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# SOLVABILITY OF A LINEAR SYSTEM WITH A NONLOCAL TERM IN A BOUNDARY CONDITION

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*Abstract.* In this paper, we analyze a linear system for the Poisson equations with a boundary condition comprising the fractional derivative in time and the time dependent right-hand sides. A system of this type arises under studying the Muskat boundary problem with surface tension in the case of subdiffusion. First, we prove existence and uniqueness of the solution to this problem in the Hölder classes, and provide the coercive estimates of the solution. Second, we apply the obtained results together with the contraction theorem to establish the one-to-one local classical solvability to the Muskat problem governed by anomalous diffusion in the case of nonzero surface tension of a free boundary.

## 1. Introduction

Boundary value problems with fractional time derivatives are among central objects of the modern theory of partial differential equations. It deals with various applications in physics (dynamical processes in fractals and viscoelastic media [10], [20], [21]), medicine [26], [28], chemistry [16], [34] and with the rich mathematical content of this subject see, for example, the monographs [11], [24] and references therein.

Note that the presence of the fractional derivative in time in equations or boundary conditions means that boundary value problems describe the anomalous diffusion (the diffusive motion cannot be modelled as the standard Brownian motion [4], [21]). The signature of the anomalous diffusion is that the mean square displacement of the diffusing species $\langle(\Delta\mathbf{x})^2\rangle$ scales as a nonlinear power law in time, i.e. $\langle(\Delta\mathbf{x})^2\rangle \sim t^{\nu}$, where $\nu$ is a nonnegative number. If $\nu \in (0,1)$, this is referred to as a subdiffusion; if $\nu = 1$, we have the case of a normal diffusion which is described with derivatives of integer orders.

In this paper we turn to solvability of the linear system with a fractional temporal derivative in the boundary condition and its application to study of a "fractional" free boundary problem.

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Let $k$, $a_0$, $c_0$ and $a_{ij}$, $i,j=\overline{1,n-1}$, be some given constants, $k\neq 1$, $k$, $a_0, a_{ij}>0$; and $a_1=\{a_1^1,\dots,a_1^{n-1}\}$, $c_1=\{c_1^1,\dots,c_1^{n-1}\}$ be given vectors; $R_+^n=\{(x',x_n):x'\in R^{n-1}, x_n>0\}$, $R_-^n=\{(x',x_n):x'\in R^{n-1}, x_n<0\}$, $R^n_{\pm T}=R^n_{\pm}\times(0,T)$, $R_T^{n-1}=R^{n-1}\times(0,T)$. Denote by $\langle\cdot,\cdot\rangle$ the inner product, and $\nabla_{x'}=(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{n-1}})$.

We look for a classical solution $(u_+(x,t),u_-(x,t),\rho(x',t))$ of the following linear system with the fractional temporal derivative in the boundary condition:

$$\Delta_x u_\pm = f_0^\pm(x,t) \ \text{ in } R^n_{\pm T}; \tag{1.1}$$

$$u_- - u_+ - \sum_{ij=1}^{n-1} a_{ij}\frac{\partial^2\rho}{\partial x_i\partial x_j} + \langle c_1,\nabla_{x'}\rho\rangle + c_0\rho = f_1(x',t) \ \text{ on } \bar{R}_T^{n-1}; \tag{1.2}$$

$$\mathbf{D}_t^\nu\rho + a_0\frac{\partial u_-}{\partial\bar{n}} = f(x',t) \ \text{ on } \bar{R}_T^{n-1},\ \nu\in(0,1); \tag{1.3}$$

$$\frac{\partial u_-}{\partial\bar{n}} - k\frac{\partial u_+}{\partial\bar{n}} - k\langle a_1,\nabla_{x'}(u_- - u_+)\rangle = f_2(x',t) \ \text{ on } \bar{R}_T^{n-1}; \tag{1.4}$$

$$\rho(x',0)=0 \ \text{ in } R^{n-1},\ u_\pm(x,0)=0 \ \text{ in } R^n_\pm, \tag{1.5}$$

where $\Delta_x=\sum\limits_{i=1}^n\frac{\partial^2}{\partial x_i^2}$, $\bar{n}$ is the unit normal to $R^{n-1}$ directed in $R^n_-$; $f_0^\pm$, $f$, $f_j$, $j=1,2$, are some given functions:

$$f_0^\pm,\ f,\ f_j\equiv 0 \quad if \ \text{ either } t=0 \text{ or } |x|>R_0, \tag{1.6}$$

for some positive number $R_0$; $\mathbf{D}_t^\nu$ denotes the Caputo fractional derivative with respect to $t$ and is defined by (see, for example, (2.4.6) in [11])

$$\mathbf{D}_t^\nu w(\cdot,t)=\frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int_0^t\frac{w(\cdot,\tau)d\tau}{(t-\tau)^\nu}-\frac{w(\cdot,0)}{\Gamma(1-\nu)t^\nu},\quad \nu\in(0,1), \tag{1.7}$$

where $\Gamma(\nu)$ is the Gamma function. If $\nu=1$, $\mathbf{D}_t^1 w(\cdot,t):=\frac{\partial w(\cdot,t)}{\partial t}$.

The considered system has two peculiarities. The first is the presence of the fractional derivative in time of the unknown function $\rho$ in boundary condition (1.3). Thus, this condition looks like a fractional dynamic boundary condition (see e.g. [12]). Moreover, as it follows from definition (1.7), the term $\mathbf{D}_t^\nu\rho$ is nonlocal. Next peculiarity deals with the structure of system (1.1)–(1.5): the unknown functions $u_+$, $u_-$, $\rho$ are connected with each other only through boundary conditions (1.2)–(1.4), and the function $\rho$ is defined only on the boundary $\bar{R}_T^{n-1}$.

Solvability of systems like (1.1)–(1.5) in the case of the normal diffusion was studied by a lot of authors (see e.g. [7], [8], [3] and references therein). As for the subdiffusion case, $\nu\in(0,1)$, Kiran and Tatar [12] analyzed existence and nonexistence of local and global solutions for systems of elliptic equations with fractional dynamic boundary conditions. In the case of absence the oldest spatial derivatives of the desired function $\rho$ (i.e., $a_{ij}=0$, $c_1^j=0$, $i,j=\overline{1,n-1}$) in boundary condition (1.2), the classical solvability of (1.1)–(1.5) was proved in [31]. To the best of our knowledge, there are no results concerned investigation of system (1.1)–(1.5) in the general case, i.e. $a_{ij} \neq 0,\ c_1^j \neq 0,\ i,j = \overline{1,n-1}$.

Moreover, analysis of problem (1.1)–(1.5) is important for investigation of the two-phase “fractional” Hele-Shaw problem (or “fractional” Muskat problem) in the case of nonzero surface tension of a free boundary. We recall that in the classical case (i.e. in the case of the normal diffusion) this free boundary problem was proposed by Muskat in 1934 [22]. This problem describes the evolution of two immiscible incompressible fluids (for instance, water and oil). The interface $\Upsilon(t)$ between these fluids is called as a free boundary (or a moving boundary). The motion of fluids is governed by the Darcy law, stating that the velocity of the moving boundary $V_{n_t} := \langle \mathbf{D}_t^1 \Upsilon(t), \bar{n}_t \rangle$, $\bar{n}_t$ is the unit normal to $\Upsilon(t)$, is proportional to the pressure gradients of fluids. In the case of the subdiffusion ($\nu \in (0,1)$), the Muskat problem governed by “fractional” Darcy low which is formulated in [33], [23] and means that the “fractional” velocity $V_{n_t}^{\nu} := \langle \mathbf{D}_t^{\nu} \Upsilon(t), \bar{n}_t \rangle$ is proportional to the pressure gradients. The mathematical model of the “fractional” Muskat problem is represented by (5.1)–(5.5) (see Subsection 5.1 in this paper).

Note that the one-phase “fractional” Hele-Shaw problem with zero and nonzero surface tension of moving boundary was analyzed in [17], [18], [32], [29], [30]. In particular, the existence and uniqueness of a classical solution to this problem locally in time were proved in [29], [30]. As for the Muskat problem subjected by the subdiffusion, its local classical solvability was obtained in the case of zero surface tension in [31]. To our knowledge, there are no advances yet about fractional Muskat problem with nonzero surface tension.

To show the solvability of such problem, we adapt the classical approach which is used for a moving boundary problem in the case of the normal diffusion (see, e.g. [1]) to the subdiffusion case. We reduce the “fractional” free boundary problem to a nonlinear problem in a fixed domain; then linearize this problem and obtain the one-to-one local classical solvability of the linearized problem. After that we use this result for the reduction of the nonlinear problem to a fixed point theorem. On this route we have to solve many technically difficulties which are related to the nonlocal behavior of the moving boundary velocity. Significant part of this research is connected with system like (1.1)–(1.5) which is the principal model problem and the nonlinear problem inherits the main features of (1.1)–(1.5).

The paper is organized as follows. In Section 2 we define the function spaces and formulate the main results related with solvability of system (1.1)–(1.5), Theorem 2.1. In Section 3, using Fourier and Laplace transformations, we obtain the integral representation for solution of (1.1)–(1.5). Section 4 is devoted to proof of Theorem 2.1. To this end we obtain the corresponding coercive estimates for the constructed solution in Section 3. Note that Lemma 4.1 plays the significant role in this process. In Section 5, following the above stated, we study the solvability of the “fractional” Muskat problem with surface tension. The principal result of this investigation is represented by Theorem 5.1.

## 2. Function spaces and main result

In order to analyze problems (1.1)–(1.4) and (5.1)–(5.5) we need some definitions and auxiliary results.

