Fractional Differential Calculus
Volume 7, Number 1 (2017), 169–184
doi:10.7153/fdc-07-05

ANALYTIC SOLUTION OF GENERALIZED SPACE TIME FRACTIONAL REACTION DIFFUSION EQUATION

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(*Communicated by R. Garrappa*)

Abstract. The aim of this paper is to investigate the solution of a generalized space-time fractional reaction-diffusion equation associated with the Hilfer-Prabhakar time fractional derivative and the space fractional Laplacian operator. The solution of the equation in terms of the three parameter Mittag-Leffler function, is obtained by applying the Laplace and Fourier transforms. The work by K. B. Kachhia and Prajapati (2015), R. Garra *et al.* (2014) and S. D. Purohit (2011) and references therein follow as particular cases of our results.

1. Introduction

Fractional calculus has gained remarkable popularity and significance during last few years, mainly due to its attractive applications in frequent, ostensibly diverse and wide spread fields of science and engineering. Fractional differential equations have been used for mathematical modeling in potential fields, viscoelastic materials, signal processing, diffusion problems, control theory, heat propagation and many others. The various type of partial differential equations occurring in the fluid mechanics are discussed by Debnath [14]. Kachhia and Prajapati [11] investigated the solutions of fractional partial differential equations, occurring in the study of heat transfer through diathermanous materials. Purohit [29] found the solutions of some fractional partial differential equations occurring in quantum mechanics. Agarwal *et al.* [20] investigated the solutions of time-space fractional advection-dispersion equation with Hilfer composite fractional derivative. Many authors like Mainardi [7], Boyadjiev and Schere [12], Saxena and Kalla [28] and Agarwal *et al.* [19] have discussed various applications of fractional differential equations in their work.

Garra *et al.* [22] analyzed and discussed Hilfer-Prabhakar derivative and its properties. Further, they showed some applications of the generalized Hilfer-Prabhakar derivative in the classical equations of mathematical physics, like the heat and the

Mathematics subject classification (2010): 35R11, 35C15, 33E12, 26A33.
Keywords and phrases: Linear space time reaction diffusion equation, Fourier transform, Laplace transform, Mittag-Leffler function, composite fractional derivative, fractional Laplace operator.

Paper FDC-07-05

free electron laser equations, and in difference-differential equations governing the dynamics of generalized renewal stochastic processes. Garrappa [21] have shown the applications of Prabhakar function in anomalous relaxation problems. He discussed fractional operators describing the time relaxation in systems governed by Havriliak-Negami laws. He proposed a formulation of Grünwald-Letnikov type which turns out to be effective not only to provide a theoretical characterization of the operators associated to Havriliak-Negami model which is obtained by inserting two independent real powers in the classical Debye model. Polito and Scalas [6] introduced a generalization of the so-called space-fractional Poisson process by extending the difference operator acting on state space present in the associated difference-differential equations by using Prabhakar derivative. Recently, Polito and Tomovski [5] studied some properties of the Prabhakar integrals and derivatives and of some of their extensions such as the regularized Prabhakar derivative or the Hilfer-Prabhakar derivative.

Reaction-diffusion equations have found many applications in applied science and engineering. In recent work, many authors have explained some significant physical issues of reaction-diffusion equations such as oscillations, stationary, spatio-temporal dissipative pattern formation, waves etc. (see, e.g., Frank [31], Gafiychuk *et al.* [33]). A reaction-diffusion equation comprises a reaction term and a diffusion term, i.e. the typical form of this equation is as follows:

$$u(x,t) = k\Delta u + f(u)$$

$u(x,t)$ is a state variable and describe density or concentration of a substance or a population at position $x \in \Omega \subset \mathbb{R}$ at time t (Ω being an open set). Δ denotes the Laplace operator. The first term on the right hand side describes the diffusion, k being diffusion coefficient. The second term, $f(u)$ is a smooth function $f : \mathbb{R} \to \mathbb{R}$ and describes processes which really change the present u, i.e. something happens to it (birth, death, chemical reaction, etc.), not just diffuse in the space. Analytical solution of generalized reaction-diffusion equation studied by Saxena *et al.* [25], [26] and [27]. Linear fractional reaction-diffusion equation on a finite domain is solved by Yildirim and Sezer [3] using homotopy perturbation method and Yu *et al.* [18] using Adomian decomposition method. Recently, Garg and Manohar [17] obtained analytical solution of linear space-time fractional reaction-diffusion equation using generalized differential transform method.

Linear space-time fractional reaction diffusion equation on finite domain $0 < x < L$, $t > 0$ with $0 < \mu \leqslant 1$ and $0 < \nu \leqslant 2$ as discussed by Yildirim and Sezer [34] and Yu et al. [35]

$$\frac{\partial^{\mu} u(x,t)}{\partial t^{\mu}} = b(x)\frac{\partial^{\nu} u(x,t)}{\partial x^{\nu}} - c(x)u(x,t) + f(x,t) \tag{1}$$

where $\frac{\partial^{\mu} u(x,t)}{\partial t^{\mu}}$ is the Caputo time fractional derivative of order $0 < \mu \leqslant 1$, $\frac{\partial^{\nu} u(x,t)}{\partial x^{\nu}}$ is the Caputo Space fractional derivative of order $1 < \nu \leqslant 2$ and $0 < b(x) \leqslant b_{\max}$ and $0 < c(x) \leqslant c_{\max}$ are continuous for $0 < x < L$ and the function $u(x,t)$ represent source or sink and $f(x,t)$ is a sufficiently well behaved function.

