j)}(a)=0,\ 0\leqslant j\leqslant m-2,\ v^{(\beta_1)}(b)=\xi v^{(\beta_1)}(\eta),\ 1\leqslant\beta_1\leqslant\beta-2,\ \text{(but fixed)} \end{aligned} \tag{2}$$

where $D^{\alpha}_{a^+}$ and $D^{\beta}_{a^+}$ are denote the Riemann-Liouville fractional derivatives of order α and β respectively, $n-1<\alpha\leqslant n,\ m-1<\beta\leqslant m,\ n,m\in\mathrm{N},n,m\geqslant 3,\ 1\leqslant\alpha_1\leqslant\alpha-2,\ 1\leqslant\beta_1\leqslant\beta-2$ are fixed integers and $\xi\in(0,\infty),\ \eta\in(a,b)$ are constants with $0<\xi(\eta-a)^{\alpha-\alpha_1-1}<(b-a)^{\alpha-\alpha_1-1}$.

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The aim of this paper is to establish some simple criteria for the existence of single and multiple solutions of the system (1)–(2) in explicit intervals for λ and μ. By a positive solution of problem (1)–(2) we mean a pair of functions $(u,v) \in C([a,b]) \times C([a,b])$ satisfying (1) and (2) with $u(t) \geqslant 0$, $v(t) \geqslant 0$ for all $t \in [a,b]$ and $(u,v) \neq (0,0)$.

The following assumptions are made to establish our results.

$(H1)$ The functions f and g are continuous and nonnegative. $f(t,u,v) \leqslant p_1(t)q_1(t,u,v)$, $g(t,u,v) \leqslant p_2(t)q_2(t,u,v)$, $(t,u,v) \in [a,b]\times[0,\infty)\times[0,\infty)$, where $q_i \in C\big[[a,b]\times[0,\infty)\times[0,\infty),[0,\infty)\big]$, and $p_i \in C\big[[a,b],[0,\infty)\big]$ satisfy $\int_a^b p_i(s)ds < \infty$, $i = 1,2$.

$(H2)$ The limits

$$f_0 = \lim_{u+v\to 0} \min_{t\in[a,b]} \frac{f(t,u,v)}{u+v}, \quad f_\infty = \lim_{u+v\to\infty} \min_{t\in[a,b]} \frac{f(t,u,v)}{u+v},$$
$$g_0 = \lim_{u+v\to 0} \min_{t\in[a,b]} \frac{g(t,u,v)}{u+v}, \quad g_\infty = \lim_{u+v\to\infty} \min_{t\in[a,b]} \frac{g(t,u,v)}{u+v},$$
$$q_{i0} = \overline{\lim_{u+v\to 0}} \max_{t\in[a,b]} \frac{q_i(t,u,v)}{u+v}, \quad q_{i\infty} = \overline{\lim_{u+v\to\infty}} \max_{t\in[a,b]} \frac{q_i(t,u,v)}{u+v},$$

exist with $f_0, f_\infty, g_0, g_\infty, q_{i0}, q_{i\infty} \in [0,\infty), i = 1,2$.

The rest of the paper is organized as follows.In Section 2, we present the definitions, some lemmas from the theory of fractional calculus and also state Krasnosel'skii fixed point theorem. In Section 3, we construct the Green's function for the fractional order boundary value problem and estimate the bounds for the Green's function. Later, we express the solution of the boundary value problem (1)–(2) into an equivalent integral equation. In Section 4, we discuss the existence of a single positive solution of the system (1)–(2). The intervals in which the parameters λ, μ can guarantee the existence of a positive solution are obtained. In Section 5, we study the existence conditions of at least two positive solutions of the system (1)–(2). Finally, we give an example as an application.

2. Preliminaries

In this section, we present here the definitions, some lemmas from the theory of fractional calculus and also state Krasnosel'skii fixed point theorem.

DEFINITION 1. The (left-sided) fractional integral of order $\alpha > 0$ of a function $f:(0,\infty) \to R$ is given by

$$(I_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \ \ t > 0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$, $\alpha > 0$.

DEFINITION 2. The Riemann-Liouville fractional derivative of order $\alpha \geqslant 0$ for a function $f:(0,\infty) \to R$ is given by

$$(D_{0+}^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^n (I_{0+}^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \ \ t>0,$$

where $n = [\alpha]+1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

The notation $[\alpha]$ stands for the largest integer not greater than α. We also denote the Riemann-Liouville fractional derivative of f by $D_{0+}^{\alpha} f(t)$. If $\alpha = m \in N$ then $D_{0+}^{m} f(t) = f^{(m)}(t)$ for $t>0$, and if $\alpha = 0$ then $D_{0+}^{0} f(t) = f(t)$ for $t>0$.

LEMMA 1. ([17]) *a) If $\alpha>0$, $\beta>0$ and $f \in L^p(0,1)$, $(1 \leqslant p \leqslant \infty)$, then the relation $(I_{0+}^{\alpha} I_{0+}^{\beta} f)(t) = (I_{0+}^{\alpha+\beta} f)(t)$ is satisfied at almost every point $t \in (0,1)$. If $\alpha + \beta > 1$, then the above relation holds at any point of $[0,1]$.*

b) If $\alpha>0$ and $f \in L^p(0,1)$, $(1 \leqslant p \leqslant \infty)$, then the relation $(D_{0+}^{\alpha} I_{0+}^{\alpha} f)(t) = f(t)$ holds almost everywhere on $(0,1)$.

c) If $\alpha > \beta > 0$ and $f \in L^p(0,1)$, $(1 \leqslant p \leqslant \infty)$, then the relation $(D_{0+}^{\beta} I_{0+}^{\alpha} f)(t) = (I_{0+}^{\alpha-\beta} f)(t)$ holds almost everywhere on $(0,1)$.