Let $D$ be a given domain in $R^n$, $D_T = D\times(0,T)$; $\bar{x}$, $x$ be any points in $\bar{D}$, $x\neq\bar{x}$, $t,\tau\in[0,T]$, $t\neq\tau$; $\alpha,\beta\in(0,1)$, $l$ be an integer nonnegative number. In this paper we will use the following function spaces: $C([0,T],C^{l+\alpha}(\bar{D}))$, $C^{l+\alpha,\beta,\alpha}(\bar{D}_T)$, $C^{l+\alpha,\frac{l+\alpha}{3}\nu}(\bar{D}_T)$. Note that the spaces $C([0,T],C^{l+\alpha}(\bar{D}))$ are used by many authors, and their definitions and properties can be found, for instance, in [19].

Let

$$[w]^{(\alpha,\beta)}_{D_T} = \sup_{(x,\bar{x})\in\bar{D},(t,\tau)\in[0,T]} \frac{|w(x,t)-w(\bar{x},t)-w(x,\tau)+w(\bar{x},\tau)|}{|x-\bar{x}|^{\alpha}|t-\tau|^{\beta}},$$

and define function spaces $C^{l+\alpha,\beta,\alpha}(\bar{D}_T)$.

DEFINITION 2.1. The Banach space $C^{l+\alpha,\beta,\alpha}(\bar{D}_T)$ is the set of functions $w(x,t)$ with the finite norm:

$$\|w\|_{C^{l+\alpha,\beta,\alpha}(\bar{D}_T)} = \sum_{|j|=0}^{l}\{\sup_{\bar{D}_T}|D^j_x w| + \langle D^j_x w\rangle^{(\alpha)}_{x,D_T} + \langle D^j_x w\rangle^{(\beta)}_{t,D_T} + [D^j_x w]^{(\alpha,\beta)}_{D_T}\},$$

where $\langle w\rangle^{(\alpha)}_{x,D_T}$ and $\langle w\rangle^{(\beta)}_{t,D_T}$ are Hölder constants of a function $w(x,t)$ in $x$ and $t$, respectively.

Note that classes $C^{l+\alpha,\frac{l+\alpha}{2}\nu}(\bar{D}_T)$, $l=\overline{0,3}$ and $\nu=1$, coincide with usual parabolic Hölder spaces (see (1.10)–(1.12) in Chapter 3 in [14]).

DEFINITION 2.2. We will say that the function $w(x,t)\in C^{l+\alpha,\frac{l+\alpha}{3}\nu}(\bar{D}_T)$, $l=\overline{0,3}$, $\nu\in(0,1)$, iff the following norms are finite:

$$\|w\|_{C^{l+\alpha,\frac{l+\alpha}{3}\nu}(\bar{D}_T)} = \|w\|_{C([0,T],C^{l+\alpha}(\bar{D}))} + \sum_{|j|=0}^{l}\langle D^j_x w\rangle^{(\frac{l+\alpha-|j|}{3}\nu)}_{t,D_T}, \text{ if } [l/3]=0,$$

where $[l/3]$ denotes the integer part of $l/3$;

$$\|w\|_{C^{3+\alpha,\frac{3+\alpha}{3}\nu}(\bar{D}_T)} = \|w\|_{C([0,T],C^{3+\alpha}(\bar{D}))} + \sum_{|j|=1}^{3}\langle D^j_x w\rangle^{(\frac{3+\alpha-|j|}{3}\nu)}_{t,D_T}$$
$$+\|\mathbf{D}^{\nu}_t w\|_{C([0,T],C^{\alpha}(\bar{D}))} + \langle \mathbf{D}^{\nu}_t w\rangle^{(\frac{\alpha}{3}\nu)}_{t,D_T}.$$

In a similar way we introduce the spaces $C^{l+\alpha,\frac{l+\alpha}{3}\nu}(\partial D_T)$, $\partial D_T = \partial D\times[0,T]$. Moreover, we will use the usual Hölder spaces $C^{l+\alpha}(\bar{D})$ and $C^{l+\alpha}(\partial D)$, their definitions can be found, for example, in [15].

We define classes $\underset{0}{C}^{k+\alpha,\beta,\alpha}(\bar{D}_T)$, as the subspaces of $C^{k+\alpha,\beta,\alpha}(\bar{D}_T)$ such that $D_x^j w(x,0)=0,\ |j|=\overline{0,k}$.

Let $d$ be a smooth surface and $d_T=d\times(0,T)$.

DEFINITION 2.3. We will say $w\in P_\nu^{4+\alpha}(d_T)$ iff $w\in C([0,T],C^{4+\alpha}(\bar{d}))$ and $D_x w\in C^{3+\alpha,\frac{3+\alpha}{3}\nu}(\bar{d}_T)$, $\mathbf{D}_t^\nu w\in C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\bar{d}_T)$:

$$\|w\|_{P_\nu^{4+\alpha}(\bar{d}_T)}:=\|w\|_{C([0,T],C^\alpha(\bar{d}))}+\|D_x w\|_{C^{3+\alpha,\frac{3+\alpha}{3}\nu}(\bar{d}_T)}+\|\mathbf{D}_t^\nu w\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\bar{d}_T)}.$$

Throughout the paper we will need in the interpolation inequality (see Corollary 1.2.18 [19]):

$$\|w\|_{C^{l_1+a(l_2-l_1)}(\bar{D})}\leqslant C\|w\|^a_{C^{l_2}(\bar{D})}\|w\|^{1-a}_{C^{l_1}(\bar{D})}, \tag{2.1}$$

where $\partial D\in C^{l_2},\ a\in(0,1),\ 0\leqslant l_1<l_2$.

One of the main results of our paper is the following.

THEOREM 2.1. *Let* $\alpha,\nu\in(0,1)$, *and condition* (1.6) *hold.*

*(i) If* $f_0^\pm\in C^{\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^n_{\pm T})$, $f_1\in C^{2+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)$, $f,f_2\in C^{1+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)$. *Then there exists a unique classical solution* $(u_+,u_-,\rho)$ *of problem* (1.1)–(1.5)*:* $u_\pm\in C^{2+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^n_{\pm T})$, $\rho\in C^{4+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)$; *and the following estimate holds*

$$\begin{aligned}&\|u_+\|_{C^{2+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^n_{+T})}+\|u_-\|_{C^{2+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^n_{-T})}+\|\rho\|_{C^{4+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)}\\&+\|\boldsymbol{D}_t^\nu\rho\|_{C^{1+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)}\\ \leqslant{}& C_1[\|f_0^+\|_{C^{\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^n_{+T})}+\|f_0^-\|_{C^{\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^n_{-T})}+\|f_1\|_{C^{2+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)}\\&+\|f\|_{C^{1+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)}+\|f_2\|_{C^{1+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar{R}^{n-1}_T)}].\end{aligned} \tag{2.2}$$

*(ii) If* $f\in C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\bar{R}^{n-1}_T)$, *and*

$$f_0^\pm,\ f_1,\ f_2\equiv 0, \tag{2.3}$$

*then there is a unique classical solution* $(u_+,u_-,\rho)$ *of problem* (1.1)–(1.5)*:* $u_\pm\in C^{2+\alpha,\frac{2+\alpha}{3}\nu}(\bar{R}^n_{\pm T})$, $\rho\in P_\nu^{4+\alpha}(R^{n-1}_T)$, *and the following estimate holds*

$$\begin{aligned}&\|u_+\|_{C^{2+\alpha,\frac{2+\alpha}{3}\nu}(\bar{R}^n_{+T})}+\|u_-\|_{C^{2+\alpha,\frac{2+\alpha}{3}\nu}(\bar{R}^n_{-T})}+\|\rho\|_{P_\nu^{4+\alpha}(R^{n-1}_T)}\\ \leqslant{}& C_2\|f\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\bar{R}^{n-1}_T)},\end{aligned} \tag{2.4}$$

*where* $C_i$, $i=1,2$, *are positive constants independent of the right-hand sides of* (1.1)–(1.5).

Note that one can easily see that statement (i) from Theorem 2.1 should be proven in the case of assumption (2.3). Indeed, applying results from [27] to the following transmission problem:

$$\Delta_x u_\pm = f_0^\pm(x,t) \text{ in } R^n_{\pm T};\ u_- - u_+ = f_1(x',t) \text{ on } \bar{R}^{n-1}_T;$$

$$\frac{\partial u_-}{\partial \bar{n}} - k\frac{\partial u_+}{\partial \bar{n}} - k\langle a_1, \nabla_{x'}(u_- - u_+)\rangle = f_2(x',t) \text{ on } \bar{R}^{n-1}_T,\ u_\pm(x,0) = 0 \ in\ R^n_\pm,$$

we can reduce problem (1.1)–(1.5) with arbitrary right-hand sides to the problem with the right-hand sides satisfying (2.3). Thus, we represent below the proof of Theorem 2.1 in the case of (2.3).

First of all we analyze problem (1.1)–(1.5) in the case of

$$c_0 = 0, \quad c_1^l = 0, \quad l = \overline{1,n-1}. \tag{2.5}$$

Henceforward the letter $C$ will be used to denote different constants encountered in our formulas.

## 3. Integral representation of the solution to problem (1.1)–(1.5) in the case of (2.3), (2.5)

Due to the quadratic form connected with the Laplace equation is positive, we can choose coordinates $x'$ such that the form $\sum_{i,j=1}^{n-1} a_{ij}\zeta_i\zeta_j$, $\forall \zeta \in R^{n-1}$, is reduced to the diagonal view. Thus, we believe that $a_{ij} = 0$, if $i \neq j$.

