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POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS OF N–DIMENSION NONLINEAR FRACTIONAL DIFFERENTIAL SYSTEM WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study existence of positive solutions to the system of three-point fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha_i} u(t) + \lambda_i a_i(t) f_i(u_1(t), \dots, u_n(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha_i \leqslant 3 \\ u_i(0) = u_i'(0) = 0 \\ u_i'(1) - \mu_i u_i'(\eta_i) = \int_0^1 \phi_i(s) u_i'(s) ds \end{cases}$$

where for $i = 1, \dots, n$, λ_i is a positive parameter, $D_{0+}^{\alpha_i}$ is the standard Riemann-Liouville differential operator of order $\alpha_i \in (2,3]$, $\eta_i \in (0,1)$, $\mu_i \geqslant 0$, $f_i : [0,+\infty)^n \to [0,+\infty)$ is a continuous function and $\phi_i : (0,1) \to (0,+\infty)$ is a continuous increasing function and $\int_0^1 s^{\alpha_i - 2} \phi_i(s) ds < +\infty$. Existence results are obtained by means of Krasnosel'skii 's fixed point theorem and the vector version of Krasnosel'skii's fixed point theorem.

1. Introduction

In this paper, we concentrate on the study of existence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations (FBVP for short) with integral boundary conditions boundary conditions of the form

$$\begin{cases} D_{0+}^{\alpha_i} u_i(t) + \lambda_i a_i(t) f_i(u_1(t), u_2(t), \dots, u_n(t)) = 0, \;\; 0 < t < 1, \\ u_i(0) = u_i'(0) = 0, \;\; u_i'(1) - \mu_i u_i'(\eta_i) = \int_0^1 \phi_i(t) u_i'(t) dt, \end{cases} \tag{1.1}$$

where, for each $i = 1, \dots, n$, $D_{0+}^{\alpha_i}$, denote the standard Riemann–Liouville fractional derivatives of order $\alpha_i \in (2,3]$, μ_i, η_i satisfy $\mu_i > 0$, $\eta_i \in (0,1)$, $f_i \in C\left(\mathbb{R}_+^n, \mathbb{R}_+\right)$, $a_i \in C([0,1], \mathbb{R}_+)$, $\mathbb{R}_+ = [0, +\infty)$ and λ_i are positive parameters.

Throughout this paper, we assume that the following conditions hold.

$\phi_i : (0,1) \to (0,+\infty)$ are continuous functions and

$$\rho_i = \int_0^1 s^{\alpha_i - 2} \phi_i(s) ds < +\infty, \;\; i = 1, \dots, n. \tag{1.2}$$

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Assume that

$$d_i = 1 - \mu_i \eta_i^{\alpha_i - 2} - \int_0^1 s^{\alpha_i - 2} \phi_i(s) ds > 0, \ \ i = 1, \ldots, n. \tag{1.3}$$

Fractional differential equations can describe many phenomena in various fields of engineering and scientific disciplines such as control theory, physics, chemistry, biology, economics, mechanics and electromagnetic. In consequence, the subject of fractional differential equations is gaining much importance and attention. In recent years, there are a large number of papers dealing with the existence and uniqueness oluof stions of boundary value problems for nonlinear differential equations of fractional order. For examples and recent development of the topic, see ([2, 6, 14, 15, 18]) and references therein. In addition, the study of coupled systems involving fractional differential equations is also of great importance. Such systems occur in various problems of applied science, for instance, we refer the reader to ([1, 4, 7, 9, 13, 16, 17, 19]) and the references therein.

The rest of this paper is organized as follows: In section 2, we present some preliminaries and lemmas that will be used to prove our main results. Section 3 is devoted to prove the existence of positive solutions for FBVP (1.1). In section 4 some examples illustrating our results are also presented.

2. Background and preliminary lemmas

In this section, we introduce some basic definitions of fractional derivative for the readers' convenience. We also state in this section the classical and the vector version of Krasnosel'skii's fixed point theorem.

DEFINITION 2.1. [11] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha - 1} f(s) ds, \tag{2.1}$$

where $\Gamma(\alpha)$ is the gamma function, provided that the right side is pointwise defined on $(0, +\infty)$.

DEFINITION 2.2. [11] The Riemann-Liouville fractional derivative of order $\alpha > 0$, of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha - n + 1}} ds, \tag{2.2}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

The following results on fractional integral and fractional derivative will be needed in establishing our main results.

LEMMA 2.3. [11] *Let* $\alpha > 0$. *If* $u \in C(0,1) \cap \mathbb{L}(0,1)$, *then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0 \tag{2.3}$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, *as unique solutions, where* $n = [\alpha]+1$.

LEMMA 2.4. [11] *Assume that* $u \in C(0,1) \cap \mathbb{L}(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap \mathbb{L}(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}, \tag{2.4}$$

for some $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, *where* $n = [\alpha]+1$.

Now we present the Green's functions for system associated with boundary value problem (1.1).