2. Mathematics prerequisites

The right-sided Riemann-Liouville fractional integral of order α, $(\Re(\alpha) > 0)$ (Samko *et al.* [30]) is defined as:

$$\mathbb{I}_a^{\alpha}(u(x,t)) = {}_a^{RL}\mathbb{D}_t^{-\alpha}(u(x,t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} u(x,\tau)\ d\tau,\ \ t > a \tag{2}$$

The right-sided Riemann-Liouville fractional derivative of order α, $(\Re(\alpha) > 0)$ can be defined as:

$$ {}_a^{RL}\mathbb{D}_t^{\alpha}(u(x,t)) = \left(\frac{d}{dt}\right)^n (\mathbb{I}_a^{n-\alpha} u(x,t)), \qquad n = [\Re(\alpha)] + 1, \tag{3}$$

where $[x]$ represents the integral part of the number x.

The following fractional derivative of order α, $\Re(\alpha) > 0$ is introduced by Caputo [15] as

$$ {}_a^{C}\mathbb{D}_t^{\alpha}(u(x,t)) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t \frac{u^{(m)}(x,\tau)}{(t-\tau)^{\alpha+1-m}}\ d\tau, & m-1 < \alpha \leqslant m \\ \dfrac{\partial^m}{\partial t^m} u(x,t), & \text{if } \alpha = m \end{cases} \tag{4}$$

where $u^{(m)}(x,t) = \frac{\partial^m}{\partial t^m} u(x,t)$, $m \in N$ is the m-th derivative of the function $u(x,t)$ with respect to t.

DEFINITION 1. [22] (Prabhakar integral). Let $f \in L^1[0,b]$, $0 < t < b \leqslant \infty$. The Prabhakar integral can be written as

$$\mathbb{P}^{\gamma}_{\rho,\mu,\omega,0^+} f(t) = \int_0^t (t-y)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-y)^{\rho}] f(y) dy = \left(f * e^{\gamma}_{\rho,\mu,\omega}\right)(t), \tag{5}$$

where ρ, μ, ω, $\gamma \in \mathbb{C}$, $t \in \mathbb{R}$ with $\Re(\rho)$, $\Re(\mu) > 0$ and the kernel is given by

$$e^{\gamma}_{\rho,\mu,\omega}(t) = t^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega t^{\rho}),$$

In 1971, Prabhakar [32] introduced the generalization of two parameter Mittag-Leffler function as

$$E^{\gamma}_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\rho n + \mu)} \frac{z^n}{n!}, \qquad \gamma, \rho, \mu \in \mathbb{C},\ \ \Re(\rho) > 0,\ \ \Re(\mu) > 0. \tag{6}$$

Taking $\gamma = 1$, (6) reduces to the two parameter Mittag-Leffler function studied by Wiman [2] and defined as

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}, \qquad \rho, \mu \in \mathbb{C},\ \ \Re(\mu) > 0. \tag{7}$$

As $\gamma \to 0$, then by virtue of the limit formula [24, Eq. 24]

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} \tag{8}$$

The fractional Prabhakar derivative was introduced and studied by Ovidio and Polito [16] as follows.

DEFINITION 2. (Prabhakar derivative). Let $f \in L^1[0,b]$, $0 < t < b \leqslant \infty$ and $f * e^{-\gamma}_{\rho,m-\mu,\omega}(\cdot) \in W^{m,1}[0,b]$, $m = \lceil \mu \rceil$. The Prabhakar derivative of the function f is given by

$$\mathbb{D}^{\gamma}_{\rho,\mu,\omega,0^+} f(t) = \frac{d^m}{dt^m} \mathbb{P}^{-\gamma}_{\rho,m-\mu,\omega,0^+} f(t) \tag{9}$$

where $t \in \mathbb{R}$, ρ, μ, ω, $\gamma \in \mathbb{C}$, $\Re(\rho)$, $\Re(\mu) > 0$.

A generalization of the Riemann-Liouville fractional derivative operator (3) and Caputo fractional derivative operator (4) is given by Hilfer [23], by introducing a fractional derivative operator of two parameters.

DEFINITION 3. (Hilfer derivative). Let $0 < \mu < 1$ and type $0 \leqslant \nu \leqslant 1$, $f \in L^1[a,b]$, $-\infty \leqslant a < t < b \leqslant \infty$, $f * K_{(1-\nu)(1-\mu)} \in AC^1[0,b]$. Then Hilfer derivative of $u(x,t)$ with respect to variable t is defined as

$$_0\mathbb{D}^{\mu,\nu}_{a+}(u(x,t)) = \mathbb{I}^{\nu(1-\mu)}_t \frac{\partial}{\partial t} \left(\mathbb{I}^{(1-\nu)(1-\mu)}_{a+} u(x,t) \right) \tag{10}$$

It is interesting to observe that for $\nu = 0$, Eq. (10) reduces to the classical Riemann-Liouville fractional derivative operator (3). On the other hand, for $\nu = 1$, it gives the Caputo fractional derivative operator defined by (4).