Let $\xi = (\xi_1, \dots, \xi_{n-1})$, $|\xi| = \sqrt{\sum_{i=1}^{n-1} \xi_i^2}$. Denote by $\tilde{w}(\xi, x_n, t)$ the Fourier transform of $w(x', x_n, t)$, and by $\hat{w}(\cdot, p)$ the Laplace transform of $w(\cdot, t)$. Throughout in the paper, we will use the notation "$*$" instead of "$\widetilde{\widehat{\ }}$".

In virtue of (1.6), we can extend the function $f(x',t)$ by $0$ for $t < 0$, and then apply the Fourier and Laplace transformations to problem (1.1)–(1.5).

Some simple calculations allow us to deduce

$$\rho^*(\xi, p) = \frac{(k+1)f^*(\xi,p)}{(k+1)p^\nu + 8a_0k\pi^3\sum_{j=1}^{n-1} a_{jj}\xi_j^2|\xi| - i8a_0k\pi^3\langle a_1, \xi\rangle}, \tag{3.1}$$

$$u_+^*(\xi, x_n, p) = \frac{4\pi^2}{k+1}\sum_{j=1}^{n-1} a_{jj}\xi_j^2\rho^*(\xi,p)e^{-2\pi|\xi|x_n}[|\xi| + ik\langle a_1, \xi\rangle]|\xi|^{-1}, \tag{3.2}$$

$$u_-^*(\xi, x_n, p) = -\frac{4\pi^2 k}{k+1}\sum_{j=1}^{n-1} a_{jj}\xi_j^2\rho^*(\xi,p)e^{2\pi|\xi|x_n}[|\xi| - i\langle a_1, \xi\rangle]|\xi|^{-1}. \tag{3.3}$$

Let us introduce the following notations:

$$A_0 = \left\{\frac{8a_0k\pi^3}{k+1}a_{11}, \dots, \frac{8a_0k\pi^3}{k+1}a_{n-1n-1}\right\}, \quad A_1 = \left\{\frac{8a_0k\pi^3}{k+1}a_1^1, \dots, \frac{8a_0k\pi^3}{k+1}a_1^{n-1}\right\},$$

$$\mathcal{K}_+(y',x_n) = \frac{1}{k+1} \int\limits_{R^{n-1}} [1+ik\langle a_1,\xi\rangle|\xi|^{-1}]exp\{-2\pi(|\xi|x_n - i\langle\xi,y'\rangle)\}d\xi, \tag{3.4}$$

$$\mathcal{K}_-(y',x_n) = \frac{-k}{k+1} \int\limits_{R^{n-1}} [1-i\langle a_1,\xi\rangle|\xi|^{-1}]exp\{2\pi(|\xi|x_n + i\langle\xi,y'\rangle)\}d\xi, \tag{3.5}$$

$$\mathcal{K}(y',\eta) = \int\limits_{R^{n-1}} exp\{-\eta\langle A_0\xi,\xi\rangle|\xi| + i\langle A_1,\xi\rangle\eta + 2i\pi\langle\xi,y'\rangle\}d\xi, \tag{3.6}$$

$$\mathcal{G}(y',t) = \int\limits_0^\infty \frac{W(-\eta t^{-\nu};-\nu,0)}{t}\mathcal{K}(y',\eta)d\eta, \tag{3.7}$$

where $W(z;b,d)$ is the Wright function (see its definition, e.g., in [6] or [20]).

After that, we apply the inverse Laplace and Fourier transformations to (3.1)–(3.3) and get

$$\rho(x',t) = \int\limits_0^t d\tau \int\limits_{R^{n-1}} f(y',t)\mathcal{G}(x'-y',t-\tau)dy', \tag{3.8}$$

$$u_\pm(x',x_n,t) = \int\limits_{R^{n-1}} \mathcal{K}_\pm(x'-y',x_n)\sum_{i=1}^{n-1} a_{ii}\frac{\partial^2\rho}{\partial y_i^2}(y',t)dy'. \tag{3.9}$$

To obtain representations (3.8) and (3.9), we used the following equality

$$\frac{1}{p^\nu + \langle A_0\xi,\xi\rangle|\xi| - i\langle A_1,\xi\rangle} = \int\limits_0^\infty e^{-\eta p^\nu - \eta\langle A_0\xi,\xi\rangle|\xi| + i\langle A_1,\xi\rangle\eta}d\eta, \text{ if } Re p^\nu > 0,$$

which is easily verified.

By virtue of the smoothness properties of the function $f(x,t)$ and its behavior at the infinity, all above performed operations are justified.

## 4. Proof of Theorem 2.1

### 4.1. Estimates of the constructed solution $(u_+(x',x_n,t), u_-(x',x_n,t), \rho(x',t))$

First we define the following Riemann-Liouville derivative of a function $w(\cdot,t)$ with respect to $t$ (see, e.g., (2.1.8) [11]) as:

$$\partial_t^\theta w(\cdot,t) := \frac{1}{\Gamma(1-\theta)}\frac{\partial}{\partial t}\int_0^t \frac{w(\cdot,\tau)}{(t-\tau)^\theta}d\tau, \ \theta\in(0,1). \tag{4.1}$$

As usual in the potential theory, to evaluate the functions $u_\pm(x,t)$ and $\rho(x',t)$ it is necessary to describe well the properties of the kernels $\mathcal{G}(x',t)$ (and hence $\mathcal{K}(x',\eta)$) and $\mathcal{K}_\pm(x',x_n)$.

LEMMA 4.1. *Let $\alpha,\ \nu \in (0,1);\ k>0;\ y' \in R^{n-1},\ y'' \in R^{n-2},\ y''=(y_1,\dots,y_{l-1}, y_{l+1},\dots,y_{n-1}),\ l=\overline{1,n-1},\ \eta \in (0,+\infty),\ t\in[0,T],\ \varepsilon$ and $A$ be some positive numbers. Then functions $\mathscr{K}(y',\eta)$ and $\mathscr{G}(y',t)$ which are given by* (3.6), (3.7) *satisfy the following inequalities:*
*(i)*

$$|D^m_{y'}\mathscr{K}(y',\eta)| \leqslant C\frac{exp\Big(-A\sum_{j=1}^{n-1}\frac{|y_j+\frac{A_1^j\eta}{2\pi}|}{\eta^{1/3}}\Big)}{\eta^{\frac{|m|+n-1}{3}}}, \quad |m|=\sum_{j=1}^{n-1}m_j,\ |m|=0,1,2,\dots; \tag{4.2}$$

*(ii)*

$$\int_{R^{n-1}}\mathscr{K}(y',\eta)dy'=1; \tag{4.3}$$

$$\int_{R^{n-1}}D^{2k+1}_{y'}\mathscr{K}(y',\eta)dy'=0, \quad k=0,1,2,\dots; \tag{4.4}$$

$$\int_{R^{n-1}}\mathscr{G}(y',t)dy'=\frac{t^{\nu-1}}{\Gamma(\nu)}; \tag{4.5}$$

*(iii)*

$$\mathscr{K}(y',0)=\prod_{j=1}^{n-1}\delta(-y_j)\ \text{ in the distributions sense}, \tag{4.6}$$

*where $\delta(y)$ is the Dirac delta function;*
*(iv)*

$$\begin{aligned}\mathscr{I}_1 :=& \int_0^t d\tau\int_{R^{n-1}_+}dy'\int_0^{+\infty}\Big|y_i-\frac{A_1^i\eta}{2\pi}\Big|^{\alpha}|D^3_{y'}\mathscr{K}(y'-A_1\eta/2\pi,\eta)||W(-\eta\tau^{-\nu};-\nu,0)|\frac{d\eta}{\tau}\\ &\leqslant C[t^{\frac{\nu\alpha}{3}}+t^{\nu\alpha}], \quad i=\overline{1,n-1};\end{aligned} \tag{4.7}$$

*(v)*

$$\begin{aligned}\mathscr{I}_2 :=& \int_0^t d\tau\int_{R^{n-2}_+}dy''\int_0^{+\infty}d\eta|W(-\eta\tau^{-\nu};-\nu,0)|\tau^{-1}\int_0^{\varepsilon}|D^3_{y'}\mathscr{K}(y'-A_1\eta/2\pi,\eta)|\\ &\times\Big|y_j-\frac{A_1^j\eta}{2\pi}\Big|^{\alpha}dy_i\leqslant C\varepsilon^{\alpha}, \quad i,j=\overline{1,n-1};\end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathscr{I}_3 :=& \int_0^t d\tau\int_{R^{n-2}_+}dy''\int_{\varepsilon}^{+\infty}dy_i\int_0^{+\infty}|W(-\eta\tau^{-\nu};-\nu,0)|\tau^{-1}|D^4_{y'}\mathscr{K}(y'-A_1\eta/2\pi,\eta)|\\ &\times\Big|y_j-\frac{A_1^j\eta}{2\pi}\Big|^{\alpha}d\eta\leqslant C\varepsilon^{\alpha-1}, \quad i,j=\overline{1,n-1};\end{aligned} \tag{4.9}$$

$$\mathscr{I}_4 := \int_0^t d\tau \int_0^{+\infty} d\eta |W(-\eta\tau^{-\nu}; -\nu, 0)|\tau^{-1} \int_{R_+^{n-2}} \left| \int_{|y_j|>\varepsilon} D_{y'}^3 \mathscr{K}(y' - A_1\eta/2\pi, \eta) dy_j \right| dy''$$
$$\leqslant C, \quad j = \overline{1, n-1}; \tag{4.10}$$