LEMMA 2.5. (see [3]) *Let* $h \in C[0,1]$ *be a given function and* $2 < \alpha \leqslant 3$ *then the unique solution of FBVP*

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \ \ 0 < t < 1, \tag{2.5}$$

$$u(0) = u'(0) = 0, \ u'(1) - \mu u'(\eta) = \int_0^1 \phi(s) u'(s) ds, \tag{2.6}$$

is given by

$$u(t) = \int_0^1 H_{\alpha}(t,s) h(s) ds + \tfrac{t^{\alpha-1}}{d} \int_0^1 \phi(s) \left(\int_0^1 G_{1\alpha}(s,\tau) h(\tau) d\tau \right) ds, \tag{2.7}$$

where

$$d = \left(1 - \mu \eta^{\alpha-2} - \int_0^1 s^{\alpha-2} \phi(s) ds \right)$$

$$H_{\alpha}(t,s) = G_{\alpha}(t,s) + \frac{\mu t^{\alpha-1}}{d} G_{1\alpha}(\eta,s) \tag{2.8}$$

$$G_{\alpha}(t,s) = \tfrac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1 \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leqslant t \leqslant s \leqslant 1 \end{cases} \tag{2.9}$$

$$G_{1\alpha}(\eta,s) = \tfrac{1}{\Gamma(\alpha)} \begin{cases} \eta^{\alpha-2}(1-s)^{\alpha-2} - (\eta-s)^{\alpha-2}, & 0 \leqslant s \leqslant \eta \leqslant 1 \\ \eta^{\alpha-2}(1-s)^{\alpha-2}, & 0 \leqslant \eta \leqslant s \leqslant 1 \end{cases} \tag{2.10}$$

The following results give some properties of the Green functions $G_{\alpha}(t,s)$, $G_{1\alpha}(\eta,s)$, and $H_{\alpha}(t,s)$ in order to discuss the existence of positive solutions.

LEMMA 2.6. (see [3]) $G_{\alpha}(t,s)$ *defined by* (2.9) *satisfies the following conditions:*

(P1) $G_{\alpha}(t,s)$ *is continuous for all* $t,s \in [0,1]$, $G_{\alpha}(t,s) > 0$, *for* $t,s \in (0,1)$,

(P2) $\dfrac{\partial G_\alpha(t,s)}{\partial t} = (\alpha-1)G_{1\alpha}(t,s)$ *for* $t,s \in (0,1)$,

(P3) $t^{\alpha-1}G_\alpha(t,s) \leqslant G_\alpha(t,s) \leqslant G_\alpha(1,s)$, $(t,s)\in[0,1]\times[0,1]$,

(P5) For $l \in (0,1)$, $\min_{l\leqslant t\leqslant 1} G_\alpha(t,s) \geqslant (l)^{\alpha-1}G_\alpha(1,s)$, *where* $G_\alpha(1,s) = \dfrac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)}$.

LEMMA 2.7. (see [3]) $G_{1\alpha}(t,s)$ *defined by* (2.10) *satisfies the following conditions:*

(P1) $G_{1\alpha}(\eta,s) > 0$ *for* $t,s\in(0,1)$,

(P2) $0 \leqslant G_{1\alpha}(\eta,s) \leqslant \frac{1}{\Gamma(\alpha)}\eta^{\alpha-2}(1-s)^{\alpha-2}$.

LEMMA 2.8. (see [3]) $H_\alpha(t,s)$ *defined by* (2.8) *satisfies the following conditions:*

(P1) $t^{\alpha-1}H_\alpha(1,s) \leqslant H_\alpha(t,s) \leqslant H_\alpha(1,s)$, $(t,s)\in[0,1]\times[0,1]$,

(P2) For $l\in(0,1)$, $\min_{l\leqslant t\leqslant 1} H_\alpha(t,s) \geqslant (l)^{\alpha-1}H_\alpha(1,s)$.

LEMMA 2.9. *The unique solution* $u(t)$ *of the FBVP* (2.5)–(2.6) *is nonnegative and satisfies*

$$\min_{l\leqslant t\leqslant 1} u(t) \geqslant (l)^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u(t)|,\quad \forall t\in[0,1],\ \ l\in(0,1). \tag{2.11}$$

Proof. It is obvious that $u(t)$ is nonnegative. For any $t\in[0,1]$, by (2.7) and Lemma 2.8, it follows that

$$\begin{aligned} u(t) &= \int_0^1 H_\alpha(t,s)h(s)ds + \frac{t^{\alpha-1}}{d}\int_0^1 \phi(s)\left(\int_0^1 G_{1\alpha}(s,\tau)h(\tau)d\tau\right)ds \\ &\leqslant \int_0^1 H_\alpha(1,s)h(s)ds + \tfrac{1}{d}\int_0^1 \phi(s)\left(\int_0^1 G_{1\alpha}(s,\tau)h(\tau)d\tau\right)ds \end{aligned}$$

and thus,

$$\max_{0\leqslant t\leqslant 1}|u(t)| \leqslant \int_0^1 H_\alpha(1,s)h(s)ds + \tfrac{1}{d}\int_0^1 \phi(s)\left(\int_0^1 G_{1\alpha}(s,\tau)h(\tau)d\tau\right)ds,$$

On the other hand, (2.7) and Lemma 2.8 imply that, for any $t\in[l,1]$,

$$\begin{aligned} u(t) &= \int_0^1 H_\alpha(t,s)h(s)ds + \frac{t^{\alpha-1}}{d}\int_0^1 \phi(s)\left(\int_0^1 G_{1\alpha}(s,\tau)h(\tau)d\tau\right)ds \\ &\geqslant t^{\alpha-1}\left[\int_0^1 H_\alpha(1,s)h(s)ds + \tfrac{1}{d}\int_0^1 \phi(s)\left(\int_0^1 G_{1\alpha}(s,\tau)h(\tau)d\tau\right)ds\right] \\ &\geqslant (l)^{\alpha-1}\max_{0\leqslant t\leqslant 1}|u(t)| \end{aligned}$$