The Laplace transform (see, e.g. Sneddon [10, Chapter 1]) for this operator is given by Hilfer [23]. Hereafter and without loss of generality, we set $a = 0$ in (10).

$$L\{\mathbb{D}^{\mu,\nu}_{0+} u(x,t); s\} = s^{\mu} L\{u(x,t)\} - s^{\nu(\mu-1)} \mathbb{I}^{(1-\nu)(1-\mu)}_{0+} u(x,0^+), \quad 0 < \mu < 1 \tag{11}$$

where the initial value term $I^{(1-\nu)(1-\mu)}_{0+} u(x,0^+)$ involves the Riemann-Liouville fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t \to 0^+$.

A generalization of the Hilfer derivative operator (10), was given by Garra *et al.* [22] as

DEFINITION 4. (Hilfer-Prabhakar derivative). Let $\mu \in (0,1)$, $\nu \in [0,1]$, $f \in L^1[0,b]$, $0 < t < b \leqslant \infty$, $f * e^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega}(\cdot p) \in AC^1[0,b]$.

Then Hilfer-Prabhakar derivative is defined by

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+} u(t) = \left(\mathbb{P}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+} \frac{d}{dt} \left\{ \mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} u \right\} \right)(t) \tag{12}$$

where γ, $\omega \in \mathbb{C}$, $\rho > 0$ and where $\mathbb{P}^{0}_{\rho,0,\omega,0^+} u = u$.

It is interesting to observe that for $\gamma = 0$, Eq. (12) reduces to the Hilfer derivative (10) and for $\gamma = 0,\ \nu = 0$, Eq. (12) reduces to the classical Riemann-Liouville fractional derivative operator (3). On the other hand, for $\gamma = 0,\ \nu = 1$, it gives the Caputo fractional derivative operator (4), respectively (see, e.g. [5]), [22]].

The Laplace Transform of Hilfer-Prabhakar derivative (12) is given by [22, Eq. 20]

$$\begin{aligned} L\left\{\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(t);s\right\} &= L\left\{\mathbb{P}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+}\frac{d}{dt}\left(\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(t)\right)\right\}(s) \\ &= s^{\mu}[1-\omega s^{-\rho}]^{\gamma}L[u](s) - s^{-\nu(1-\mu)}[1-\omega s^{-\rho}]^{\gamma\nu}\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(t)\right]_{t=0^+} \end{aligned} \tag{13}$$

A symmetric fractional Laplace operator of order λ is defined by Brockmann and Sokolov [4, Eq. A.7–A.9] as

$$\Delta^{\frac{\lambda}{2}} \equiv \frac{1}{2\cos\left(\frac{\pi\lambda}{2}\right)}\{{}_{-\infty}D^{\lambda}_{x} + {}_{x}D^{\lambda}_{\infty}\}, \quad 0<\lambda\leqslant 2 \tag{14}$$

where

$${}_{-\infty}D^{\lambda}_{x}(u(x)) = \frac{1}{k-\lambda}\int_{-\infty}^{x}\frac{u^{(k)}(t)}{(x-t)^{\lambda+1-k}}\,dt, \quad k=[\lambda]+1$$

and

$${}_{x}D^{\lambda}_{\infty}(u(x)) = \frac{1}{k-\lambda}\int_{x}^{\infty}\frac{u^{(k)}(t)}{(x-t)^{\lambda+1-k}}\,dt, \quad k=[\lambda]+1.$$

Further, the Fourier transform of fractional Laplace operator [4, Eq. A.19] is given by

$$F\{\Delta^{\frac{\lambda}{2}}(u(x,t));\eta\} = -|\eta|^{\lambda}F\{u(x,t);\eta\}, \quad 0<\lambda\leqslant 2 \tag{15}$$

where, Fourier transform (see, e.g. Debnath [13, chapter 2]) of function $u(x,t)$ with respect to variable x is defined as

$$F\{u(x,t);\eta\} = u^{*}(\eta,t) = \int_{-\infty}^{\infty} e^{i\eta x}u(x,t)\,dx, \ -\infty<\eta<\infty. \tag{16}$$

The inverse Fourier Transform of function $u^{*}(\eta,t)$ is defined as

$$F^{-1}\{u^{*}(\eta,t)\} = u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x}u^{*}(\eta,t)\,d\eta. \tag{17}$$

3. Unified generalized space time fractional reaction diffusion equation

In this section, we investigate the analytic solution of the generalized space-time fractional reaction-diffusion equation involving fractional Laplace operator contained in the following theorem:

THEOREM 1. *Consider the generalized Cauchy type problem for unified generalized linear space-time reaction-diffusion equation*

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(x,t) = k\Delta^{\frac{\lambda}{2}}(u(x,t)) + cu(x,t) + b\varphi(x,t), \;\; t>0, \;\; x\in\mathbb{R}, \tag{18}$$

with initial condition

$$\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(x,0^+)\right] = g(x) \tag{19}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \;\; t>0 \tag{20}$$

with $\mu\in(0,1)$, $\nu\in[0,1]$, $\omega\in\mathbb{R}$, $\rho>0$, $\gamma\geqslant 0$ and $k>0$ is diffusion coefficient.