*(vi)*

$$\mathscr{I}_5 := \int_0^t d\tau \int_{R_+^{n-1}} dy' \int_0^{+\infty} \left| y_i - \frac{A_1^i \eta}{2\pi} \right|^\alpha |\mathscr{K}(y' - A_1\eta/2\pi, \eta)||W(-\eta\tau^{-\nu}; -\nu, -\nu)| \frac{d\eta}{\tau^{1+\nu}}$$
$$\leqslant C[t^{\frac{\nu\alpha}{3}} + t^{\nu\alpha}], \quad i = \overline{1, n-1}; \tag{4.11}$$

*(vii)*

$$\mathscr{I}_6 := \int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_0^{+\infty} d\eta |W(-\eta\tau^{-\nu}; -\nu, -\nu)|\tau^{-1-\nu} \int_0^{\varepsilon} |\mathscr{K}(y' - A_1\eta/2\pi, \eta)|$$
$$\times \left| y_j - \frac{A_1^j \eta}{2\pi} \right|^\alpha dy_i \leqslant C\varepsilon^\alpha, \quad i, j = \overline{1, n-1}; \tag{4.12}$$

$$\mathscr{I}_7 := \int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_{\varepsilon}^{+\infty} dy_i \int_0^{+\infty} |W(-\eta\tau^{-\nu}; -\nu, -\nu)|\tau^{-1-\nu} |D_{y'} \mathscr{K}(y' - A_1\eta/2\pi, \eta)|$$
$$\times \left| y_j - \frac{A_1^j \eta}{2\pi} \right|^\alpha d\eta \leqslant C\varepsilon^{\alpha-1}, \quad i, j = \overline{1, n-1}; \tag{4.13}$$

$$\mathscr{I}_8 := \int_0^t d\tau \int_0^{+\infty} d\eta |W(-\eta\tau^{-\nu}; -\nu, -\nu)|\tau^{-1-\nu} \int_{R_+^{n-2}} \left| \int_{|y_j|>\varepsilon} \mathscr{K}(y' - A_1\eta/2\pi, \eta) dy_j \right| dy''$$
$$\leqslant C, \quad j = \overline{1, n-1}. \tag{4.14}$$

*Proof.* Repeating the arguments from the proof of Lemma 4.2 [29], one can easily obtain statements $(i)-(iii)$ from this Lemma. To get inequality (4.7), we use (4.2) and deduce

$$\mathscr{I}_1 \leqslant C \int_0^t d\tau \int_{R_+^{n-1}} dy' \int_0^{+\infty} [y_i^\alpha + \eta^\alpha] \frac{|W(-\eta\tau^{-\nu}; -\nu, 0)|}{\tau\eta^{\frac{2+n}{3}}} exp\left\{ -A \sum_{j=1}^{n-1} \frac{y_j}{\eta^{1/3}} \right\} d\eta. \tag{4.15}$$

After that performing the consecutive change of variables:

$$r_k = y_k/\eta^{1/3}, \quad k = \overline{1, n-1}; \tag{4.16}$$

$$\eta\tau^{-\nu} = \rho, \tag{4.17}$$

in the right-hand sides of (4.15), we deduce

$$\mathscr{I}_1 \leqslant C \int\limits_{R_+^{n-1}} exp\{-A\sum_{j=1}^{n-1} r_j\}dr' \int\limits_0^t d\tau \int\limits_0^{+\infty} [r_i^{\alpha}\tau^{\frac{\alpha\nu}{3}-1}\rho^{\alpha/3}+\rho^{\alpha}\tau^{\alpha\nu-1}] \frac{|W(-\rho;-\nu,0)|d\rho}{\rho}$$
$$\leqslant C\int\limits_0^t d\tau \int\limits_0^{+\infty} [\tau^{\frac{\alpha\nu}{3}-1}\rho^{\alpha/3}+\rho^{\alpha}\tau^{\alpha\nu-1}]W(-\rho;-\nu,1-\nu)d\rho. \tag{4.18}$$

Note that the last inequality in (4.18) follows from the next equality for the Wright functions (see, e.g., (2.2.5) in [24] or (F7) in [20]):

$$W(-z;-b,d-1)+(d-1)W(-z;-b,d) = bzW(-z;-b,d-b). \tag{4.19}$$

Then, applying inequality (20) from [29] to the inner integral in the last inequality of (4.18), we get (4.7).

As for (4.8), simple calculations lead to

$$\mathscr{I}_2 \leqslant C\Big[\int\limits_0^t d\tau \int\limits_{R_+^{n-2}} dy'' \int\limits_0^{+\infty} d\eta \frac{|W(-\eta\tau^{-\nu};-\nu,0)|}{\tau} \int\limits_0^{\varepsilon} y_j^{\alpha}|D_{y'}^3\mathscr{K}(y'-A_1\eta/2\pi,\eta)|dy_i$$
$$+\int\limits_0^t d\tau \int\limits_{R_+^{n-2}} dy'' \int\limits_0^{+\infty} d\eta |W(-\eta\tau^{-\nu};-\nu,0)|\tau^{-1} \int\limits_0^{\varepsilon} |D_{y'}^3\mathscr{K}(y'-A_1\eta/2\pi,\eta)|\eta^{\alpha}dy_i\Big]$$
$$\equiv C[\mathscr{I}_{21}+\mathscr{I}_{22}]. \tag{4.20}$$

Inequality (4.2) together with the arguments from the proof of Lemma 4.2 [29] allow us to conclude

$$|\mathscr{I}_{21}| \leqslant C\varepsilon^{\alpha}. \tag{4.21}$$

To estimate $\mathscr{I}_{22}$ we apply inequality (4.2) together with change of variables (4.16) for $k \neq i$ and obtain

$$\mathscr{I}_{22} \leqslant C\int\limits_0^t d\tau \int\limits_{R_+^{n-2}} dr'' exp\Big\{-A\sum_{j=1, j\neq i}^{n-1} r_j\Big\} \int\limits_0^{\varepsilon} dy_i \int\limits_0^{+\infty} \eta^{\alpha-4/3} exp\{-Ay_i/\eta^{1/3}\}$$
$$\times \frac{|W(-\eta\tau^{-\nu};-\nu,0)|}{\tau} d\eta. \tag{4.22}$$

Then equality (4.19) together with simple inequality:

$$0 < x^{\gamma}e^{-Ax} \leqslant C, \quad \gamma > 0, \quad x \in R_+^1, \tag{4.23}$$

lead to

$$\mathscr{I}_{22} \leqslant const. \int\limits_0^{\varepsilon} y_i^{\alpha-1}dy_i \int\limits_0^t d\tau \int\limits_0^{+\infty} \eta^{2\alpha/3}\tau^{-1-\nu}W(-\eta\tau^{-\nu};-\nu,1-\nu)d\eta. \tag{4.24}$$

At last, using change of variables (4.17) together with equality (22) from [29], one can deduce (4.8) from (4.20)–(4.24).

Note that the estimate of the term

$$\mathscr{I}_{31} := \int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_{\varepsilon}^{+\infty} dy_i \int_0^{+\infty} \frac{|W(-\eta\tau^{-\nu};-\nu,0)|}{\tau} |D_{y'}^4 \mathscr{K}(y'-A_1\eta/2\pi,\eta)| y_j^{\alpha} d\eta$$

has been obtained in Lemma 4.2 [29]:

$$|\mathscr{I}_{31}| \leqslant C\varepsilon^{\alpha-1}. \tag{4.25}$$

Thus, to prove inequality (4.9), it is enough to evaluate the term

$$\mathscr{I}_{32} := \int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_{\varepsilon}^{+\infty} dy_i \int_0^{+\infty} \frac{|W(-\eta\tau^{-\nu};-\nu,0)|}{\tau} |D_{y'}^4 \mathscr{K}(y'-A_1\eta/2\pi,\eta)| \eta^{\alpha} d\eta.$$

To this end, we apply again inequality (4.2) together with the change of variables (4.16) with $k \neq i$ and get

$$\begin{aligned}
\mathscr{I}_{32} &\leqslant C \int_{R_+^{n-2}} exp\Big\{-A\sum_{j=1,j\neq i}^{n-1} r_j\Big\} dr'' \int_0^t d\tau \int_0^{+\infty} d\eta \int_{\varepsilon}^{+\infty} \eta^{\alpha-5/3}\tau^{-1} e^{-Ay_i/\eta^{1/3}} \\
&\quad \times |W(-\eta\tau^{-\nu};-\nu,0)| dy_i \\
&\leqslant C \int_0^t d\tau \int_0^{+\infty} \eta^{\alpha-4/3}\tau^{-1} e^{-A\varepsilon/\eta^{1/3}} |W(-\eta\tau^{-\nu};-\nu,0)| d\eta \\
&\leqslant C \int_0^t d\tau \int_0^{+\infty} [\varepsilon/\eta^{1/3}]^{\alpha-1} \eta^{\alpha-4/3}\tau^{-1} |W(-\eta\tau^{-\nu};-\nu,0)| d\eta.
\end{aligned} \tag{4.26}$$

Here we again applied inequality (4.23).

Next equality (4.19) together with change of variable (4.17) allow us to deduce

$$\mathscr{I}_{32} \leqslant C\varepsilon^{\alpha-1} \int_0^t \tau^{-1+2\alpha\nu/3} d\tau \int_0^{+\infty} \rho^{2\alpha/3} W(-\rho;-\nu,1-\nu) d\rho \leqslant C\varepsilon^{\alpha-1}. \tag{4.27}$$

Note that to get the last inequality in (4.27), we used estimate (22) from [29]. Thus, (4.25) and (4.27) provide (4.9).

As for inequality (4.10), its proof is analogous the arguments of estimate (84) from [29].