Therefore,

$$\min_{l\leqslant t\leqslant 1} u(t) \geqslant (l)^{\alpha-1} \max_{0\leqslant t\leqslant 1} |u(t)|.$$

This complete the proof of Lemma 2.9. □

In the rest of this paper we shall use the following notions and notations. If $(X,\|\cdot\|)$ is a normed linear space, by a cone of X we mean a closed convex subset $K\subset X$ with $K\setminus\{0\}\neq\emptyset$, $\lambda K\subset K$ for every $\lambda\in\mathbb{R}_+$ and $K\cap(-K)=\{0\}$. Any cone K induces a partial order relation in X denoted by $\leqslant$, that is $u\leqslant v$ if and only if $v-u\in K$.

We shall say that $u<v$ if $v-u\in K\setminus\{0\}$ and $u\nless u\notin K\setminus\{0\}$. Finally $u\leqslant v$ means $v\geqslant u$.

Now, we cite the classical Krasnosel'skii's cone fixed point theorem (see [8, 10]) to be used in Section 3.

THEOREM 2.10. *Let $(X,\|\cdot\|)$ be a normed linear space, $K\subset X$ a cone, $0<r<R$ two real numbers and $K_{r,R}=\{u\in K: r\leqslant u\leqslant R\}$. Let $\mathscr{T}: K_{r,R}\to K$ be a compact map such that one of the following conditions is satisfied:*

(a) $\mathscr{T}u\leqslant u$ if $\|u\|=r$ and $\mathscr{T}u\geqslant u$ if $\|u\|=R$;

(b) $\mathscr{T}u\geqslant u$ if $\|u\|=r$ and $\mathscr{T}u\leqslant u$ if $\|u\|=R$.

Then $\mathscr{T}$ has a fixed point u in $K_{r,R}$.

Next we recall the recent vector version of Krasnosel'skii's cone fixed point theorem (see [5, 12]).

Before to state it, we introduce a few notations. We shall consider n cones K_i, $(i=1,\dots,n)$ of X and the corresponding $K=K_1\times K_2\times\ldots\times K_n$ of X^n.

For $r,R\in\mathbb{R}_+^n, r=(r_1,r_2,\dots,r_n)$, $R=(R_1,R_2,\dots,R_n)$, we write $0<r<R$ if $0<r_i<R_i$ $(i=1,\dots,n)$, and we use the notations:

$$(K_i)_{r_i,R_i} := \{u_i\in K_i : r_i\leqslant\|u\|\leqslant R_i\}(i=1,2,\dots,n),$$
$$K_{r,R} := \{u=(u_1,u_2,\dots,u_n)\in K : r_i\leqslant\|u_i\|\leqslant R_i \text{ for } i=1,2,\dots,n\}.$$

Clearly, $K_{r,R} := (K_1)_{r_1,R_1}\times(K_2)_{r_2,R_2}\times\ldots\times(K_n)_{r_n,R_n}$

THEOREM 2.11. *Let $(E,\|\cdot\|)$ be a normed linear space, $K_1,K_2,\dots,K_n\subset E$ n cones, $K=K_1\times K_2\ \times\ldots\times K_n$ and $r=(r_1,r_2,\dots,r_n)$, $R=(R_1,R_2,\dots,R_n)\in(\mathbb{R}_+)^n$ with $0<r<R$. Let $\mathscr{T}=(\mathscr{T}_1,\mathscr{T}_2,\dots,\mathscr{T}_n): K_{r,R}\to K$ be a compact map. Assume that for each $i\in\{1,2,\dots,n\}$, one of the following conditions is satisfied:*

(a) $\mathscr{T}_iu_i\nless u_i$ if $\|u_i\|=r_i$ and $\mathscr{T}_iu_i\ngtr u_i$ if $\|u_i\|=R_i$;

(b) $\mathscr{T}_iu_i\ngtr u_i$ if $\|u_i\|=r_i$ and $\mathscr{T}_iu_i\nless u_i$ if $\|u_i\|=R_i$.

Then $\mathscr{T}$ has a fixed point $u=(u_1,u_2,\dots,u_n)\in K_{r,R}$.

Now we write the system of boundary value problem (1.1) as an equivalent system of integral equations.

DEFINITION 2.12. The vector-valued function $u=(u_1,u_2,\dots,u_n)$ is called a positive solution of system (1.1) if and only if u satisfies (1.1) and for all $i\in\{1,2,\dots,n\}$, $u_i(t)>0$ for $t\in(0,1)$.