Here, $\mathbb{P}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}$ is the Hilfer-Prabhakar fractional derivative operator as defined in (12). $\Delta^{\frac{\lambda}{2}}$ is the fractional generalized Laplace operator of order λ, where $0<\lambda\leqslant 2$, $u(x,t)$ represent source or sink. $\varphi(x,t)$ and $g(x)$ are both sufficiently well behaved functions and b, c are arbitrary constants.

Then the solution of Eq. (18), subject to the above constraints, is given by

$$\begin{aligned} u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty} t^{\mu(n+1)-\nu(\mu-1)-1}E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho})\int_{-\infty}^{\infty}(c-k|\eta|^{\lambda})^{n}e^{-i\eta x}g^*(\eta)d\eta \\ + \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t (c-k|\eta|^{\lambda})^{n}\tau^{\mu(n+1)-1}E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\,\varphi^*(\eta,t-\tau)e^{-i\eta x}d\eta\, d\tau, \end{aligned} \tag{21}$$

where $g^*(\eta)$ and $\varphi^*(\eta,t)$ are Fourier transforms of the functions $g(x)$ and $\varphi(x,t)$ respectively.

Proof. In order to prove the theorem, we take the Fourier transform of Eq. (18) with respect to the space variable x and using boundary condition (20) and Eq. (15) therein, to obtain

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}(u^*(\eta,t)) = -k|\eta|^{\lambda}(u^*(\eta,t)) + cu^*(\eta,t) + b\varphi^*(\eta,t), \;\; t>0 \tag{22}$$

where $u^*(\eta,t)$ is the Fourier transform of the function $u(x,t)$.

Now, taking Laplace transform of (22) with respect to variable t and making use of the Eq. (13), we get

$$\begin{aligned} s^{\mu}[1-\omega s^{-\rho}]^{\gamma}\overline{u^*}(\eta,s) - s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu}\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u^*(\eta,0^+)\right] \\ = -k|\eta|^{\lambda}\overline{u^*}(\eta,s) + c\overline{u^*}(\eta,s) + b\overline{\varphi^*}(\eta,s) \end{aligned} \tag{23}$$

where $L[u(\eta,t);s]=\bar{u}(\eta,s)$.

Next, taking the Fourier transform of the initial condition (19) and putting in (33), we get

$$s^{\mu}[1-\omega s^{-\rho}]^{\gamma}\bar{u^*}(\eta,s)-s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu}g^*(\eta)=-k|\eta|^{\lambda}\bar{u^*}(\eta,s)+c\bar{u^*}(\eta,s)+b\bar{\varphi^*}(\eta,s).$$

Simplifying,

$$\left[s^{\mu}(1-\omega s^{-\rho})^{\gamma}+k|\eta|^{\lambda}-c\right]\bar{u^*}(\eta,s)=s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu}g^*(\eta)+b\bar{\varphi^*}(\eta,s),$$

which gives

$$\bar{u^*}(\eta,s)=\frac{s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu}g^*(\eta)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}+k|\eta|^{\lambda}-c}+\frac{b\bar{\varphi^*}(\eta,s)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}+k|\eta|^{\lambda}-c}. \tag{24}$$

Hence,

$$\begin{aligned}\bar{u^*}(\eta,s)=&s^{-\mu+\nu(\mu-1)}(1-\omega s^{-\rho})^{-\gamma(1-\nu)}g^*(\eta)\left[1+\frac{k|\eta|^{\lambda}-c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\right]^{-1}\\&+bs^{-\mu}(1-\omega s^{-\rho})^{-\gamma}\bar{\varphi^*}(\eta,s)\left[1+\frac{k|\eta|^{\lambda}-c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\right]^{-1}.\end{aligned} \tag{25}$$

Finally,

$$\begin{aligned}\bar{u^*}(\eta,s)=&\sum_{n=0}^{\infty}(c-k|\eta|^{\lambda})^n s^{-\mu(n+1)+\nu(\mu-1)}(1-\omega s^{-\rho})^{-\gamma[(n+1)-\nu]}g^*(\eta)\\&+b\sum_{n=0}^{\infty}(c-k|\eta|^{\lambda})^n s^{-\mu n}(1-\omega s^{-\rho})^{-\gamma n}\bar{\varphi^*}(\eta,s),\quad\left(\left|\frac{k|\eta|^{\lambda}-c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\right|<1\right).\end{aligned} \tag{26}$$

On taking inverse Laplace transform of Eq. (26) and using convolution theorem, we get

$$\begin{aligned}u^*(\eta,t)=&\sum_{n=0}^{\infty}(c-k|\eta|^{\lambda})^n t^{\mu(n+1)-\nu(\mu-1)-1}E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho})g^*(\eta)\\&+b\int_0^t(c-k|\eta|^{\lambda})^n\tau^{\mu(n+1)-1}E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\varphi^*(\eta,t-\tau)d\tau.\end{aligned} \tag{27}$$