At last, we remark that statements (**vi**) and (**vii**) of this lemma are proved as well as statements (**iv**) and (**v**). Thus, the proof of Lemma 4.1 is finished. □

Next we repeat the arguments from Section 7 [30], Lemma 4.3 [29] and apply results of Lemma 4.1 to establish the following statements.

PROPOSITION 4.1. *Let $\alpha,\ \nu\in(0,1)$, conditions* (1.6), (2.3), (2.5) *hold, and $f\in C([0,T],\ C^{1+\alpha}(R^{n-1}))$. Then the functions $\rho(x',t)$ and $u_-(x,t),\ u_+(x,t)$ which are represented with* (3.8) *and* (3.9) *satisfy conditions* (1.2)–(1.5). *Moreover, there are the following representations:*

$$\mathbf{D}_t^{\nu}\rho(x',t)=f(x',t)+\int_0^t d\tau\int_{R^{n-1}}[f(x'-y',t-\tau)-f(x',t-\tau)]\partial_\tau^{\nu}\mathcal{G}(y',\tau)dy',\quad (4.28)$$

*where*

$$\partial_\tau^{\nu}\mathcal{G}(y',\tau)=\int\limits_0^{+\infty}\tau^{-1-\nu}W(-\eta\tau^{-\nu};-\nu,-\nu)\mathcal{K}(y',\eta)d\eta;$$

$$D^4_{x'}\rho(x',t)=\int_0^t d\tau\int_{R^{n-1}}[D_{x'-y'}f(x'-y',t-\tau)-D_{x'}f(x',t-\tau)]D^3_{y'}\mathcal{G}(y',\tau)dy';\quad (4.29)$$

$$\frac{\partial^2u_\pm}{\partial x_l\partial x_j}=\begin{cases}\sum\limits_{i=1}^{n-1}a_{ii}\int_{R^{n-1}}\left[\dfrac{\partial^3\rho(x'-y',t)}{\partial y_i^2\partial y_l}-\dfrac{\partial^3\rho(x',t)}{\partial x_i^2\partial x_l}\right]\dfrac{\partial\mathcal{K}_\pm}{\partial y_j}(y',x_n)dy',\ l,j\neq n;\\ \sum\limits_{i=1}^{n-1}a_{ii}\int_{R^{n-1}}\left[\dfrac{\partial^3\rho(x'-y',t)}{\partial y_i^2\partial y_l}-\dfrac{\partial^3\rho(x',t)}{\partial x_i^2\partial x_l}\right]\dfrac{\partial\mathcal{K}_\pm}{\partial x_n}(y',x_n)dy',\ l\neq n,\ j=n;\\ \sum\limits_{i,m=1}^{n-1}a_{ii}\int_{R^{n-1}}\left[\dfrac{\partial^3\rho(x'-y',t)}{\partial y_i^2\partial y_m}-\dfrac{\partial^3\rho(x',t)}{\partial x_i^2\partial x_m}\right]\dfrac{\partial\mathcal{K}_\pm}{\partial y_m}(y',x_n)dy',\ j=l=n,\end{cases}\quad (4.30)$$

*for every $j,l=\overline{1,n}$.*

Then Lemma 4.1, Proposition 4.1 and Lemma 3.1 [30] together with results of Chapter 3 [15] allow us to deduce the following.

LEMMA 4.2. *Let conditions of Proposition* 4.1 *hold. Then there is the following inequality*

$$\|\mathbf{D}_t^{\nu}\rho\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}+\|\rho\|_{C([0,T];C^{4+\alpha}(R^{n-1}))}\leqslant C\|f\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}.\quad (4.31)$$

Note that to evaluate the term $\|\mathbf{D}_t^{\nu}\rho\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}$ we used statements (**vi**) and (**vii**) from Lemma 4.1. As for the estimate of the second term in the left-hand side in (4.31), we applied statement (**iv**) and (**v**) from Lemma 4.1.

To infer the same result for the functions $u_\pm$ which are given with (3.9), we use Lemma 4.2 together with results from [27] and deduce.

LEMMA 4.3. *Let conditions of Proposition* 4.1 *hold. Then*

$$\|u_+\|_{C([0,T];C^{2+\alpha}(\bar{R}^n_+))}+\|u_-\|_{C([0,T];C^{2+\alpha}(\bar{R}^n_-))}\leqslant C\|f\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}.\quad (4.32)$$

Next step of our investigation is a proof of the corresponding estimates to the functions $\rho(x',t),\ \mathbf{D}_t^{\nu}\rho(x',t),\ u_\pm(x,t)$ with respect to time.

PROPOSITION 4.2. *Let conditions of Proposition 4.1 hold. Then there is the following estimate*

$$\sum_{|m|=2}^{4} \langle D_{x'}^{m}\rho\rangle_{t,R_T^{n-1}}^{(\frac{4+\alpha-|m|}{3}\nu)} \leqslant C\|f\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}. \tag{4.33}$$

*Proof.* Estimate (4.33) is a simple consequence of interpolation inequality (2.1) and Lemma 4.2. Indeed, let us evaluate the term $\langle D_{x'}^{4}\rho\rangle_{t,R_T^{n-1}}^{(\frac{\alpha}{3}\nu)}$. To this end, we put in (2.1)

$$W(x') := \rho(x',t_1)-\rho(x',t_2), \quad l_1 := 1+\alpha, \quad l_2 := 4+\alpha, \quad l_1+a(l_2-l_1) := 4,$$

and get for every $t_1,\ t_2 \in [0,T]$:

$$\begin{aligned}\|D_{x'}^{4}\rho(\cdot,t_1)-D_{x'}^{4}\rho(\cdot,t_2)\|_{C(R^{n-1})} &\leqslant C[\|\rho(\cdot,t_1)-\rho(\cdot,t_2)\|_{C^{4+\alpha}(R^{n-1})}]^{\frac{3-\alpha}{3}}\\ &\times[\|\rho(\cdot,t_1)-\rho(\cdot,t_2)\|_{C^{1+\alpha}(R^{n-1})}]^{\alpha/3}.\end{aligned} \tag{4.34}$$

Then, inequalities (11), (12) from [29] together with (4.31) lead to

$$\begin{aligned}\|D_{x'}^{4}\rho(\cdot,t_1)-D_{x'}^{4}\rho(\cdot,t_2)\|_{C(R^{n-1})} &\leqslant C[2\|\rho\|_{C([0,T];C^{4+\alpha}(R^{n-1}))}]^{\frac{3-\alpha}{3}}|t_1-t_2|^{\alpha\nu/3}\\ &\times[\|\mathbf{D}_t^{\nu}\rho\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}]^{\alpha/3}\\ &\leqslant C|t_1-t_2|^{\alpha\nu/3}\|f\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}.\end{aligned} \tag{4.35}$$

The last inequality in (4.35) guarantees estimate (4.33) for $\langle D_{x'}^{4}\rho\rangle_{t,R_T^{n-1}}^{(\frac{\alpha}{3}\nu)}$.

As for evaluating of the rest term in the left-hand side of (4.33), they are estimated with the same way. □

We need the following result in order to get the estimate of $\mathbf{D}_t^{\nu}\rho,\ u_{\pm}$ with respect to time.

PROPOSITION 4.3. *Let* $\alpha,\ \nu \in (0,1)$, *then the functions* $\mathscr{K}_{\pm}(y',x_n)$ *given by* (3.4) *and* (3.5) *satisfy estimates*

$$\int_{R^{n-1}} |\mathscr{K}_{\pm}(y',x_n)|dy' \leqslant C. \tag{4.36}$$

*Proof.* Let us show (4.36) for the function $\mathscr{K}_{+}(y',x_n)$. The change of variables: $\xi_i x_n = \mu_i,\ i=\overline{1,n-1}$, in (3.4) leads to

$$\mathscr{K}_{+}(y',x_n) = \frac{1}{x_n^{n-1}(k+1)}\int_{R^{n-1}}\left[1+\frac{ik\langle a_1,\mu\rangle}{|\mu|}\right]exp\{-2\pi[|\mu|-i\langle\mu,y'/x_n\rangle]\}d\mu. \tag{4.37}$$

Since the function $\left[1+\frac{ik\langle a_1,\mu\rangle}{|\mu|}\right]exp\{-2\pi[|\mu|-i\langle\mu,y'/x_n\rangle]\}$ vanishes if $|\mu_j|\to+\infty$, $j=\overline{1,n-1}$; and one does not have any poles, we can calculate the integrals in (4.37) along the shifted contours: $\mu_j=\lambda_j\pm i\tilde{A}$, $j=\overline{1,n-1}$, $\tilde{A}$ is a positive constant. Then we have

$$\mathscr{K}_+(y',x_n)=const.\frac{\exp\{-A\sum_{j=1}^{n-1}|y_j|/x_n\}}{x_n^{n-1}}\int_{R^{n-1}}\left[1+\frac{ik\langle a_1,\lambda\pm i\tilde{A}\rangle}{|\lambda\pm i\tilde{A}|}\right]$$
$$\times\exp\{-2\pi[|\lambda\pm i\tilde{A}|+i\langle\lambda,y'/x_n\rangle]\}d\lambda,\ \ A=\tilde{A}2\pi. \tag{4.38}$$

One can easily verify that the module of the integrals in (4.38) is bounded uniformly in $y'$ and $x_n$. Thus, we have got

$$\int_{R^{n-1}}|\mathscr{K}_+(y',x_n)|dy'\leqslant const.\int_{R^{n-1}}\frac{\exp\{-A\sum_{j=1}^{n-1}|y_j|/x_n\}}{x_n^{n-1}}dy'. \tag{4.39}$$

Finally, performing the change of variables: $y_j/x_n=z_j$, $j=\overline{1,n-1}$, we deduce estimate (4.36) for the function $\mathscr{K}_+(y',x_n)$.