LEMMA 2. ([17]) *Let $\alpha>0$ and $n = [\alpha]+1$ for $\alpha \notin N$ and $n=\alpha$ for $\alpha \in N$; that is, n is the smallest integer greater than or equal to α. Then, the solutions of the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$, $0<t<1$, are*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}, \ \ 0<t<1,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.

LEMMA 3. ([17]) *Let $\alpha>0$, n be the smallest integer greater than or equal to α $(n-1<\alpha \leqslant n)$ and $y \in L^1(0,1)$. The solutions of the fractional equation $D_{0+}^{\alpha} u(t) + y(t) = 0$, $0<t<1$, are*

$$u(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + \ldots + c_n t^{\alpha-n}, \ \ 0<t<1,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.

Proof. By Lemma 1 b), the equation $D_{0+}^{\alpha} u(t) + y(t) = 0$ can be written as

$$D_{0+}^{\alpha} u(t) + D_{0+}^{\alpha} (I_{0+}^{\alpha} y)(t) = 0 \quad \text{or} \quad D_{0+}^{\alpha} (u + I_{0+}^{\alpha} y)(t) = 0.$$

By using Lemma 2, the solutions for the above equation are

$$\begin{aligned} & u(t) + I_{0+}^{\alpha} y(t) = c_1 t^{\alpha-1} + \ldots + c_n t^{\alpha-n} \Leftrightarrow \\ & u(t) = -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + \ldots + c_n t^{\alpha-n} \\ & \quad = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + \ldots + c_n t^{\alpha-n}, \ \ 0<t<1, \end{aligned}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants. $\square$

THEOREM 1. [Krasnosel'skii] *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous operator such that either*

(i) $\| Tu \| \leqslant \| u \|$, $u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \geqslant \| u \|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\| Tu \| \geqslant \| u \|$, $u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \leqslant \| u \|$, $u \in K \cap \partial\Omega_2$

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Green's function and bounds

In this section, we construct the Green function for the homogeneous BVP corresponding to (1)–(2) and estimate the bounds for the Green function which are needed to establish the main results.

$$-D^{\alpha}_{a^+}u(t) = 0, \;\; a < t < b, \tag{3}$$

$$u^{(i)}(a) = 0, \;\; 0 \leqslant i \leqslant n-2, \;\; u^{(\alpha_1)}(b) = \xi u^{(\alpha_1)}(\eta) \tag{4}$$

LEMMA 4. *If $y \in C[a,b]$, then the fractional order BVP*

$$D^{\alpha}_{a^+}u(t) + y(t) = 0, \;\; a < t < b, \tag{5}$$

with (4)*, has a unique solution, $u(t) = \int_a^b G_\lambda(t,s)y(s)ds$ where $G_\lambda(t,s)$ is the Green function for the BVP* (5)–(4) *and is given by*

$$G_\lambda(t,s) = G_1(t,s) + \frac{\xi(t-a)^{\alpha-1}}{[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}]}G_2(\eta,s) \tag{6}$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (t-s)^{\alpha-1}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}, & a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{7}$$

$$G_2(\eta,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \frac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (\eta-s)^{\alpha-\alpha_1-1}, & a \leqslant s \leqslant \eta, \\ \frac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}, & \eta \leqslant s \leqslant b. \end{cases} \tag{8}$$

Proof. Assume that $u \in C^{[\alpha]+1}[a,b]$ is a solution of fractional order BVP (5)–(4) and is uniquely expressed as $I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(t) = -I^{\alpha}_{a^+}y(t)$, so that

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n}.$$

From $u^{(i)}(a) = 0$, $0 \leqslant i \leqslant n-2$, we have $c_n = c_{n-1} = c_{n-2} = \cdots = c_2 = 0$. Then

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds + c_1(t-a)^{\alpha-1},$$

$$u^{(\alpha_1)}(t) = c_1 \prod_{i=1}^{\alpha_1}(\alpha - i)(t-a)^{\alpha-\alpha_1-1} - \prod_{i=1}^{\alpha_1}(\alpha - i)\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\alpha_1-1}y(s)ds.$$

From $u^{(\alpha_1)}(b) = \xi u^{(\alpha_1)}(\eta)$, we have

$$\begin{aligned}
&c_1 \prod_{i=1}^{\alpha_1}(\alpha - i)(b-a)^{\alpha-\alpha_1-1} - \prod_{i=1}^{\alpha_1}(\alpha - i)\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-\alpha_1-1}y(s)ds \\
&= \xi\Big[c_1 \prod_{i=1}^{\alpha_1}(\alpha - i)(\eta-a)^{\alpha-\alpha_1-1} - \prod_{i=1}^{\alpha_1}(\alpha - i)\frac{1}{\Gamma(\alpha)}\int_a^\eta (\eta-s)^{\alpha-\alpha_1-1}y(s)ds\Big].
\end{aligned}$$

Therefore

$$\begin{aligned}
c_1 &= \frac{1}{\Gamma(\alpha)\Big[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\Big]} \\
&\quad \times \Big[\int_a^b (b-s)^{\alpha-\alpha_1-1}y(s)ds - \xi\int_a^\xi (\eta-s)^{\alpha-\alpha_1-1}y(s)ds\Big] \\
&= \frac{1}{\Gamma(\alpha)\Big[(b-a)^{\alpha-\alpha_1-1}\Big]}\int_a^b (b-s)^{\alpha-\alpha_1-1}y(s)ds \\
&\quad + \frac{\xi(\eta-a)^{\alpha-\alpha_1-1}}{\Gamma(\alpha)(b-a)^{\alpha-\alpha_1-1}\Big[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\Big]}\int_a^b (b-s)^{\alpha-\alpha_1-1}y(s)ds \\
&\quad - \frac{\xi}{\Gamma(\alpha)\Big[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\Big]}\int_a^\eta (\eta-s)^{\alpha-\alpha_1-1}y(s)ds \\
&= \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-\alpha_1-1}}\int_a^b (b-s)^{\alpha-\alpha_1-1}y(s)ds \\
&\quad + \frac{\xi}{\Big[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\Big]}\int_a^b G_2(\eta,s)y(s)ds.
\end{aligned}$$