Further, taking the inverse Fourier transform of (27), we get

$$\begin{aligned}u(x,t)=&\frac{1}{2\pi}\sum_{n=0}^{\infty}t^{\mu(n+1)-\nu(\mu-1)-1}E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho})\int_{-\infty}^{\infty}(c-k|\eta|^{\lambda})^n e^{-i\eta x}g^*(\eta)d\eta\\&+\frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t(c-k|\eta|^{\lambda})^n\tau^{\mu(n+1)-1}E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\varphi^*(\eta,t-\tau)e^{-i\eta x}d\eta\,d\tau\end{aligned}$$

where $g^*(\eta)$ and $\varphi^*(\eta,t)$ are Fourier transforms of the functions $g(x)$ and $\varphi(x,t)$, respectively. □

It is interesting to observe that as an particular case of Theorem 1, we can obtain solution of homogeneous Schrödinger equation occurring in the quantum mechanics, solution of non homogeneous fractional generalized diffussion wave equation and the solution of fractional partial differential equation that arises in the study of heat transfer through diathermanous materials.

(1) If we set $\gamma = 0$ then the Hilfer-Prabhakar fractional derivative (12) reduces to a Hilfer fractional derivative (10) and we get the following result:

THEOREM 2. *Consider the generalized Cauchy type problem for fractional linear space-time reaction-diffusion equation*

$$\mathbb{D}_t^{\mu,\nu} u(x,t) = k\Delta^{\frac{\lambda}{2}}(u(x,t)) + cu(x,t) + b\varphi(x,t), \;\; t>0, \;\; x\in\mathbb{R}, \tag{28}$$

with initial condition

$$\mathbb{I}_{0+}^{(1-\nu)(1-\mu)} u(x,0^+) = \left[\mathbb{P}^{0}_{\rho,(1-\nu)(1-\mu),\omega,0^+} u(x,0^+)\right] = g(x), \;\; x\in\mathbb{R} \tag{29}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \;\; t>0 \tag{30}$$

with $\mu\in(0,1),\ \nu\in[0,1],\ \omega\in\mathbb{R},\ \rho>0,\ 0<\lambda\leqslant 2.$

Then, the solution of (28) is given by

$$\begin{aligned} u(x,t) = &\frac{t^{(\mu-1)(1-\nu)}}{2\pi}\int_{-\infty}^{\infty} g^*(\eta)E_{\mu,\mu+\nu(1-\nu)}(c-k|\eta|^{\lambda})t^{\mu}e^{-i\eta x}d\eta \\ &+\frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t \xi^{\mu-1}E_{\mu,\mu}(c-k|\eta|^{\lambda})t^{\mu}\varphi^*(\eta,t-\xi)d\xi d\eta \end{aligned} \tag{31}$$

where $g^*(\eta)$ *and* $\varphi^*(\eta,t)$ *are Fourier transforms of the functions* $g(x)$ *and* $\varphi(x,t)$*, respectively and* $E_{\rho,\mu}(\cdot)$ *is the two parameter Mittag-Leffler function.*

Proof. In order to prove the theorem, we take the Fourier transform of Eq. (18) with respect to the space variable x and using boundary condition (30) and Eq. (15) therein, to obtain

$$\mathbb{D}_t^{\mu,\nu}(u^*(\eta,t)) = -k|\eta|^{\lambda}(u^*(\eta,t)) + cu^*(\eta,t) + b\varphi^*(\eta,t), \;\; t>0 \tag{32}$$

where $u^*(\eta,t)$ is the Fourier transform of the function $u(x,t)$.

Now, taking Laplace transform of (22) with respect to variable t and making use of the Eq. (11), we get

$$s^{\mu}\bar{u}^*(\eta,s) - s^{\nu(\mu-1)}I_{0+}^{(1-\nu)(1-\mu)}u(\eta,0+) = -k|\eta|^{\lambda}\bar{u}^*(\eta,s) + c\bar{u}^*(\eta,s) + b\bar{\varphi}^*(\eta,s) \tag{33}$$

where $L[u(\eta,t);s] = \bar{u}(\eta,s)$.

Next, taking the Fourier transform of the initial condition (29) and putting in (33), we get

$$s^{\mu}\bar{u^{*}}(\eta,s) - s^{\nu(\mu-1)}u(\eta,0+)g^{*}(\eta) = -k|\eta|^{\lambda}\bar{u^{*}}(\eta,s) + c\bar{u^{*}}(\eta,s) + b\bar{\varphi^{*}}(\eta,s)$$

Simplifying,

$$\left[s^{\mu} + k|\eta|^{\lambda} - c\right]\bar{u^{*}}(\eta,s) = s^{\nu(\mu-1)}g^{*}(\eta) + b\bar{\varphi^{*}}(\eta,s),$$

which gives

$$\bar{u^{*}}(\eta,s) = \frac{s^{\nu(\mu-1)}g^{*}(\eta)}{s^{\mu} + k|\eta|^{\lambda} - c} + \frac{b\bar{\varphi^{*}}(\eta,s)}{s^{\mu} + k|\eta|^{\lambda} - c}. \tag{34}$$