As for $\mathscr{K}_-(y',x_n)$, its estimate is obtained with the same way. $\square$

LEMMA 4.4. *Let conditions of Proposition 4.1 hold.*

*(i) If* $f\in C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\bar{R}_T^{n-1})$, *then there is the following estimate:*

$$\sum_{|m|=0}^{2}\langle D_x^m u_\pm\rangle_{t,R_{\pm T}^n}^{(\frac{2+\alpha-|m|}{3}\nu)}+\sum_{|m|=0}^{1}\langle D_{x'}^m\mathbf{D}_t^\nu\rho\rangle_{t,R_T^{n-1}}^{(\frac{1+\alpha-|m|}{3}\nu)}\leqslant C\|f\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\bar{R}_T^{n-1})}. \tag{4.40}$$

*(ii) If* $f\in C^{1+\alpha,\alpha\nu/3,\alpha}(\bar{R}_T^{n-1})$, *then*

$$\sum_{|m|=0}^{2}[D_x^m u_\pm]_{R_{\pm T}^n}^{(\alpha,\alpha\nu/3)}+\sum_{|m|=0}^{1}[D_{x'}^m\mathbf{D}_t^\nu\rho]_{R_T^{n-1}}^{(\alpha,\alpha\nu/3)}+\sum_{|m|=0}^{4}[D_{x'}^m\rho]_{R_T^{n-1}}^{(\alpha,\alpha\nu/3)}$$
$$\leqslant C\|f\|_{C^{1+\alpha,\alpha\nu/3,\alpha}(\bar{R}_T^{n-1})}. \tag{4.41}$$

*Proof.* Interpolation inequality (2.1) together with estimate (4.32) lead to

$$\langle D_x^2u_\pm\rangle_{t,R_{\pm T}^n}^{(\alpha\nu/3)}+\langle D_xu_\pm\rangle_{t,R_{\pm T}^n}^{(\frac{1+\alpha}{3}\nu)}\leqslant C\Big\{\|f\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}^{\frac{2}{2+\alpha}}\left[\langle u_\pm\rangle_{t,R_{\pm T}^n}^{(\frac{2+\alpha}{3}\nu)}\right]^{\frac{\alpha}{2+\alpha}}$$
$$+\|f\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}^{\frac{1}{2+\alpha}}\left[\langle u_\pm\rangle_{t,R_{\pm T}^n}^{(\frac{2+\alpha}{3}\nu)}\right]^{\frac{1+\alpha}{2+\alpha}}\Big\}. \tag{4.42}$$

Then based on representations (3.9), Propositions 4.2 and 4.3, one can easily deduce inequality

$$\langle u_\pm\rangle_{t,R^n_{\pm T}}^{(\frac{2+\alpha}{3}\nu)} \leqslant C\langle D^2_{x'}\rho\rangle_{t,R^{n-1}_T}^{(\frac{2+\alpha}{3}\nu)} \leqslant C\|f\|_{C([0,T];C^{1+\alpha}(R^{n-1}))}. \tag{4.43}$$

After that, we joint inequalities (4.42) and (4.43) and get

$$\sum_{|m|=0}^{2}\langle D^m_x u_\pm\rangle_{t,R^n_{\pm T}}^{(\frac{2+\alpha-|m|}{3}\nu)} \leqslant C\|f\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\bar{R}^{n-1}_T)}. \tag{4.44}$$

Finally, boundary condition (1.3) together with Proposition 4.1 and inequality (4.44) allow us to infer estimate (4.40) for the term $\langle D^m_{x'}\mathbf{D}^\nu_t\rho\rangle_{t,R^{n-1}_T}^{(\frac{1+\alpha-|m|}{3}\nu)}$, $|m|=0,1$.

As for estimate (4.41), it follows from inequalities (4.31) and (4.32). $\square$

### 4.2. Completion of the proof of Theorem 2.1

Note that estimates (2.2) and (2.4) follow immediately from Lemmas 4.2–4.4 and Proposition 4.2. The uniqueness of the constructed solution $(u_+,u_-,\rho)$ follows from (2.2). Proposition 4.1 together with results from [27] ensure that functions $u_+,u_-,\rho$ satisfy conditions (1.1)–(1.5). Thus, we prove Theorem 2.1 under condition (2.5). To remove this restriction, we apply method of a parameter extension together with the results of Theorem 2.1 in the case of (2.5).

Using the arguments above one can easily verify the following result, which will be essentially use to prove Theorem 5.2 in Subsection 5.4.

REMARK 4.1. Let conditions of Theorem 2.1 hold and boundary conditions (1.3), (1.4) be changed by

$$\mathbf{D}^\nu_t\rho + a_0\frac{\partial u_+}{\partial \bar{n}} = f(x',t) \ \text{ on } \ \bar{R}^{n-1}_T,$$

$$\frac{\partial u_-}{\partial \bar{n}} - k\delta\frac{\partial u_+}{\partial \bar{n}} = f_2(x',t) \ \text{ on } \ \bar{R}^{n-1}_T, \tag{4.45}$$

where $\delta\in[0,1]$. Then problem (1.1), (1.2), (4.45) and (1.5) has a unique classical solution $(u_+,u_-,\rho)$ which satisfies inequalities (2.2) and (2.4) with the constants independent of $\delta$.

## 5. Nonlinear free boundary problem subjected by anomalous diffusion

In this Section we apply the results of Theorem 2.1 to investigate the Muskat problem with surface tension on a free boundary in the case of subdiffusion (“fractional” Muskat problem).

### 5.1. Statement of the problem

Let $\Omega$ be a double-connected bounded open domain in $R^n$, $n \geqslant 2$ with the boundary $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$, $\Gamma_1 \bigcap \Gamma_2 = \emptyset$. Let $\Upsilon(t)$, for each $t \in [0,T]$, be a surface: $\Upsilon(t) \subset \Omega$, that separates $\Omega$ into two subdomains $\Omega_1(t)$ and $\Omega_2(t)$ such that $\Omega = \Omega_1(t) \bigcup \Upsilon(t) \bigcup \Omega_2(t)$, and $\partial\Omega_i = \Gamma_i \bigcup \Upsilon(t)$, $i = 1,2$.

The mathematical model of the "fractional" Muskat problem in the case of nonzero surface tension is determining functions $p_i(y,t)$, $y \in \Omega_i(t)$, $t \in [0,T]$, $i = 1,2$, and an unknown boundary $\Upsilon(t)$ by the following conditions:

$$-\Delta_y p_i = 0 \;\text{ in }\; \Omega_i(t), \quad i = 1,2,\, t \in [0,T], \tag{5.1}$$

$$p_1 - p_2 = \gamma\kappa(\Upsilon(t)) \;\text{ on }\; \Upsilon(t), \tag{5.2}$$

$$V_{n_t}^{\nu} = -k_1 \frac{\partial p_1}{\partial n_t} = -k_2 \frac{\partial p_2}{\partial n_t} \;\text{ on }\; \Upsilon(t), \quad \nu \in (0,1); \tag{5.3}$$

$$p_i = \psi_i(y) \;\text{ on }\; \Gamma_{iT} = \Gamma_i \times [0,T], \tag{5.4}$$

$$\Omega_i(t)|_{t=0} = \Omega_i, \;\text{ and, hence, }\; \Upsilon(t)|_{t=0} = \Upsilon \;\text{ are given.} \tag{5.5}$$

Here $\gamma$, $k_i$, $i = 1,2$, are given positive constants, $k_1 \neq k_2$, the quantity $\kappa(\Upsilon(t))$ defined on $\Upsilon(t)$ denotes the mean curvature of this surface; $n_t$ is the unit normal to $\Upsilon(t)$ directed in $\Omega_1(t)$; $\psi_i(y)$, $i = 1,2$, are given positive functions; $V_{n_t}^{\nu}$ is the fractional velocity of the boundary $\Upsilon(t)$ in the direction of the normal $n_t$ and is represented by (see, e.g., [33]):

$$V_{n_t}^{\nu} = \langle \mathbf{D}_t^{\nu} \Upsilon(t), n_t \rangle.$$

We assume that

$$\Upsilon \in C^{k+\alpha}, \quad \Gamma_i \in C^{k-2+\alpha}, \quad k \geqslant 7, \quad i = 1,2, \quad \alpha \in (0,1). \tag{5.6}$$

### 5.2. The nonlinear mapping and main results

We apply Hanzawa approach [9] to reduce free boundary problem (5.1)–(5.5) to a problem in a fixed domain.