LEMMA 2.13. *Assume that* $f_i : \mathbb{R}_+^n \to \mathbb{R}_+$, $i = 1,\dots,n$, *are continuous. Then* $u = (u_1, u_2, \dots, u_n) \in E$ *is a solution of FBVP* (1.1) *if and only if for all* $i \in \{1,2,\dots,n\}$

$$u_i(t) = \lambda_i \left[\int_0^1 H_{\alpha_i}(t,s)a_i(s)f_i(u(s))ds \right.$$
$$\left. + \frac{t^{\alpha_i - 1}}{d_i}\int_0^1 \phi_i(s)\left(\int_0^1 G_{1\alpha_i}(s,\tau)\,a_i(\tau)f_i(u(\tau))d\tau\right)ds\right],$$

Proof. The proof is immediate from the discussion above, so we omit it. □

Let the Banach space $E = C\left([0,1], \mathbb{R}_+^n\right)$ endowed with the sum-norm

$$\|u\|_E = \sum_{i=1}^n \max_{0\leqslant t\leqslant 1} |u_i(t)| \quad \text{for} \quad u = (u_1, u_2, \dots, u_n),$$

where $\|u\|_{C([0,1],\mathbb{R}_+)} = \max_{0\leqslant t\leqslant 1} |u_i(t)|$.

Based on the estimation (2.11) we define the cones:

$$K_i = \left\{ u_i \in C([0,1], \mathbb{R}_+) : \min_{t\in[l,1]} u_i(t) \geqslant \delta_i \|u_i\| \right\} \quad (i = 1,2,\dots,n),$$

where $\delta_i = (l)^{\alpha_i - 1}, l \in (0,1)$ and the product cone $K = K_1 \times K_2 \times \dots \times K_n$ in E.

Let $\mathscr{T} : E \to E$ be the operator defined as

$$\mathscr{T}(u) = (\mathscr{T}_1 u, \mathscr{T}_2 u, \dots, \mathscr{T}_n u),$$

where for all $t \in [0,1]$,

$$(\mathscr{T}_i u)(t) = \lambda_i \left[\int_0^1 H_{\alpha_i}(t,s)f_i(u(s))ds + \frac{t^{\alpha_i-1}}{d_i}\int_0^1 \phi_i(s)\left(\int_0^1 G_{1\alpha_i}(s,\tau)\,f_i(u(\tau))d\tau\right)ds\right],$$

We can write Lemma 2.9 in the following form: $u = (u_1, u_2, \dots, u_n)$ is a solution of (1.1) if and only if u is a fixed point of the operator $\mathscr{T}$.

LEMMA 2.14. *The operator operator* $\mathscr{T} : E \to E$ *is completely continuous.*

Proof. Due to (2.11) we have the invariance property $\mathscr{T}(K) \subset K$. Moreover, the operator $\mathscr{T}$ is completely continuous since, by standard arguments, the components $\mathscr{T}_i$ are completely continuous. □

3. Existence of solutions

In this section, we will discuss the existence of positive solutions for boundary value problem (1.1). For convenience we introduce the following notations:

($\mathscr{H}_1$) For $\nu = 0$ or $+\infty$ there exist nonnegative constants f_i^ν $(i = 1,2,\dots,n)$ defined as

$$f_i^\nu = \lim_{\|x\|\to\nu} \frac{f_i(x_1,x_2,\dots,x_n)}{\|x\|}, \quad x = (x_1,x_2,\dots,x_n)$$

($\mathscr{H}_2$) For $i = 1,2,\dots,n$ and $l \in (0,1)$. Let

$$\Gamma_i^0 = \begin{cases} \left(\delta^2 B_{\alpha_i} f_i^0\right)^{-1} & \text{if } \ 0 < f_i^0 < +\infty \\ 0 \ \text{ if } \ f_i^0 = +\infty \\ +\infty \ \text{ if } \ f_i^0 = 0 \end{cases}$$

and

$$\Gamma_i^\infty = \begin{cases} (A_{\alpha_i} f_i^\infty)^{-1} & \text{if } \ 0 < f_i^\infty < +\infty \\ +\infty \ \text{ if } \ f_i^\infty = 0 \\ 0 \ \text{ if } \ f_i^\infty = +\infty \end{cases}$$

where

$$A_{\alpha_i} = \int_0^1 H_{\alpha_i}(1,s)a_i(s)ds + \frac{1}{d_i}\int_0^1 \phi_i(s)\left(\int_0^1 G_{1\alpha_i}(s,\tau)\,a_i(\tau)d\tau\right)ds$$
$$B_{\alpha_i} = \int_l^1 H_{\alpha_i}(1,s)a_i(s)ds + \frac{1}{d_i}\int_l^1 \phi_i(s)\left(\int_l^1 G_{1\alpha_i}(s,\tau)\,a_i(\tau)d\tau\right)ds.$$

($\mathscr{H}_3$) For $\nu = 0$ or $\nu = \infty$ there exist nonnegative constants F_i^ν $(i = 1,2,\dots,n)$ defined as

$$F_i^\nu = \lim_{x_i\to\upsilon} \frac{f_i(x_1,x_2,\dots,x_n)}{x_i}$$

uniformly with respect to $(x_1,x_2,\dots,x_{i-1},x_{i+1},\dots,x_n)$ on compact subsets of $(\mathbb{R}_+)^{n-1}$.