On taking inverse Laplace transform of equation (51), by means of the following result by haubold et al ([8], Eq. 18) and using convolution theorem,

$$L^{-1}\left\{\frac{s^{\beta-1}}{s^{\alpha}+a}\right\} = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-at^{\alpha}) \tag{35}$$

where $\Re(s) > 0$, $\Re(\alpha) > 0$, $\Re(\alpha-\beta) > -1$, we obtain

$$\begin{aligned} u^{*}(\eta,s) =& t^{(\mu-1)(1-\nu)}E_{\mu,\mu+\nu(1-\nu)}(c-k|\eta|^{\lambda})t^{\mu}g^{*}(\eta) \\ &+ \int_{0}^{t}\xi^{\mu-1}E_{\mu,\mu}(c-k|\eta|^{\lambda})t^{\mu}\varphi^{*}(\eta,t-\xi)d\xi \end{aligned} \tag{36}$$

Further, taking the inverse Fourier transform of (27), we get

$$\begin{aligned} u(x,t) = & \frac{t^{(\mu-1)(1-\nu)}}{2\pi}\int_{-\infty}^{\infty}g^{*}(\eta)E_{\mu,\mu+\nu(1-\nu)}(c-k|\eta|^{\lambda})t^{\mu}e^{-i\eta x}d\eta \\ &+ \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t}\xi^{\mu-1}E_{\mu,\mu}(c-k|\eta|^{\lambda})t^{\mu}\varphi^{*}(\eta,t-\xi)d\xi d\eta. \end{aligned}$$

(2) Further, on taking, $c = 0$ and $k = \frac{ih}{2m}$, the above result yields the solution of the non-homogenous fractional generalized Schrödinger equation considered in Corollary 3.1 by Purohit [29]. □

COROLLARY 1. *Consider the following one dimensional non-homogenous generalized fractional Schrödinger equation of a particle of mass m, defined by*

$$\mathbb{D}_{t}^{\mu,\nu}u(x,t) = \left(\frac{ih}{2m}\right)\Delta^{\frac{\lambda}{2}}u(x,t) + b\varphi(x,t), \quad t > 0,\ 0 < \lambda \leqslant 2\ x \in \mathbb{R}, \tag{37}$$

with initial condition

$$\mathbb{I}_{0+}^{(1-\nu)(1-\mu)}u(x,0+) = g(x), \quad -\infty < x < \infty, \quad 0 < \mu < 1, \quad 0 \leqslant \nu \leqslant 1 \tag{38}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \ \ t > 0, \tag{39}$$

where b is arbitrary, $h = 2\pi\bar{h}$ is the Plank constant and $g(x)$ and $\varphi(x,t)$ are given functions.

Then, the solution of (37), under the given conditions, is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t)g(\xi)\,d\xi + b\int_0^t (t-\tau)\left[\int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)\varphi(\xi,\tau)d\xi\right]d\tau, \tag{40}$$

where the Green's function $G_1(x,t)$ is given by

$$G_1(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\lambda|x|} H_{3,3}^{2,1}\left[\frac{|x|}{a^{\frac{1}{\lambda}} t^{\frac{\mu}{\lambda}}}\middle|\begin{matrix}(1,\frac{1}{\lambda}),(\mu+\nu(1-\mu),\frac{\mu}{\lambda}),(1,\frac{1}{2})\\ (1,\frac{1}{\lambda}),(1,1),(1,\frac{1}{2})\end{matrix}\right] \tag{41}$$

and the function $G_2(x,t)$ is given by

$$G_2(x,t) = \frac{1}{\lambda|x|} H_{3,3}^{2,1}\left[\frac{|x|}{a^{\frac{1}{\lambda}} t^{\frac{\mu}{\lambda}}}\middle|\begin{matrix}(1,\frac{1}{\lambda}),(\mu,\frac{\mu}{\lambda}),(1,\frac{1}{2})\\ (1,\frac{1}{\lambda}),(1,1),(1,\frac{1}{2})\end{matrix}\right] \tag{42}$$

where $a = \frac{ih}{2m}$ and $H_{p,q}^{m,n}$ is well known H-function defined by (see, e.g. Mathai et al. [1, Eq. Chapter 1]).

(3) On taking $c = 0$ and $k = \psi^2$, in Eq. (28) we get the solution of non-homogenous fractional generalized diffusion wave equation considered in Corollary 3.2 by Purohit [29].

COROLLARY 2. *Consider the following one dimensional non-homogenous generalized fractional diffusion wave equation, defined by*

$$\mathbb{D}_t^{\mu,\nu} u(x,t) = \psi^2 \Delta^{\frac{\lambda}{2}} u(x,t) + b\varphi(x,t), \ \ t > 0, \ \ 0 < \lambda \leqslant 2, \ \ x \in \mathbb{R}, \tag{43}$$

with initial condition

$$\mathbb{I}_{0+}^{(1-\nu)(1-\mu)} u(x,0^+) = g(x), \quad -\infty < x < \infty, \quad 0 < \mu < 1, \quad 0 \leqslant \nu \leqslant 1 \tag{44}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \ \ t > 0 \tag{45}$$

where b is arbitrary constant and $g(x)$ and $\varphi(x,t)$ are given functions.