Thus, the unique solution of (5)–(4) is

$$\begin{aligned}
u(t) &= \frac{-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}y(s)ds + \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-\alpha_1-1}}\int_a^b (b-s)^{\alpha-\alpha_1-1}y(s)ds \\
&\quad + \frac{\xi(t-a)^{\alpha-1}}{\Big[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\Big]}\int_a^b G_2(\eta,s)y(s)ds \\
&= \int_a^b G_1(t,s)y(s)ds + \frac{\xi(t-a)^{\alpha-1}}{\Big[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\Big]}\int_a^b G_2(\eta,s)y(s)ds \\
&= \int_a^b G_\lambda(t,s)y(s)ds
\end{aligned}$$

where $G_\lambda(t,s)$ is given in (6) □

LEMMA 5. *The Green's function $G_\lambda(t,s)$ satisfies the following inequalities*

(i) $G_\lambda(t,s) \leqslant G_\lambda(b,s)$, *for all* $(t,s) \in [a,b]\times[a,b]$,

(ii) $G_\lambda(t,s) \geqslant \gamma_1 G_\lambda(b,s)$, *for all* $(t,s) \in [\eta,b]\times[a,b]$,

where $\gamma_1 = \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}$.

Proof. The Green's function $G_1(t,s)$ is given (7).

For $a \leqslant t \leqslant s \leqslant b$,

$$\frac{\partial G_1(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha-1)}\Big[\frac{(t-a)^{\alpha-2}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\Big] \geqslant 0. \tag{9}$$

For $a \leqslant s \leqslant t \leqslant b$

$$\begin{aligned}\frac{\partial G_1(t,s)}{\partial t} &= \frac{1}{\Gamma(\alpha-1)}\Big[\frac{(t-a)^{\alpha-2}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (t-s)^{\alpha-2}\Big]\\ &\geqslant \frac{1}{\Gamma(\alpha-1)}\Big[\frac{(t-a)^{\alpha-2}(b-s)^{\alpha-\alpha_1-1} - (t-ts)^{\alpha-2}(b-a)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\Big] \geqslant 0.\end{aligned}$$

Now we prove

$$G_2(\eta,s) \geqslant 0 \;\; s \in [a,b]. \tag{10}$$

In fact, if $s \geqslant \eta$, obviously, (10) holds. If $s \leqslant \eta$ one has

$$\begin{aligned}G_2(\eta,s) &= \frac{1}{\Gamma(\alpha)}\Big[\frac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (\eta-s)^{\alpha-\alpha_1-1}\Big]\\ &\geqslant \frac{1}{\Gamma(\alpha)}\Big[\frac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1} - (\eta-\eta s)^{\alpha-\alpha_1-1}(b-a)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\Big] \geqslant 0.\end{aligned}$$

That implies that (10) is also true. Therefore, by (6), (9) and (10) we find

$$\frac{\partial G_\lambda(t,s)}{\partial t} = \frac{\partial G_1(t,s)}{\partial t} + \frac{(\alpha-1)\xi(t-a)^{\alpha-2}}{(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}} G_2(\eta,s) \geqslant 0.$$

Therefore $G_\lambda(t,s)$ is increasing with respect to $t \in [a,b]$. Hence the inequality (i) is proved. Now, we establish the inequality (ii).

On the other hand, if $s \leqslant t$ then

$$\begin{aligned}G_1(t,s) &= \frac{1}{\Gamma(\alpha)}\Big[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (t-s)^{\alpha-1}\Big]\\ &= \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}}\Big[\frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (b-s)^{\alpha-1}\Big]\\ &\quad + \frac{1}{\Gamma(\alpha)}\Big[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-\alpha_1-1}} - (t-s)^{\alpha-1}\Big]\\ &\geqslant \frac{1}{\Gamma(\alpha)}\Big(\frac{t-a}{b-a}\Big)^{\alpha-1}\Big[\frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (b-s)^{\alpha-1}\Big]\\ &\geqslant \Big(\frac{t-a}{b-a}\Big)^{\alpha-1} G_1(b,s).\end{aligned}$$

If $t \leqslant s$, we have

$$
\begin{aligned}
G_1(t,s) &= \frac{1}{\Gamma(\alpha)}\Big[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\Big] \\
&= \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}}\Big[\frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (b-s)^{\alpha-1}\Big] \\
&\quad + \frac{1}{\Gamma(\alpha)}\Big[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\Big] \\
&\geqslant \frac{1}{\Gamma(\alpha)}\Big(\frac{t-a}{b-a}\Big)^{\alpha-1}\Big[\frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-p-1}} - (b-s)^{\alpha-1}\Big] \\
&\geqslant \Big(\frac{t-a}{b-a}\Big)^{\alpha-1} G_1(b,s).
\end{aligned}
$$

Therefore

$$
G_1(t,s) \geqslant \Big(\frac{t-a}{b-a}\Big)^{\alpha-1} G_1(b,s). \tag{11}
$$

From (6) and (11) we have

$$
\begin{aligned}
G_\lambda(t,s) &= G_1(t,s) + \frac{\xi(t-a)^{\alpha-1}}{[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}]} G_2(\eta,s) \\
&\geqslant \Big(\frac{t-a}{b-a}\Big)^{\alpha-1} G_1(b,s) + \frac{\xi(t-a)^{\alpha-1}}{[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}]} G_2(\eta,s) \\
&\geqslant \Big(\frac{\eta-a}{b-a}\Big)^{\alpha-1} G_\lambda(b,s).
\end{aligned}
$$