($\mathscr{H}_4$) For $i = 1,2,\dots,n$ and $l \in (0,1)$. Let

$$\Lambda_i^0 = \begin{cases} \left(A_{\alpha_i} F_i^0\right)^{-1} & \text{if } \ 0 < F_i^0 < +\infty \\ +\infty \ \text{ if } \ F_i^0 = 0 \\ 0 \ \text{ if } \ F_i^0 = +\infty \end{cases}$$

and

$$\Lambda_i^\infty = \begin{cases} (\delta B_{\alpha_i} F_i^\infty)^{-1} & \text{if } \ 0 < F_i^\infty < +\infty \\ 0 \ \text{ if } \ F_i^\infty = +\infty \\ +\infty \ \text{ if } \ F_i^\infty = 0 \end{cases}$$

The classical Krasnoselk'skii fixed point theorem of cone compression and expansion (Theorem 2.10) is now used in order to prove a first existence result of positive solutions

THEOREM 3.1. *With the notations in* $(\mathscr{H}_1)-(\mathscr{H}_2)$. *Assume that hypothesis* (1.3) *holds and* $\Gamma_i^\infty < \Gamma_i^0$. *Then the FBVP* (1.1) *has at least one positive for* $(\lambda_1,\lambda_2,\dots,\lambda_n) \in \mathscr{R}_1$,

$$\mathscr{R}_1 = \left\{(\lambda_1,\lambda_2,\dots,\lambda_n) : \tfrac{\Gamma_i^\infty}{n} < \lambda_i < \tfrac{\Gamma_i^0}{n}\right\}.$$

Proof. Let

$$\tfrac{\Gamma_i^\infty}{n} < \lambda_i < \tfrac{\Gamma_i^0}{n}, i = 1,\dots,n$$

and choose $\varepsilon > 0$ such that $\frac{1}{n(f_i^\infty-\varepsilon)\delta B_{\alpha_i}} \leqslant \lambda_i \leqslant \frac{1}{nA_{\alpha_i}(f_i^0+\varepsilon)}$, $i = 1,\dots,n$.

By definition of f_i^0 $(i = 1,2,\dots,n)$, there exists $H_1 > 0$ such that

$$f_i(u) = f_i(u_1,\dots,u_n) \leqslant \left(f_i^0+\varepsilon\right)\|u\| \quad \text{for } u_i \geqslant 0,\ \ \|u\| \in [0,H_1].$$

Let $u = (u_1,\dots,u_n) \in K$ with $\|u\| = \|(u_1,\dots,u_n)\| = H_1$. For $i = 1,2,\dots,n$ and any $t \in [0,1]$ we have:

$$\begin{aligned}
&\mathscr{T}_i(u)(t)\\
&= \lambda_i\left[\int_0^1 H_{\alpha_i}(t,s)a_i(s)f_i(u(s))ds + \tfrac{t^{\alpha_i-1}}{d_i}\int_0^1 \phi_i(s)\left(\int_0^1 G_{1\alpha_i}(s,\tau)\,a_i(\tau)f_i(u(\tau))d\tau\right)ds\right]\\
&\leqslant \lambda_i\left[\int_0^1 H_{\alpha_i}(t,s)a_i(s)f_i(u(s))ds + \tfrac{1}{d_i}\int_0^1 \phi_i(s)\left(\int_0^1 G_{1\alpha_i}(s,\tau)\,a_i(\tau)f_i(u(\tau))d\tau\right)ds\right]\\
&\leqslant \lambda_i\left(f_i^0+\varepsilon\right)\|u\|\left[\int_0^1 H_{\alpha_i}(1,s)a_i(s)ds + \tfrac{1}{d_i}\int_0^1 \phi_i(s)\left(\int_0^1 G_{1\alpha_i}(s,\tau)\,a_i(\tau)d\tau\right)ds\right]\\
&\leqslant \lambda_i\left(f_i^0+\varepsilon\right)A_{\alpha_i}\|u\| \leqslant \frac{1}{nA_{\alpha_i}\left(f_i^0+\varepsilon\right)}A_{\alpha_i}\left(f_i^0+\varepsilon\right)\|u\| = \frac{1}{n}\|u\|.
\end{aligned}$$

Then $\|\mathscr{T}_i(u)\| \leqslant \frac{1}{n}\|u\|$ for each $i = 1,\dots,n$. As a consequence, we deduce that

$$\|\mathscr{T}(u)\| = \sum_{i=1}^{i=n}\|\mathscr{T}_i(u)\| \leqslant \|u\|.$$

Let $r = H_1$. Then $\|\mathscr{T}(u)\| \leqslant \|u\|$ for any $u \in K$ with $\|u\| = r$.

Furthermore, from the definition of f_i^∞ $(i = 1,2,\dots,n)$, there exists $H_2 > 0$ such that

$$f_i(u) = f_i(u_1,\dots,u_n) \geqslant (f_i^\infty-\varepsilon)\|u\| \quad \text{for} \quad u_i \geqslant 0, \quad \|u\| \in [H_2,+\infty).$$

Let $R = H_3 = \max\left\{2H_1, \frac{1}{\delta}H_2\right\}$. If $u \in K$ with $\|u\| = R$. Then

$$\min_{t\in[l,1]}\left(\sum_{i=1}^{i=n}u_i\right) \geqslant \delta\sum_{i=1}^{i=n}\|u_i\| = \delta\|u\| \geqslant H_2, \text{ where } \delta = \min_{1\leqslant i\leqslant n}\delta_i$$