Then, the solution of (43) under the given conditions, is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t)g(\xi)\,d\xi + b\int_0^t (t-\tau)\left[\int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)\varphi(\xi,\tau)d\xi\right]d\tau, \tag{46}$$

where the Green's function $G_1(x,t)$ and $G_2(x,t)$ are, respectively, given by (41) and (42) with $a = \psi^2$.

(4) On taking $b = 0$, $c = 0$ and $\lambda = 2$ in Theorem 1, we arrive at the following result by Garra *et al.* [22, Theorem 5.1]:

COROLLARY 3. *Consider the Cauchy problem*

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+} u(x,t) = k\frac{\partial^2}{\partial x^2} u(x,t), \ \ t > 0, \ \ x \in \mathbb{R} \tag{47}$$

with initial condition

$$\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu),(1-\mu),\omega,0^+} u(x,0^+)\right] = g(x) \tag{48}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0 \tag{49}$$

with $\mu \in (0,1), \ \nu \in [0,1], \ \omega \in \mathbb{R}, \ \rho > 0, \ \gamma \geqslant 0.$

Then, the solution of equation (47) is given by

$$u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-k)^n t^{\mu(n+1)-\nu(\mu-1)-1} E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho}) \int_{-\infty}^{\infty} \eta^{2n}\cos\eta x g^*(\eta)d\eta, \tag{50}$$

where $g^*(\eta)$ is the Fourier transform of the function $g(x)$.

(5) Further, if we take $\gamma = 0$, $c = 0$, $k = \alpha$, $b = \beta$ and $\varphi(x,t) = e^{-\tau x}$, Theorem 2 yields the solution of fractional partial differential equation arising in the study of heat transfer through diathermanous materials considered by Kachhia and Prajapati [11].

COROLLARY 4. *Consider the fractional partial differential equation that arise in the study of heat transfer through diathermanous materials as*

$$\mathbb{D}^{\mu,\nu}_t u(x,t) = \alpha\Delta^{\frac{\lambda}{2}} u(x,t) + \beta e^{-\tau x}, \ \ 0 < \lambda \leqslant 2 \tag{51}$$

with initial condition

$$\mathbb{I}^{(1-\nu)(1-\mu)}_{0+} u(x,0^+) = 0, \tag{52}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \ \ t > 0. \tag{53}$$

with $\mu \in (0,1), \ \nu \in [0,1], \ \alpha > 0.$

Then, the solution of (51) under the given conditions, is given by

$$u(x,t) = \frac{\beta t^{\mu} e^{-\tau x}}{\lambda}\int_{-\infty}^{\infty}\frac{e^{\tau\mu}}{|\xi|} H^{2,1}_{3,3}\left[\frac{|\xi|}{\alpha^{\frac{1}{\tau}} t^{\frac{\mu}{\lambda}}}\middle|\begin{matrix}(1,\frac{1}{\lambda}),(\mu+1,\frac{\mu}{\lambda}),(1,\frac{1}{2})\\(1,\frac{1}{\lambda}),(1,1),(1,\frac{1}{2})\end{matrix}\right] d\xi \tag{54}$$

4. Illustrative examples

EXAMPLE 1. Consider the generalized Cauchy type problem for unified generalized linear space-time reaction-diffusion equation

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(x,t) = k\Delta^{\frac{\lambda}{2}}(u(x,t)) + cu(x,t) + b\varphi(x,t), \; t > 0, \; x \in \mathbb{R}, \tag{55}$$

with initial condition

$$\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(x,0^+)\right] = e^{-x} \tag{56}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \; t > 0 \tag{57}$$

with $\mu \in (0,1)$, $\nu \in [0,1]$, ω, $k \in \mathbb{R}$, k, $\rho > 0$, $\gamma \geqslant 0$, $0 < \lambda \leqslant 2$.

In view of Theorem 1, the solution of equation (55) is given by

$$\begin{aligned} u(x,t) = &\frac{1}{2\pi}\sum_{n=0}^{\infty} t^{\mu(n+1)-\nu(\mu-1)-1}E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho})\int_{-\infty}^{\infty}(c-k|\eta|^{\lambda})^n e^{-i\eta x}G(\eta)d\eta \\ &+\frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t (c-k|\eta|^{\lambda})^n \tau^{\mu(n+1)-1}E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\varphi^*(\eta,t-\tau)e^{-i\eta x}d\eta\, d\tau \end{aligned} \tag{58}$$

where $\varphi^*(\eta,t)$ is Fourier transform of the functions $\varphi(x,t)$ and $G(\eta) = F\{e^{-x};\eta\} = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(1+i\eta)}-1}{1+i\eta}\right]$.

Next, we take an example where, in the initial condition, we put $g(x) = \delta(x)$, the Dirac delta function.