Therefore

$$
G_\lambda(t,s) \geqslant \gamma_1 G_\lambda(b,s) \ \text{ for all } \ (t,s) \in [\eta,b] \times [a,b]
$$

where $\gamma_1 = \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}$. Hence the inequality (ii) is proved. □

We can also formulate similar results as Lemma 4–Lemma 5 above, for the fractional boundary value problem,

$$
D^{\beta}_{a^+} v(t) + h(t) = 0,\ a < t < b,\ \ m-1 < \beta \leqslant m, \tag{12}
$$

$$
v^{(j)}(a) = 0,\ 0 \leqslant j \leqslant m-2, \quad v^{(\beta_1)}(b) = \xi v^{(\beta_1)}(\eta),\ 1 \leqslant \beta_1 \leqslant \beta-2,\ \text{(but fixed)} \tag{13}
$$

where $m \geqslant 3$, $1 \leqslant \beta_1 \leqslant \beta - 2$ is a fixed integer. We denote by G_μ and γ_2 the corresponding Green's function and constant for the problem (12)–(13) defined in a similar manner as G_λ and γ_1 respectively.

By using the Green's function G_λ and G_μ, our problem (1)–(2) can be written equivalently as the following nonlinear system of integral equations

$$
\begin{cases}
u(t) = \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds,\ a \leqslant t \leqslant b, \\
v(t) = \mu \int_a^b G_\mu(t,s) g(s,u(s),v(s))ds,\ a \leqslant t \leqslant b.
\end{cases}
$$

We consider the Banach space $X = C[a,b]$ with supremum norm $\| \cdot \|$, and the Banach space $Y = X \times X$ with the norm $\| (u,v) \| = \| u \| + \| v \|$. It is easy to show that $(Y, \| \cdot \|)$ is a real Banach space. We define the cone $\kappa \subset Y$ by

$$\kappa = \left\{ (u,v) \in Y : u(t) \geqslant 0,\ v(t) \geqslant 0, \forall\, t \in [a,b] \text{ and } \min_{t \in\ [\eta,b]} (u(t)+v(t)) \geqslant \gamma \| (u,v) \| \right\},$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$.

For $\lambda,\ \mu > 0$, we define the operators $Q_\lambda, Q_\mu : Y \to X$ as

$$Q_\lambda(u,v)(t) = \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds, \quad a \leqslant t \leqslant b,$$
$$Q_\mu(u,v)(t) = \mu \int_a^b G_\mu(t,s) g(s,u(s),v(s))ds, \quad a \leqslant t \leqslant b,$$

and an operator $Q : Y \to Y$ as

$$Q(u,v) = \Big(Q_\lambda(u,v), Q_\mu(u,v) \Big), \ \ (u,v) \in Y.$$

It is clear that the existence of a positive solution to the system (1)–(2) is equivalent to the existence of a fixed points of the operator Q.

LEMMA 6. *If $(H1)$ and $(H2)$ hold, then $Q : \kappa \to \kappa$ is completely continuous.*

Proof. By using standard arguments, we can easily show that, the operator Q is completely continuous, we need only to prove $Q(\kappa) \subset \kappa$. Let $(u,v) \in \kappa$ clearly, $Q_1(u,v)(t) \geqslant 0$ and $Q_2(u,v)(t) \geqslant 0$ for $t \in [a,b]$. Also, for $(u,v) \in \kappa$,

$$\| Q_\lambda(u,v)(t) \| \leqslant \lambda \int_a^b G_\lambda(b,s) f(s,u(s),v(s))ds,$$
$$\| Q_\mu(u,v)(t) \| \leqslant \mu \int_a^b G_\mu(b,s) g(s,u(s),v(s))ds.$$

In fact, for any $(t,s) \in\ [\eta,b] \times [a,b]$, we have from Lemma 5

$$\begin{aligned}
&\min_{t\in[\eta,b]} \Big[Q_\lambda(u,v)(t) + Q_\mu(u,v)(t) \Big] \\
&= \min_{t\in[\eta,b]} \Big[\lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds + \mu \int_a^b G_\mu(t,s) g(s,u(s),v(s))ds \Big] \\
&\geqslant \lambda\gamma \int_a^b G_\lambda(b,s) f(s,u(s),v(s))ds + \mu\gamma \int_a^b G_\mu(b,s) g(s,u(s),v(s))ds \\
&= \gamma \| Q_\lambda(u,v) \| + \gamma \| Q_\mu(u,v) \| = \gamma \| Q(u,v) \|,
\end{aligned}$$

hence,

$$\min_{t\in[\eta,b]} \Big[Q_\lambda(u,v)(t) + Q_\mu(u,v)(t) \Big] \geqslant \gamma \| Q(u,v) \| .$$

Therefore, $Q(\kappa) \subset \kappa$. Let $(u,v) \in \kappa$ and $\varepsilon > 0$ be given. By the continuity of f and g, there exists $\delta > 0$ such that

$$| f(t,u,v) - f(t,u',v') | < \varepsilon, | g(t,u,v) - g(t,u',v') | < \varepsilon,$$

whenever $| u - u' | < \delta$, $| v - v' | < \delta$ for all $t \in [a,b]$.

$$\begin{aligned} | Q_\lambda(u,v)(t) - Q_\lambda(u,v)(t) | &= \lambda \int_a^b G_\lambda(t,s) \, | f(s,u,v) - f(s,u',v') | \, ds \\ &\leqslant \varepsilon\lambda \int_a^b G_\lambda(t,s)ds. \end{aligned}$$