Thus, for $t \in [l,1], l \in (0,1)$ we have

$$\begin{aligned}
&\mathscr{T}_i(u)(t)\\
&= \lambda_i \left[\int_0^1 H_{\alpha_i}(t,s) a_i(s) f_i(u(s)) ds + \tfrac{t^{\alpha_i-1}}{d_i} \int_0^1 \phi_i(s) \left(\int_0^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau \right) ds \right]\\
&\geqslant \lambda_i \left[\int_l^1 H_{\alpha_i}(t,s) a_i(s) f_i(u(s)) ds + \tfrac{t^{\alpha_i-1}}{d_i} \int_l^1 \phi_i(s) \left(\int_l^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau \right) ds \right]\\
&\geqslant \lambda_i (l)^{\alpha_i-1} \left[\int_l^1 H_{\alpha_i}(1,s) a_i(s) f_i(u(s)) ds + \tfrac{1}{d_i} \int_l^1 \phi_i(s) \left(\int_l^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau \right) ds \right]\\
&\geqslant \lambda_i (l)^{\alpha_i-1} (f_i^\infty - \varepsilon) \|u\| \left[\int_l^1 H_{\alpha_i}(1,s) a_i(s) ds + \tfrac{1}{d_i} \int_l^1 \phi_i(s) \left(\int_l^1 G_{1\alpha_i}(s,\tau) a_i(\tau) d\tau \right) ds \right]\\
&\geqslant \lambda_i (f_i^\infty - \varepsilon) \delta \|u\| B_{\alpha_i} \geqslant \tfrac{1}{n(f_i^\infty - \varepsilon)\delta B_{\alpha_i}} (f_i^\infty - \varepsilon) \delta B_{\alpha_i} \|u\| = \frac{1}{n} \|u\|.
\end{aligned}$$

Then

$$\|\mathscr{T}(u)\| = \sum_{i=1}^{i=n} \|\mathscr{T}_i(u)\| \geqslant \|u\|.$$

Therefore, the first condition in Theorem 2.10 is fulfilled. Consequently, the operator $\mathscr{T}$ has a fixed point $u = (u_1, \ldots, u_n) \in K$ with $r \leqslant \|u\| \leqslant R$, then from Lemma 2.9 and Lemma 2.13, u is a positive solution to FBVP (1.1). The proof is complete. □

In a similar way, we can prove the following results.

THEOREM 3.2. *With the notations in* $(\mathscr{H}_1)$–$(\mathscr{H}_2)$. *Assume that hypothesis* (1.3) *holds and* $\Gamma_i^0 < \Gamma_i^\infty$. *Then the FBVP* (1.1) *has at least one positive for* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathscr{R}_2$,

$$\mathscr{R}_2 = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) : \tfrac{\Gamma_i^0}{n} < \lambda_i < \tfrac{\Gamma_i^\infty}{n} \right\}.$$

In this part of this paper, the vector version of the Krasnosel'skii's fixed point theorem (Theorem 2.11) is applied to Problem (1.1) providing new results for the existence of positive solutions to (1.1) and their component-wise localization. In this section our main result is the following theorem.

THEOREM 3.3. *With the notations in* $(\mathscr{H}_3)$–$(\mathscr{H}_4)$. *Assume that hypothesis* (1.3) *holds and* $\Lambda_i^\infty < \Lambda_i^0$. *Then the FBVP* (1.1) *has at least one positive for* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathscr{Q}_1$,

$$\mathscr{Q}_1 = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) : \Lambda_i^\infty < \lambda_i < \Lambda_i^0 \right\}.$$

Proof. Let

$$\Lambda_i^\infty < \lambda_i < \Lambda_i^0, \quad i = 1, \ldots, n.$$

and choose $\varepsilon > 0$ such that

$$\frac{1}{\delta^2 B_{\alpha_i} (F_i^\infty - \varepsilon)} \leqslant \lambda_i \leqslant \frac{1}{A_{\alpha_i} (F_i^0 + \varepsilon)}, \quad i = 1, \ldots, n.$$

By the definition of F_i^0, there exists $H_1 > 0$ such that

$$f_i(u) = f_i(u_1, \dots, u_n) \leqslant \left(F_i^0 + \varepsilon\right) u_i \text{ for } u_i \in [0, H_1], (i = 1, 2, \dots, n)$$

Thus, $\mathscr{T}_i u \nless u_i$ if $\|u_i\| = H_1$ for each $i \in \{1, 2, \dots, n\}$. To prove this end, assume the contrary, i.e.

$$u_i \prec \mathscr{T}_i u(t), \quad \text{for} \quad \|u_i\| = H_1$$

then

$$\begin{aligned}
&u_i(t) < \mathscr{T}_i(u)(t)\\
&= \lambda_i \left[\int_0^1 H_{\alpha_i}(t,s) a_i(s) f_i(u(s)) ds + \tfrac{t^{\alpha_i - 1}}{d_i} \int_0^1 \phi_i(s) \left(\int_0^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau\right) ds\right]\\
&\leqslant \lambda_i \left[\int_0^1 H_{\alpha_i}(1,s) a_i(s) f_i(u(s)) ds + \tfrac{1}{d_i} \int_0^1 \phi_i(s) \left(\int_0^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau\right) ds\right]\\
&\leqslant \lambda_i \left(F_i^0 + \varepsilon\right) \|u_i\| \left[\int_0^1 H_{\alpha_i}(1,s) a_i(s) ds + \tfrac{1}{d_i} \int_0^1 \phi_i(s) \left(\int_0^1 G_{1\alpha_i}(s,\tau) a_i(\tau) d\tau\right) ds\right]\\
&= \lambda_i \left(F_i^0 + \varepsilon\right) A_{\alpha_i} \|u_i\| \leqslant \frac{\left(F_i^0 + \varepsilon\right) A_{\alpha_i}}{A_{\alpha_i}\left(F_i^0 + \varepsilon\right)} H_1 = H_1, \forall t \in (0,1).
\end{aligned}$$

So the contradiction $\|u_i\| = H_1 < H_1$.