Let $\omega = (\omega_1, \ldots, \omega_{n-1})$ be some coordinates on $\Upsilon$. We represent $\Upsilon$ in the form $y = \bar{m}(\omega)$ and denote by $\bar{n}(\omega)$ the normal to $\Upsilon$ directed into $\Omega_1$. For sufficiently small $\gamma_0 > 0$, $\omega-$surfaces: $\bar{m}(\omega) + \eta\bar{n}(\omega)$, $|\eta| < 2\gamma_0$, do not intersect each other and $\Gamma_1 \bigcup \Gamma_2$. On the set $N = \{y \in R^n : \; dist(y,\Upsilon) < 3\gamma_0/2\}$ we introduce the local coordinates $(\omega,\eta)$ by

$$y = (y_1, \ldots, y_n) = \bar{m}(\omega) + \eta\bar{n}(\omega), \; \bar{m}(\omega) \subset \Upsilon.$$

The free boundary in problem (5.1)–(5.5) can be represented as

$$\Upsilon(t) = \{(y,t) : y(\omega,t) = \bar{m}(\omega) + \wp(\omega,t)\bar{n}(\omega), \; t \in [0,T]\}, \tag{5.7}$$

where $\wp(\omega,t)$ is an unknown function, and

$$|\wp(\omega,t)| < \gamma_0 C_0, \quad 0 < C_0 < 1, \quad \wp(\omega,0) = 0. \tag{5.8}$$

Thus, the surface $\Upsilon(t)$ in the local variables is given by

$$\Phi_{\wp}(y,t) = \eta(y) - \wp(\omega(y),t) = 0. \tag{5.9}$$

As well as in [31] and [29], we can rewrite boundary conditions (5.2) and (5.3) as

$$\begin{aligned}\mathbf{D}_t^{v}\wp\frac{\langle\nabla_y\Phi_{\wp}(y,0),\nabla_y\Phi_{\wp}(y,t)\rangle}{|\nabla_y\Phi_{\wp}(y,0)|} &= -k_1\langle\nabla_y p_1,\nabla_y\Phi_{\wp}(y,t)\rangle\\ &= -k_2\langle\nabla_y p_2,\nabla_y\Phi_{\wp}(y,t)\rangle;\end{aligned} \tag{5.10}$$

$$p_1 - p_2 = \gamma\nabla_y\left(\frac{\nabla_y\Phi_{\wp}(y,t)}{|\nabla_y\Phi_{\wp}(y,t)|}\right)\Bigg|_{\Upsilon(t)}. \tag{5.11}$$

Let $\chi(\lambda)\in C_0^{\infty}(R^1)$, $\chi(\lambda)=1$ if $|\lambda|<\gamma_0/4$ and $\chi(\lambda)=0$ if $|\lambda|>3\gamma_0/4$, $|\chi^{(k)}|\leqslant C_1/\gamma_0^k$, $k=\overline{1,3}$. We choose the constant $C_0$ in (5.8) such that $C_0<\frac{1}{2C_1}$ then $1+\chi\prime(\lambda)\mu\geqslant 1/2$ if $|\mu|\leqslant\gamma_0 C_0$. We will use the coordinates $(\omega,\eta)$ to define the diffeomorphism

$$e_{\wp}:(x,t)\to(y,t)$$

from $X_T=R^n\times[0,T]$ onto $Y_T=R^n\times[0,T]$ by setting

$$\begin{cases} y=x, \text{ if } dist(x,\Upsilon)>3\gamma_0/4,\\ \omega(y)=\omega(x),\eta(y)=\lambda(x)+\chi(\lambda(x))\wp(\omega(x),t) \text{ otherwise,}\end{cases} \tag{5.12}$$

such that the transform $e_{\wp}^{-1}$ maps $\Omega_i(t)$ onto $\Omega_i\times(0,T):=\Omega_{iT}$, $i=1,2$; and $\Upsilon(t)$ onto $\Upsilon\times[0,T]:=\Upsilon_T$; the free boundary is given by

$$e_{\wp}(\{\lambda(x)=0\}), \tag{5.13}$$

and $\omega(x)$, $\lambda(x)$ are the coordinates in $X_T$ similar to the coordinates $\omega(y)$, $\eta(y)$ in $Y_T$. After the change of variables (5.12), we have the new desired functions

$$v_i(x_1,\ldots,x_n,t) = p_i(y_1,\ldots,y_n,t)\circ e_{\wp}(x,t), \quad i=1,2. \tag{5.14}$$

Denote by $\nabla_{\wp}=(E_{\wp}^*)^{-1}\nabla_x$ where $E_{\wp}$ is the Jacobi matrix of the mapping $y=e_{\wp}(x,t)$ so that

$$\nabla_y=\nabla_{\wp} \text{ and } \Delta_y=\nabla_{\wp}^2.$$

Taking into account that $y=x$ near $\Gamma_{iT}$, $i=1,2$, we reduce free boundary problem (5.1)–(5.5) to the following nonlinear problem in the fixed domain:

$$\nabla_{\wp}^2 v_i(x,t)=0 \text{ in } \Omega_{iT} \quad i=1,2; \tag{5.15}$$

$$v_1(x,t)-v_2(x,t)=\gamma\sum_{ij=1}^{n-1}\mathscr{A}_{ij}(\omega,\wp,\nabla_{\omega}\wp)\frac{\partial^2\wp}{\partial\omega_i\partial\omega_j}+\mathscr{A}_0(\omega,\wp,\nabla_{\omega}\wp) \quad on\ \Upsilon_T; \tag{5.16}$$

$$-\mathbf{D}_t^{v}\wp = k_1\left[S(\omega,\wp,\nabla_{\omega}\wp)\frac{\partial v_1}{\partial\lambda}+\sum_{i=1}^{n-1}S_i(\omega,\wp,\nabla_{\omega}\wp)\frac{\partial v_1}{\partial\omega_i}\right]$$

$$= k_2\left[S(\omega,\wp,\nabla_\omega\wp)\frac{\partial v_2}{\partial\lambda}+\sum_{i=1}^{n-1}S_i(\omega,\wp,\nabla_\omega\wp)\frac{\partial v_2}{\partial\omega_i}\right] \text{ on } \Upsilon_T; \tag{5.17}$$

$$v_i=\psi_i(x) \text{ on } \Gamma_{iT}; \tag{5.18}$$

$$\wp(\omega,0)=0, \quad \omega\in\Upsilon, \tag{5.19}$$

where $\nabla_\omega\wp=\left(\frac{\partial\wp}{\partial\omega_1},\dots,\frac{\partial\wp}{\partial\omega_{n-1}}\right)$; $S(\omega,\wp,\nabla_\omega\wp)$, $S_i(\omega,\wp,\nabla_\omega\wp)$, $\mathscr{A}_{ij}(\omega,\wp,\nabla_\omega\wp)$, $i,j=\overline{1,n-1}$, $\mathscr{A}_0(\omega,\wp,\nabla_\omega\wp)$ are some specific smooth functions (their representations and properties can be found in [1], [5] and [30]) such that

$$S(\omega,0,0)=1, \quad \frac{\partial S}{\partial\wp_{\omega_i}}(\omega,0,0)=0, \quad S_i(\omega,0,0)=0, \quad i=\overline{1,n-1},$$

$$\det\{\mathscr{A}_{ij}(\omega,0,0)\}\geqslant\varepsilon_0>0. \tag{5.20}$$

As easily verified,

$$\nabla^2_\wp|_{t=0}=\Delta_x. \tag{5.21}$$

We look for the functions $v_{i0}(x)$ as a solution of the following transmission problem:

$$\Delta_x v_{i0}=0 \text{ in } \Omega_i, \ i=1,2; \quad v_{i0}|_{\Gamma_i}=\psi_i(x);$$

$$v_{10}-v_{20}=\gamma\kappa(\Upsilon) \text{ and } k_1\frac{\partial v_{10}}{\partial\bar n(\omega)}=k_2\frac{\partial v_{20}}{\partial\bar n(\omega)} \quad \text{on } \Upsilon. \tag{5.22}$$

We assume that conditions (5.6) hold, and

$$\psi_i(x)\in C^{5+\alpha}(\Gamma_i), \quad i=1,2. \tag{5.23}$$

Then, the regularity theory for transmission problems (see, e.g., [27]) can be applied to problem (5.22) that yields the one-valued solvability of this problem and

$$\sum_{i=1}^{2}\|v_{i0}\|_{C^{5+\alpha}(\bar\Omega_i)}\leqslant C\Big(\sum_{i=1}^{2}\|\psi_i\|_{C^{5+\alpha}(\Gamma_i)}+\|\kappa(\Upsilon)\|_{C^{5+\alpha}(\Upsilon)}\Big), \quad v_{i0}(x)=v_i(x,0) \text{ in } \bar\Omega_i. \tag{5.24}$$

THEOREM 5.1. *Let $\alpha$, $\nu\in(0,1)$; $\Gamma_i$ and $\Upsilon$ satisfy assumptions mentioned in Subsection 5.1 and conditions (5.6), (5.23) hold. Then for some small $T$, there exists a unique solution $(v_1(x,t),v_2(x,t),\wp(\omega,t))$ of nonlinear problem (5.15)–(5.19) for $t\in[0,T]$ such that*

$$v_i(x,t)\in C^{2+\alpha,\frac{\alpha\nu}{3},\alpha}(\bar\Omega_{iT}), \quad \wp(\omega,t)\in C^{4+\alpha,\frac{\alpha\nu}{3},\alpha}(\Upsilon_T), \quad \mathbf{D}_t^\nu\wp(\omega,t)\in C^{1+\alpha,\frac{\alpha\nu}{3},\alpha}(\Upsilon_T),$$

*and $v_i(x,0)$ is given with (5.22) and (5.24).*

The proof of Theorem 5.1 consists in two steps. The first is the linearization of nonlinear problem (5.15)–(5.19) on the initial data $(v_{10},\ v_{20})$, and then proving that the linear problem has a unique solution. On the next step we show that the corresponding nonlinear mapping is a contraction, and so it has a unique fixed point.