Thus, $\| Q_\lambda(u,v)(t) - Q_\lambda(u',v')(t) \| \leqslant \varepsilon\lambda \int_a^b G_\lambda(t,s)ds$. In a similar manner $\| Q_\mu(u,v)(t) - Q_\mu(u',v')(t) \| \leqslant \varepsilon\mu \int_a^b G_\mu(t,s)ds$ and Q is continuous. Now, let $\{(u_n,v_n)\}$ be a bounded sequence in κ. Since f and g are continuous, there exists $N > 0$ such that $| f(t,u_n,v_n) | \leqslant N$, $| g(t,u_n,v_n) | \leqslant N$ for all $u_n, v_n \in [0,\infty)$. Then, for each $t \in [a,b]$ and for each n,

$$\begin{aligned} | Q_\lambda(u_n,v_n)(t) | &= | \lambda \int_a^b G_\lambda(t,s) f(s,u_n,v_n) ds | \\ &\leqslant \lambda \int_a^b | G_\lambda(b,s) || f(s,u_n,v_n) | \, ds \\ &\leqslant N\lambda \int_a^b G_\lambda(b,s)ds. \end{aligned}$$

In a similar manner $| Q_\mu(u_n,v_n)(t) | \leqslant N\mu \int_a^b G_\mu(b,s)ds$. By choosing successive subsequences, there exists a subsequence $\{Q(u_{n_j}, v_{n_j})\}$ which converges uniformly on $[a,b]$. Hence Q is completely continuous. $\square$

If $y \in \kappa$ is a fixed point of Q, then y satisfies (6) and hence y is a positive solution of the BVP (1)–(2). We seek the fixed points of the operator Q in the cone κ.

4. Existence results

In this section, we discuss the existence of at least one positive solution to the system (1)–(2). We use the following notation for simplicity.

$$\begin{aligned} A_1 &= \int_a^b G_\lambda(b,s) p_1(s) ds, \quad B_1 = \int_\eta^b G_\lambda(b,s) ds, \\ A_2 &= \int_a^b G_\mu(b,s) p_2(s) ds, \quad B_2 = \int_\eta^b G_\mu(b,s) ds. \end{aligned}$$

Our approach is based on the following Guo-Krasnosel'skii fixed point theorem [11, 16].

THEOREM 2. *Suppose* $(H1), (H2)$ *hold, and* $0 < \delta < 1$ *then we have the following results:*

(1) If $0<q_{10},f_\infty,q_{20},g_\infty<\infty,\ A_1q_{10}<\delta\gamma^2B_1f_\infty,$ *then for each* $\lambda\in\left(\frac{1}{\gamma^2B_1f_\infty},\frac{\delta}{A_1q_{10}}\right)$ *and* $\mu\in\left(0,\frac{1-\delta}{A_2q_{20}}\right)$, *the system* (1)–(2)*has at least one positive solution.*

(2) If $0<q_{10},f_\infty,q_{20},g_\infty<\infty,\ A_2q_{20}<(1-\delta)\gamma^2B_2g_\infty,$ *then for each* $\lambda\in\left(0,\frac{\delta}{A_1q_{10}}\right)$ *and* $\mu\in\left(\frac{1}{\gamma^2B_2g_\infty},\frac{1-\delta}{A_2q_{20}}\right)$, *the system* (1)–(2) *has at least one positive solution.*

Proof. We only prove case (1). The other case can be proved similarly. We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2. Let

$$\lambda\in\left(\frac{1}{\gamma^2B_1f_\infty},\frac{\delta}{A_1q_{10}}\right),\quad \mu\in\left(0,\frac{1-\delta}{A_2q_{20}}\right),$$

and we choose $\varepsilon>0$ such that

$$\frac{1}{\gamma^2B_1(f_\infty-\varepsilon)}\leqslant\lambda\leqslant\frac{\delta}{A_1(q_{10}+\varepsilon)},\quad 0<\mu\leqslant\frac{1-\delta}{A_2(q_{20}+\varepsilon)}.$$

By the definition of q_{10} and q_{20}, there exists $R_1>0$ such that $q_1(t,u,v)\leqslant(q_{10}+\varepsilon)(u+v)$, $q_2(t,u,v)\leqslant(q_{20}+\varepsilon)(u+v)$, for $u+v\in[0,R_1]$. Choosing $(u,v)\in\kappa$ with $\|(u,v)\|=R_1$ we have

$$\begin{aligned}
Q_\lambda(u,v)(t)&=\lambda\int_a^b G_\lambda(t,s)f(s,u(s),v(s))ds\\
&\leqslant\lambda\int_a^b G_\lambda(b,s)p_1(s)q_1(s,u(s),v(s))ds\\
&\leqslant\lambda\int_a^b G_\lambda(b,s)p_1(s)(q_{10}+\varepsilon)(u+v)ds\\
&\leqslant\lambda(q_{10}+\varepsilon)\|(u,v)\|\int_a^b G_\lambda(b,s)p_1(s)ds\\
&\leqslant\lambda(q_{10}+\varepsilon)\|(u,v)\|A_1\leqslant\delta\|(u,v)\|.
\end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned}
Q_\mu(u,v)(t)&=\mu\int_a^b G_\mu(t,s)g(s,u(s),v(s))ds\\
&\leqslant\mu\int_a^b G_\mu(b,s)p_2(s)q_2(s,u(s),v(s))ds\\
&\leqslant\mu\int_a^b G_\mu(b,s)p_2(s)(q_{20}+\varepsilon)(u+v)ds\\
&\leqslant\mu(q_{20}+\varepsilon)\|(u,v)\|\int_a^b G_\mu(b,s)p_2(s)ds\\
&\leqslant\mu(q_{20}+\varepsilon)\|(u,v)\|A_2\leqslant(1-\delta)\|(u,v)\|,
\end{aligned}$$

then $\| Q(u,v) \| \leqslant \delta \| (u,v) \| + (1-\delta) \| (u,v) \| = \| (u,v) \|$. Consequently, if we set $\Omega_1 = \{(u,v) \in \kappa : \| (u,v) \| < R_1\}$, then

$$\| Q(u,v) \| \leqslant \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_1. \tag{14}$$