Next from the definition of F_i^∞, there exists $H_2 > 0$ such that

$$f_i(u) = f_i(u_1, \dots, u_n) \geqslant (F_i^\infty - \varepsilon) u_i \quad \text{for} \quad u_i \in [H_2, +\infty), (i = 1, \dots, n).$$

Let $H_3 = \max\left\{2H_1, \frac{1}{\delta} H_2\right\}$. If $u = (u_1, \dots, u_n) \in K$ with $\|u_i\| = H_3$, then

$$\min_{t \in [l,1]} u_i(t) \geqslant \delta \|u_i\| \geqslant H_2, \quad i = 1, \dots, n.$$

Thus, $\mathscr{T} u_i \ngtr \|u_i\|$ if $\|u_i\| = H_3$ for each $i \in \{1, 2, \dots, n\}$. To prove this end, assume the contrary, i.e.

$$u_i(t) \succ \mathscr{T}_i u(t), \text{ for some } u = (u_1, \dots, u_n) \in K \text{ with } \|u_i\| = H_3.$$

Then we would obtain

$$\begin{aligned}
&u_i(t) > \mathscr{T}_i(u)(t)\\
&= \lambda_i \left[\int_0^1 H_{\alpha_i}(t,s) a_i(s) f_i(u(s)) ds + \tfrac{t^{\alpha_i - 1}}{d_i} \int_0^1 \phi_i(s) \left(\int_0^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau\right) ds\right]\\
&\geqslant \lambda_i \left[\int_l^1 H_{\alpha_i}(t,s) a_i(s) f_i(u(s)) ds + \tfrac{t^{\alpha_i - 1}}{d_i} \int_l^1 \phi_i(s) \left(\int_l^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau\right) ds\right]\\
&\geqslant \lambda_i t^{\alpha_i - 1} \left[\int_l^1 H_{\alpha_i}(1,s) a_i(s) f_i(u(s)) ds + \tfrac{1}{d_i} \int_l^1 \phi_i(s) \left(\int_l^1 G_{1\alpha_i}(s,\tau) a_i(\tau) f_i(u(\tau)) d\tau\right) ds\right]\\
&\geqslant \lambda_i (l)^{\alpha_i - 1} (F_i^\infty - \varepsilon) \left[\int_l^1 H_{\alpha_i}(1,s) a_i(s) u_i(s) ds + \tfrac{1}{d_i} \int_l^1 \phi_i(s) \left(\int_l^1 G_{1\alpha_i}(s,\tau) a_i(\tau) u_i(\tau) d\tau\right) ds\right]
\end{aligned}$$

$$\geqslant \lambda_i (l)^{2\alpha_i-2} (F_i^\infty - \varepsilon) \|u_i\| \left[\int_l^1 H_{\alpha_i}(1,s) a_i(s) ds + \tfrac{1}{d_i} \int_l^1 \phi_i(s) \left(\int_l^1 G_{1\alpha_i}(s,\tau) a_i(\tau) d\tau \right) ds \right]$$
$$\geqslant \lambda_i \delta^2 (F_i^\infty - \varepsilon) \|u_i\| B_{\alpha_i} = \frac{1}{\delta^2 B_{\alpha_i} (F_i^\infty - \varepsilon)} \delta^2 (F_i^\infty - \varepsilon) B_{\alpha_i} H_3 = H_3.$$

So the contradiction

$$\|u_i\| = H_3 > H_3.$$

Therefore, the first condition in Theorem 2.10 is fulfilled. Consequently, the operator $\mathscr{T}$ has a fixed point $u = (u_1, \ldots, u_n)$ with $u_i \in K_{H_1,H_3}$, $(i = 1, \ldots, n)$. □

The following existence result can be proved in an analogous way. The proof is omitted.

THEOREM 3.4. *With the notations in* $(\mathscr{H}_3)$–$(\mathscr{H}_4)$. *Assume that hypothesis* (1.3) *holds and* $\Lambda_i^0 < \Lambda_i^\infty$. *Then the FBVP* (1.1) *has at least one positive for* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathscr{Q}_2$,

$$\mathscr{Q}_2 = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) : \Lambda_i^0 < \lambda_i < \Lambda_i^\infty \right\}.$$

REMARK 3.5. Let $f_1(x,y) = x^2 \sin(y)$ and $f_2(x,y) = y^2 \sin(x)$. Then f_1 and f_2 verify Assumption of Theorem 3.4. We claim that, $F_1^0 = 0$, $F_2^0 = 0$, $F_1^\infty = +\infty$, $F_2^\infty = +\infty$, consequently $\Lambda_i^0 < \Lambda_i^\infty$, $(i = 1, 2)$. Then by Theorem 3.4, Problem (1.1) has at least one solution for every positive λ_1, λ_2.