EXAMPLE 2. Consider the generalized Cauchy type problem for unified generalized linear space-time reaction-diffusion equation

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(x,t) = k\Delta^{\frac{\lambda}{2}}(u(x,t)) + cu(x,t) + b\varphi(x,t), \quad t > 0, \; x \in \mathbb{R}, \tag{59}$$

with initial condition

$$\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(x,0^+)\right] = \delta(x), \tag{60}$$

where $\delta(x)$ is the Dirac delta function and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \; t > 0 \tag{61}$$

with $\mu \in (0,1)$, $\nu \in [0,1]$, $\omega \in \mathbb{R}$, k, $\rho > 0$, $\gamma \geqslant 0$, $0 < \lambda \leqslant 2$.

In view of Theorem 1, the solution of equation (59), is given by

$$u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty} t^{\mu(n+1)-\nu(\mu-1)-1} E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho}) \int_{-\infty}^{\infty} (c-k|\eta|^{\lambda})^{n} e^{-i\eta x} d\eta$$
$$+ \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} (c-k|\eta|^{\lambda})^{n} \tau^{\mu(n+1)-1} E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho}) \varphi^{*}(\eta,t-\tau) e^{-i\eta x} d\eta\, d\tau \tag{62}$$

where $\varphi^{*}(\eta,t)$ is Fourier transform of the function $\varphi(x,t)$ and $F\{\delta(x);\eta\}=1$.

5. Concrete applications

When $\gamma=0$, $\nu=1$, the Hilfer-Prabhakar fractional space derivative (12) get reduced to Caputo fractional derivative (4) and it yields the following result:

COROLLARY 5. *Consider the generalized Cauchy type problem for fractional linear space-time reaction-diffusion equation*

$${}^{C}_{0}\mathbb{D}^{\mu}_{t} u(x,t) = k\Delta^{\frac{\lambda}{2}}(u(x,t)) + cu(x,t) + b\varphi(x,t), \;\; t>0, \;\; x\in\mathbb{R}, \tag{63}$$

with initial condition

$$\mathbb{I}^{(1-\mu)}_{t} u(x,0^{+}) = g(x), \tag{64}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \;\; t>0 \tag{65}$$

with $\mu\in(0,1)$, $0<\lambda\leqslant 2$.

Then the solution of equation (63), is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g^{*}(\eta) E_{\mu,0}(c-k|\eta|^{\lambda}) t^{\mu} e^{-i\eta x} d\eta$$
$$+ \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} \xi^{\mu-1} E_{\mu,\mu}(c-k|\eta|^{\lambda}) t^{\mu} \varphi^{*}(\eta,t-\xi) d\xi\, d\eta.$$

where $g^{*}(\eta)$ *and* $\varphi^{*}(\eta,t)$ *are Fourier transform of the functions* $g(x)$ *and* $\varphi(x,t)$ *respectively and* $E_{\rho,\mu}(\cdot)$ *is the two parameter Mittag- Leffler function.*

On taking $\gamma=0$, $\nu=0$, the Hilfer-Prabhakar fractional derivative (12) reduces to a Riemann-Liouville fractional derivative (3) and the Theorem 2 yields the following corollary:

COROLLARY 6. *Consider the generalized Cauchy type problem for fractional linear space-time reaction-diffusion equation*

$${}^{RL}_{0}\mathbb{D}^{\mu}_{t} u(x,t) = k\Delta^{\frac{\lambda}{2}}(u(x,t)) + cu(x,t) + b\varphi(x,t), \;\; t>0, \;\; x\in\mathbb{R}, \tag{66}$$

with initial condition

$$\mathbb{I}_t^{(1-\mu)} u(x,0^+) = g(x), \tag{67}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0, \;\; t > 0 \tag{68}$$

with $\mu \in (0,1), \; 0 < \lambda \leqslant 2$.

Then, the solution of equation (66) is given by

$$\begin{aligned} u(x,t) = {} & \frac{t^{(\mu-1)}}{2\pi} \int_{-\infty}^{\infty} g^*(\eta) E_{\mu,\mu}(c - k|\eta|^{\lambda}) t^{\mu} e^{-i\eta x} d\eta \\ & + \frac{b}{2\pi} \int_{-\infty}^{\infty} \int_0^t \xi^{\mu-1} E_{\mu,\mu}(c - k|\eta|^{\lambda}) t^{\mu} \varphi^*(\eta, t-\xi) d\xi d\eta. \end{aligned} \tag{69}$$

where $g^*(\eta)$ and $\varphi^*(\eta,t)$ are Fourier transforms of the functions $g(x)$ and $\varphi(x,t)$, respectively and $E_{\rho,\mu}(\cdot)$ is the two parameter Mittag-Leffler function.

6. Conclusion

The solution of a unified generalized linear space-time fractional reaction-diffusion equation involving Hilfer-Prabhakar time fractional derivative and the space fractional generalized Laplace operators is obtained in terms of Mittag-Leffler function by using Laplace transform and Fourier transform. This Method is very useful for studying the various problems arising in fluid dynamics, control theory, aerodynamics and applied sciences. The analytic solutions are the exact solutions. Efficient numerical technique can be developed to find the solution of the fractional PDE by considering their analytic solutions as base.

Acknowledgement. Authors are thankful to the referees for their useful suggestions for the improvement of the paper.

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(Received March 22, 2016)

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