On the other hand, by the definition of f_∞, there exists $\overline{R}_2 > 0$, such that $f(t,u,v) \geqslant (f_\infty - \varepsilon)(u+v)$, for all $u+v \in [\overline{R}_2, \infty)$. Let $R_2 = \max\{2R_1, \frac{\overline{R}_2}{\gamma}\}$ and $\Omega_2 = \{(u,v) \in \kappa : \| (u,v) \| < R_2\}$. If $(u,v) \in \kappa$ with $\| (u,v) \| = R_2$, then $\min_{t\in[\eta,b]}(u+v) \geqslant \gamma \| (u,v) \| \geqslant \overline{R}_2$, thus we have

$$\begin{aligned}
Q_\lambda(u,v)(t) &= \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds \\
&\geqslant \lambda\gamma \int_\eta^b G_\lambda(b,s)(f_\infty - \varepsilon)(u+v)ds \\
&\geqslant \lambda\gamma^2 (f_\infty - \varepsilon) \| (u,v) \| \int_\eta^b G_\lambda(b,s)ds \\
&\geqslant \lambda\gamma^2 B_1 (f_\infty - \varepsilon) \| (u,v) \| \\
&\geqslant \| (u,v) \|, \text{ for all } t \in [\eta,b],
\end{aligned}$$

then

$$\| Q(u,v) \| \geqslant \| Q_\lambda(u,v) \| \geqslant \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_2. \tag{15}$$

Therefore, it follows from (14),(15) and Theorem 1, Q has a fixed points in $\kappa \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$, which is a positive solution of (1)–(2). □

Similarly, we can also obtain the following theorem that is in some way of duality of Theorem 2.

THEOREM 3. *Suppose $(H1),(H2)$ hold, and $0 < \delta < 1$ then we have the following results:*

(1) If $0 < f_0, q_{1\infty}, g_0, q_{2\infty} < \infty$, $A_1 q_{1\infty} < \delta\gamma^2 B_1 f_0$, then for each $\lambda \in \left(\frac{1}{\gamma^2 B_1 f_0}, \frac{\delta}{A_1 q_{1\infty}}\right)$ and $\mu \in \left(0, \frac{1-\delta}{A_2 q_{2\infty}}\right)$, the system (1)–(2)has at least one positive solution.

(2) If $0 < f_0, q_{1\infty}, g_0, q_{2\infty} < \infty$, $A_2 q_{2\infty} < (1-\delta)\gamma^2 B_2 g_0$, then for each $\lambda \in \left(0, \frac{\delta}{A_1 q_{1\infty}}\right)$ and $\mu \in \left(\frac{1}{\gamma^2 B_2 g_0}, \frac{1-\delta}{A_2 q_{2\infty}}\right)$, the system (1)–(2) has at least one positive solution.

5. Multiplicity results

In this section, we prove the existence of at least two positive solutions for the system (1)–(2).

THEOREM 4. *Suppose $(H1)$, $(H2)$ hold. In addition, assume that there exist four constants r_1, M, K, δ where K is sufficient small, $0 < \delta < 1$, with $\delta\gamma B_1 M > A_1 K$, $(1-\delta)\gamma B_2 M > A_2 K$, such that:*

(1) $q_{10} = q_{1\infty} = 0,\ q_{20} = q_{2\infty} = 0;$

(2) $f(t,u,v) \geqslant Mr_1$, *or* $g(t,u,v) \geqslant Mr_1$, *for* $\gamma r_1 \leqslant \| (u,v) \| < r_1$. *Then for any* $\lambda \in \left(\frac{1}{\gamma B_1 M}, \frac{\delta}{A_1 K}\right]$, $\mu \in \left(0, \frac{1-\delta}{A_2 K}\right]$ *or* $\lambda \in \left(0, \frac{\delta}{A_1 K}\right], \mu \in \left(\frac{1}{\gamma B_2 M}, \frac{1-\delta}{A_2 K}\right]$, *the system* (1)–(2) *has at least two positive solutions.*

Proof. We only prove the case of $\lambda \in \left(\frac{1}{\gamma B_1 M}, \frac{\delta}{A_1 K}\right]$, $\mu \in \left(0, \frac{1-\delta}{A_2 K}\right]$. The other case is similar.

Step 1. By the definition of $q_{10} = q_{20} = 0$, there exists $H_1 \in (0, r_1)$ such that $q_1(t,u,v) \leqslant K(u+v)$, $q_2(t,u,v) \leqslant K(u+v)$, for $u+v \in (0, H_1)$. Then we have

$$\begin{aligned} Q_\lambda(u,v)(t) &= \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds \\ &\leqslant \lambda \int_a^b G_\lambda(b,s) p_1(s) q_1(s,u(s),v(s))ds \\ &\leqslant \lambda \int_a^b G_\lambda(b,s) p_1(s) K(u+v)ds \\ &\leqslant \lambda K \| (u,v) \| A_1 \leqslant \delta \| (u,v) \| . \end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned} Q_\mu(u,v)(t) &= \mu \int_a^b G_\mu(t,s) g(s,u(s),v(s))ds \\ &\leqslant \mu \int_a^b G_\mu(b,s) p_2(s) q_2(s,u(s),v(s))ds \\ &\leqslant \mu \int_a^b G_\mu(b,s) p_2(s) K(u+v)ds \\ &\leqslant \mu K \| (u,v) \| A_2 \leqslant (1-\delta) \| (u,v) \|, \end{aligned}$$

Hence, $\| Q(u,v) \| = \| Q_\lambda(u,v) \| + \| Q_\mu(u,v) \| \leqslant \| (u,v) \|$.