However f_1 and f_2 do not satisfy Assumptions of Theorems 3.1 and 3.2. We claim that

$$f_1^0 = 0, \quad f_2^0 = 0, \quad f_1^\infty \neq +\infty, \quad f_2^\infty \neq +\infty,$$

for the reason that for $x = \frac{1}{y}$, $\lim_{x+y\to+\infty} f_1(x,y) = \lim_{y\to+\infty} f_1(\frac{1}{y}, y) = \lim_{y\to+\infty} \frac{\sin(y)}{y^2} = 0$, and for $y = \frac{1}{x}$, $\lim_{x+y\to+\infty} f_2(x,y) := \lim_{x\to+\infty} f_2(x, \frac{1}{x}) = \lim_{x\to+\infty} \frac{\sin(x)}{x^2} = 0$.

Then $\Gamma_i^\infty = \Gamma_i^0 = 0$, $(i = 1, 2)$. Therefore the Theorems 3.1 and 3.2 do not give results.

4. Application

In the following examples 4.1–4.4 we select $n = 2$, $\alpha_1 = \alpha_2 = \frac{5}{2}$, $\mu_1 = 4$, $\mu_2 = 2$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \dfrac{1}{4}$, $\phi_1(t) = \exp(-2t)$, $\phi_2(t) = \exp(-3t)$ in the system (1.1). It is easy to see that, $d_1 = 0.59182$, $d_2 = 0.73737$. Choose $a_1(t) = a_2(t) = 1$, then $A_{\alpha_1} = 0.29440$, $B_{\alpha_1} = 0.19521$, $A_{\alpha_2} = 0.33059$, $B_{\alpha_2} = 0.15843$.

EXAMPLE 4.1. Let

$$f_1(u,v) = (u+v) \left(1 + 19 \exp\left(\frac{-1}{u+v} \right) \right),$$

and

$$f_2(u,v) = (u+v) \left(1 + \frac{99}{1 + 98 \exp(-u-v)} \right).$$

By simple computation, we have $f_1^0 = 1$, $f_1^\infty = 20$, $f_2^0 = 2$, $f_2^\infty = 100$.

Therefore, $4^{\alpha_1 - 1} A_{\alpha_1} f_1^0 = 2.3552 < B_{\alpha_1} f_1^\infty = 3.9042$ and $4^{\alpha_2 - 1} A_{\alpha_2} f_2^0 = 5.2894 < B_{\alpha_2} f_2^\infty = 15.843$. Thus, by Theorem 3.1, the FBVP (1.1) has at least one positive solution for each $(\lambda_1, \lambda_2) \in (2.0491, 3.3967) \times (0.50495, 1.5124)$.

EXAMPLE 4.2. Let

$$f_1(u,v) = (u+v)(2 + 38\exp(-u-v))$$

and

$$f_2(u,v) = (u+v)\left(1 + \frac{50}{1+\exp(u+v)}\right).$$

By simple computation, we have $f_1^0 = 40$, $f_1^\infty = 2$, $f_2^0 = 26$, $f_2^\infty = 1$.

Therefore, $4^{\alpha_1 - 1} A_{\alpha_1} f_1^\infty = 4.7104 < B_{\alpha_1} f_1^0 = 7.8084$ and $4^{\alpha_2 - 1} A_{\alpha_2} f_2^\infty = 2.6447 < B_{\alpha_2} f_2^0 = 4.1192$. Thus, by Theorem 3.2,the FBVP (1.1) has at least one positive solution for each $(\lambda_1, \lambda_2) \in (1.0245, 1.6984) \times (1.9421, 3.0249)$.

EXAMPLE 4.3. Let

$$f_1(u,v) = u\left(\exp(-uv) + 20\exp\left(\frac{-1}{uv}\right)\right)$$

and

$$f_2(u,v) = v\left(\exp(-uv) + 20\exp\left(\frac{-1}{uv}\right)\right).$$

By simple computation, we have $F_1^0 = 1$, $F_1^\infty = 20$, $F_2^0 = 1$, $F_2^\infty = 20$.

Therefore, $4^{\alpha_1 - 1} A_{\alpha_1} F_1^0 = 2.3552 < B_{\alpha_1} F_1^\infty = 3.9042$ and $4^{\alpha_2 - 1} A_{\alpha_2} F_2^0 = 2.6447 < B_{\alpha_2} F_2^\infty = 3.1686$ Thus, by Theorem 3.3, the FBVP (1.1) has at least one positive solution for each $(\lambda_1, \lambda_2) \in (2.0491, 3.3967) \times (2.5248, 3.0249)$.

EXAMPLE 4.4. Let

$$f_1(u,v) = u\left(2 + \frac{38}{1+uv}\right)$$

and

$$f_2(u,v) = v\left(2 + \frac{38}{1+uv}\right).$$

By simple computation, we have $F_1^0 = 40$, $F_1^\infty = 2$, $F_2^0 = 40$, $F_2^\infty = 2$.

Therefore, $4^{\alpha_1 - 1} A_{\alpha_1} F_1^\infty = 4.7104 < B_{\alpha_1} F_1^0 = 7.8084$ and $4^{\alpha_2 - 1} A_{\alpha_2} F_2^\infty = 5.2894 < B_{\alpha_2} F_2^0 = 6.3372$. Thus, by Theorem 3.4, the FBVP (1.1) has at least one positive solution for each $(\lambda_1, \lambda_2) \in (1.0245, 1.6984) \times (1.2624, 1.5124)$.

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