Set $\Omega_1 = \left\{(u,v) \in \kappa : \| (u,v) \| < H_1\right\}$, then

$$\| Q(u,v) \| \leqslant \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_1. \tag{16}$$

Step 2. By the definition of $q_{1\infty} = q_{2\infty} = 0$, there exist $H_2 > r_1$ such that $q_1(t,u,v) \leqslant K(u+v)$, $q_2(t,u,v) \leqslant (u+v)$, for $u+v \in [H_2, \infty)$. Similarly, set $\Omega_2 = \left\{(u,v) \in \kappa : \| (u,v) \| < H_2\right\}$, then

$$\| Q(u,v) \| \leqslant \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_2. \tag{17}$$

Step 3. Set $\Omega_3 = \{(u,v) \in \kappa : \| (u,v) \| < r_1\}$, then $\forall (u,v) \in \kappa$ with $\| (u,v) \| = r_1$, we have

$$\begin{aligned} Q_\lambda(u,v)(t) &= \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds \\ &\geqslant \lambda\gamma \int_\eta^b G_\lambda(b,s) f(s,u(s),v(s))ds \\ &\geqslant \lambda\gamma \int_\eta^b G_\lambda(b,s) M r_1 ds \\ &\geqslant \lambda\gamma M r_1 B_1 > r_1 \ \ \forall\, t \in [\eta, b], \end{aligned}$$

then

$$\| Q(u,v) \| > \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_3. \tag{18}$$

Consequently, from (16)–(18) and Theorem 1, the system has at least two positive solutions $(u_1,v_1) \in \kappa, (u_2,v_2) \in \kappa$ with $0 \leqslant \| (u_1,v_1) \| < r_1 < \| (u_2,v_2) \|$. □

The following result is an antithesis of Theorem 4.

THEOREM 5. *Suppose $(H1),(H2)$ hold. In addition, assume that there exist four constants r_1, M, K, δ where K is sufficient large, $0 < \delta < 1$, with $\gamma B_1 K > A_1 M$, $(1-\delta)\gamma B_2 K > A_2 M$, such that*

(3) $q_1(t,u,v) \leqslant M r_1$, or $q_2(t,u,v) \leqslant M r_1$, for $0 \leqslant \| (u,v) \| \leqslant r_1$;

(4) $f_0 = f_\infty = \infty$ or $g_0 = g_\infty = \infty$. Then for any $\lambda \in \left(\frac{1}{\gamma B_1 K}, \frac{\delta}{A_1 M}\right]$ and $\mu \in \left(0, \frac{1-\delta}{A_2 M}\right]$ or $\lambda \in \left(0, \frac{\delta}{A_1 M}\right]$ and $\mu \in \left(\frac{1}{\gamma B_2 K}, \frac{1-\delta}{A_2 M}\right]$, the system (1)–(2) has at least two positive solutions.

For the convenience of the discussion of at least two positive solutions for the system (1)–(2), we study the problem under a more general case than the assumption of Theorem 4 and Theorem 5.

$$\begin{aligned} \varphi_i(r) &= \sup\{q_i(t,u,v) : t \in [a,b],\ \gamma r \leqslant u+v \leqslant r,\},\ i = 1,2. \\ \psi_1(r) &= \inf\{f(t,u,v) : t \in [\eta,b],\ \gamma r \leqslant u+v \leqslant r,\}. \\ \psi_2(r) &= \inf\{g(t,u,v) : t \in [\eta,b],\ \gamma r \leqslant u+v \leqslant r,\}. \\ \varphi(r) &= \max\{\varphi_1(r), \varphi_2(r)\},\ \psi(r) = \min\{\psi_1(r), \psi_2(r)\}. \end{aligned}$$

Then, we can obtain the following result.

THEOREM 6. *Suppose $(H1)$ hold. In addition, assume that there exist three constants $M, K, \delta, 0 < \delta < 1$ with $\delta\gamma B_1 M > A_1 K, (1-\delta)\gamma B_2 M > A_2 K$ and three constants d_1, d_2, d_3 with $0 < d_1 < d_2 < d_3$, such that one of the following two conditions is satisfied:*

(1) $\varphi(d_1) \leqslant d_1 K, \psi(d_2) > d_2 M$, and $\varphi(d_3) \leqslant K d_3$,

(2) $\psi(d_1) \geqslant d_1M, \varphi(d_2) < d_2K,$ *and* $\psi(d_3) \geqslant Md_3,$ *then for any* $\lambda \in \left(\frac{1}{B_1M\gamma}, \frac{\delta}{A_1K}\right]$ *and* $\mu \in \left(0, \frac{1-\delta}{A_2K}\right]$ *or* $\lambda \in \left(0, \frac{\delta}{A_1K}\right]$ *and* $\mu \in \left(\frac{1}{\gamma B_2M}, \frac{1-\delta}{A_2K}\right]$, *the system* (1)–(2) *has at least two positive solutions* $(u_1^\star, v_1^\star)$, $(u_2^\star, v_2^\star)$ *and* $d_1 \leqslant \| (u_1^\star, v_1^\star) \| < d_2 < \| (u_2^\star, v_2^\star) \| \leqslant d_3$.

Proof. We only prove the case of (1) and $\lambda \in \left(\frac{1}{B_1M\gamma}, \frac{\delta}{A_1K}\right]$, $\mu \in \left(0, \frac{1-\delta}{A_2K}\right]$. The other cases are similar. Let $\Omega_{d_1} = \left\{(u,v) \in \kappa : \| (u,v) \| < d_1\right\}$. If $(u,v) \in \partial\Omega_{d_1}$, then $\| (u,v) \| = d_1$. Since $\gamma d_1 \leqslant u+v \leqslant d_1, a \leqslant t \leqslant b$, then we have

$$\begin{aligned}
Q_\lambda(u,v)(t) &= \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds \\
&\leqslant \lambda \int_a^b G_\lambda(b,s) p_1(s) q_1(s,u(s),v(s))ds \\
&\leqslant \lambda \int_a^b G_\lambda(b,s) p_1(s) \varphi(d_1)ds \\
&\leqslant \lambda d_1 K \int_a^b G_\lambda(b,s) p_1(s)ds \\
&\leqslant \lambda d_1 K A_1 \leqslant d_1\delta = \delta \| (u,v) \| .
\end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned}
Q_\mu(u,v)(t) &= \mu \int_a^b G_\mu(t,s) g(s,u(s),v(s))ds \\
&\leqslant \mu \int_a^b G_\mu(b,s) p_2(s) q_2(s,u(s),v(s))ds \\
&\leqslant \mu \int_a^b G_\mu(b,s) p_2(s) \varphi(d_1)ds \\
&\leqslant \mu K d_1 A_2 \leqslant (1-\delta)d_1 = (1-\delta) \| (u,v) \| .
\end{aligned}$$

Then

$$\| Q(u,v) \| \leqslant \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_{d_1}. \tag{19}$$

Let $\Omega_{d_2} = \left\{(u,v) \in \kappa : \| (u,v) \| < d_2\right\}$. If $(u,v) \in \partial\Omega_{d_2}$, then $\| (u,v) \| = d_2$. Since $\gamma d_2 \leqslant u+v \leqslant d_2, t \in [\eta, b]$, then we have

$$\begin{aligned}
Q_\lambda(u,v)(t) &= \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds \\
&\geqslant \lambda\gamma \int_\eta^b G_\lambda(b,s) \psi(d_2)ds \\
&> \lambda\gamma d_2 M B_1 \geqslant d_2 = \| (u,v) \| .
\end{aligned}$$

Then

$$\| Q(u,v) \| > \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_{d_2}. \tag{20}$$

Let $\Omega_{d_3} = \left\{(u,v) \in \kappa : \| (u,v) \| < d_3\right\}$. If $(u,v) \in \partial\Omega_{d_3}$, then $\| (u,v) \| = d_3$. Since $\gamma d_3 \leqslant u+v \leqslant d_3, a \leqslant t \leqslant b$, then we have

$$\begin{aligned} Q_\lambda(u,v)(t) &= \lambda \int_a^b G_\lambda(t,s) f(s,u(s),v(s))ds \\ &\leqslant \lambda \int_a^b G_\lambda(b,s) p_1(s) q_1(s,u(s),v(s))ds \\ &\leqslant \lambda \int_a^b G_\lambda(b,s) p_1(s) \varphi(d_3)ds \\ &\leqslant \lambda d_3 K A_1 \leqslant d_3 \delta = \delta \| (u,v) \| . \end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned} Q_\mu(u,v)(t) &= \mu \int_a^b G_\mu(t,s) g(s,u(s),v(s))ds \\ &\leqslant \mu \int_a^b G_\mu(b,s) p_2(s) q_2(s,u(s),v(s))ds \\ &\leqslant \mu \int_a^b G_\mu(b,s) p_2(s) \varphi(d_3)ds \\ &\leqslant \mu K d_3 A_2 \leqslant (1-\delta) d_1 = (1-\delta) \| (u,v) \| . \end{aligned}$$

Then

$$\| Q(u,v) \| \leqslant \| (u,v) \|, \text{ for all } (u,v) \in \kappa \cap \partial\Omega_{d_3}. \tag{21}$$

From (19), (20), (21) and Theorem 1, the system has at least two positive solutions $(u_1^\star, v_1^\star) \in \kappa, (u_2^\star, v_2^\star) \in \kappa$ and $d_1 \leqslant \| (u_1^\star, v_1^\star) \| < d_2 < \| (u_2^\star, v_2^\star) \| \leqslant d_3$. □

6. Example

In this section, we demonstrate our main results with an example.
Let $a = 0,\ b = 1,\ \alpha = \frac{5}{2},\ n = 3,\ \beta = \frac{7}{2},\ m = 4,\ \eta = \frac{1}{2},\ \xi = \frac{1}{3},\ \alpha_1 = 1,\ \beta_1 = 1.$
We consider the system of fractional differential equations

$$\begin{aligned} &D_{0^+}^{2.5} u(t) + \lambda\left[(u+v)^3 + (u+v)^{\frac{1}{3}}\right] = 0, \ \ 0 < t < 1, \\ &D_{0^+}^{3.5} v(t) + \mu\left[(u+v)^2 + (u+v)^{\frac{1}{2}}\right] = 0, \ \ 0 < t < 1, \end{aligned} \tag{22}$$

with the three-point boundary conditions

$$\begin{aligned} &u(0) = 0,\ u'(0) = 0, \quad \text{and} \quad u'(1) = \frac{1}{3} u'\left(\frac{1}{2}\right), \\ &v(0) = 0,\ v'(0) = 0,\ v''(0) = 0, \quad \text{and} \quad v'(1) = \frac{1}{3} v'\left(\frac{1}{2}\right). \end{aligned} \tag{23}$$

We choose $r_1 = 1,\ M = 3,\ K = \frac{16^4 \times 2^{14} \times 2}{11^5},\ \delta = \frac{1}{2}$, then all the conditions in Theorem 5 are satisfied. Therefore, for any $\lambda \in (\frac{95}{126}, 8]$ and $\mu \in (0,32]$ or $\lambda \in (0,8]$

and $\mu \in (28,32]$, the system (22)–(23) has at least two positive solutions $(u_1(t),v_1(t))$, $(u_2(t),v_2(t))$ with $0 <\| (u_1(t),v_1(t)) \|< 1 <\| (u_2(t),v_2(t)) \|$.

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