### 5.3. Linearization of system (5.15)–(5.19)

Based on boundary conditions (5.17) and relations (5.20), (5.24), we can conclude

$$\mathbf{D}_t^\nu \wp(\omega,0) = -k_1 S(\omega,0,0)\frac{\partial v_{10}}{\partial\lambda}|_\Upsilon = -k_2 S(\omega,0,0)\frac{\partial v_{20}}{\partial\lambda}|_\Upsilon. \tag{5.25}$$

Let us define a function $s(\omega,t)$ as

$$s(\omega,0)=0, \quad \mathbf{D}_t^\nu s(\omega,0) = \mathbf{D}_t^\nu \wp(\omega,0) \ \text{ on } \ \Upsilon. \tag{5.26}$$

In virtue of $\mathbf{D}_t^\nu \frac{t^\nu}{\Gamma(1+\nu)} = 1$, we may choose the function $s(\omega,t)$ as

$$s(\omega,t) = \frac{t^\nu}{\Gamma(1+\nu)}\mathbf{D}_t^\nu \wp(\omega,0)\Big|_\Upsilon. \tag{5.27}$$

Using (5.24)–(5.27), one can easily verify the following:

COROLLARY 5.1. *The function $s(\omega,t)$ given by (5.27) satisfies (5.26) and*

$$\|s\|_{C([0,T],C^{4+\alpha}(\Upsilon))} + \|\boldsymbol{D}_t^\nu s\|_{C([0,T],C^{4+\alpha}(\Upsilon))} + \sum_{|l|=0}^{4} \langle \boldsymbol{D}_t^\nu D_\omega^l s\rangle_{t,\Upsilon_T}^{(\beta)}$$
$$\leqslant C\Big(\sum_{j=1}^{2}\|\psi_j\|_{C^{5+\alpha}(\Gamma_j)} + \|\kappa(\Upsilon)\|_{C^{5+\alpha}(\Upsilon)}\Big), \quad \beta\in(0,1).$$

Then we introduce the new unknown functions $w_i(x,t)$, $i=1,2$, and $\sigma$ as

$$\sigma(\omega,t) = \wp(\omega,t) - s(\omega,t); \qquad w_i(x,t) = v_i(x,t) - v_{i0}(x), \quad i=1,2. \tag{5.28}$$

After some tedious calculations, we get next problem:

$$\Delta_x w_i = \mathscr{F}_{i0}(w_i,\sigma) \ \text{ in } \ \Omega_{iT}, \quad i=1,2;$$

$$w_1 - w_2 - \sum_{lj=1}^{n-1} b_{lj}(\omega,t)\frac{\partial^2\sigma}{\partial\omega_l\partial\omega_j} + \sum_{l=1}^{n-1} b_l(\omega,t)\frac{\partial\sigma}{\partial\omega_l} = \mathscr{F}_1(\sigma) \quad \text{on } \ \Upsilon_T;$$

$$\begin{aligned}\mathbf{D}_t^\nu\sigma &= -k_1 b_0(\omega)\frac{\partial w_1}{\partial\bar{n}(\omega)} + \mathscr{F}_2(w_1,\sigma)\\ &= -k_2 b_0(\omega)\frac{\partial w_2}{\partial\bar{n}(\omega)} + \mathscr{F}_3(w_2,\sigma) \quad \text{on } \ \Upsilon_T;\end{aligned}$$

$$w_i = 0 \ \text{ on } \ \Gamma_{iT}; \ \sigma(\omega,0) = \mathbf{D}_t^\nu\sigma(\omega,0) = 0, \ \omega\in\Upsilon, \tag{5.29}$$

where the representations of the functions $\mathscr{F}_{i0}(w_i,\sigma)$, $i=1,2$, $\mathscr{F}_1(\sigma)$, $\mathscr{F}_2(w_1,\sigma)$, $\mathscr{F}_3(w_2,\sigma)$, $b_{lj}$, $b_l$, $l,j=\overline{1,n-1}$, $b_0$ can be found in [1], [31].

System (5.29) can be written briefly as

$$\mathscr{A}\mathbf{z}=\mathscr{F}(\mathbf{z}), \quad \text{where} \quad \mathbf{z}=(w_1,w_2,\sigma), \tag{5.30}$$

where a linear operator $\mathscr{A}$ is defined by the left-hand side of (5.29), and $\mathscr{F}$ is a nonlinear operator, $\mathscr{F}(\mathbf{z})=\{\mathscr{F}_{10}(\mathbf{z}),\mathscr{F}_{20}(\mathbf{z}),\mathscr{F}_1(\mathbf{z}),\mathscr{F}_2(\mathbf{z}),\mathscr{F}_3(\mathbf{z})\}$. Following the arguments from Subsection 3.2 [31], we can deduce the following result.

COROLLARY 5.2. *The functions $\mathscr{F}_{i0}$, $i=1,2$, $\mathscr{F}_j$, $j=\overline{1,3}$, contain the higher derivatives of $w_i(x,t)$ and $\sigma(\omega,t)$ with the coefficients that tend to zero as $t\to 0$, the "quadratic" terms of minor differential orders of unknown functions. Moreover,*

$$\mathscr{F}_{i0}|_{t=0}=0, \quad i=1,2, \ \mathscr{F}_j|_{t=0}=0 \quad j=1,2,3; \tag{5.31}$$

$$b_{lj}(\omega,t),\ b_j(\omega,t)\in C^{3+\alpha,\nu,\alpha}(\Upsilon_T);\ b_0(\omega)\in C^{5+\alpha}(\Upsilon_T), \quad l,j=\overline{1,n-1}; \tag{5.32}$$

$$b_0(\omega)>0, \beta_0|\xi|^2\leqslant\sum_{lj=1}^{n-1}b_{lj}\xi_l\xi_j\leqslant\beta_0^{-1}|\xi|^2, \quad \xi\in R^{n-1}, \quad (\omega,t)\in\Upsilon_T; \tag{5.33}$$

*where $\beta_0$ is a positive constant.*

Note that results of Corollary 5.2 together with conditions (5.29) provide

$$w_i(x,0)=0, \quad x\in\Omega_i, \quad i=1,2. \tag{5.34}$$

## 5.4. Proof of Theorem 5.1

First of all we prove the boundedness of the linear operator $\mathscr{A}$ in the corresponding functional spaces. To this end, we freeze the functional arguments in the functions and put

$$\begin{gathered}\mathscr{F}_{i0}(w_i,\sigma):=F_{i0}(x,t), \quad i=1,2, \qquad \mathscr{F}_1(\sigma):=F_1(x,t),\\ \mathscr{F}_2(w_1,\sigma):=F_2(x,t), \qquad \mathscr{F}_3(w_2,\sigma):=F_3(x,t).\end{gathered} \tag{5.35}$$

Then system (5.30) can be considered as a linear system with variable coefficients. The solvability of this system will be proved under the weaker assumptions on the coefficients then in Corollary 5.2.

THEOREM 5.2. *Let $\alpha$, $\nu\in(0,1)$, $\Upsilon$, $\Gamma_i\in C^{3+\alpha}$, $k_1$, $k_2$ be positive and $k_1\neq k_2$ condition (5.35) hold and*

$$F_{i0}\in C^{\alpha,\frac{\nu\alpha}{3},\alpha}(\bar{\Omega}_{iT}),\ i=1,2,\ F_1\in C^{2+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T),\ F_j\in C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T),\ j=2,3,$$

$$F_j(x,0)=0, \quad j=\overline{1,3} \quad F_{i0}(x,0)=0; \tag{5.36}$$

$$\beta_0|\xi|^2\leqslant\sum_{lm=1}^{n-1}b_{lm}(\omega,t)\xi_l\xi_m\leqslant\beta_0^{-1}|\xi|^2, \quad (\omega,t)\in\Upsilon_T, \quad \xi\in R^{n-1}, \tag{5.37}$$

$$b_0(\omega)>0; \quad b_0\in C^{1+\alpha}(\Upsilon), \quad b_{lm},\ b_l\in C^{2+\alpha,\nu,\alpha}(\Upsilon_T), \quad l,m=\overline{1,n-1}. \tag{5.38}$$

*Then for some small $T$, there exists a unique solution $(w_1, w_2, \sigma)$ of linear problem* (5.30)*, such that*

$$
\begin{aligned}
&\sum_{i=1}^{2} \|w_i\|_{C^{2+\alpha,\frac{\nu\alpha}{3},\alpha}(\bar{\Omega}_{iT})} + \|\sigma\|_{C^{4+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \|\mathbf{D}_t^{\nu}\sigma\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} \\
&\leqslant C\Bigg[\sum_{i=1}^{2} \|F_{i0}\|_{C^{\alpha,\frac{\nu\alpha}{3},\alpha}(\bar{\Omega}_{iT})} + \|F_1\|_{C^{2+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \|F_2\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \|F_3\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)}\Bigg],
\end{aligned}
\tag{5.39}
$$

*where $C$ is a positive constant independent of the right-hand sides of* (5.30) *and depends only on the measure of $\Omega_i$, $\Gamma_i$, $\Upsilon$, and $\|b_0\|_{C^{1+\alpha}(\Upsilon)}$, $\|b_{lm}\|_{C^{2+\alpha,\nu,\alpha}(\Upsilon_T)}$, $\|b_l\|_{C^{2+\alpha,\nu,\alpha}(\Upsilon_T)}$.*

*Proof.* At first, we prove Theorem 5.2 under condition:

$$
F_{i0},\ F_1 \equiv 0, \quad i = 1,2, \quad F_2 = F_3. \tag{5.40}
$$

We apply the method of parameter extension to solve problem (5.30) and rewrite it as

$$
\begin{gathered}
\Delta_x w_i = 0 \ \text{ in } \ \Omega_{iT}, \quad i = 1,2; \\
w_1 - w_2 - \sum_{lm=1}^{n-1} b_{lm}(\omega,t)\frac{\partial^2\sigma}{\partial\omega_l\partial\omega_m} + \sum_{l=1}^{n-1} b_l(\omega,t)\frac{\partial\sigma}{\partial\omega_l} = 0 \ \text{ on } \ \Upsilon_T; \\
\mathbf{D}_t^{\nu}\sigma + k_2 b_0(\omega)\frac{\partial w_2}{\partial\bar{n}(\omega)} = F_2 \ \text{ on } \ \Upsilon_T; \\
\frac{\partial w_1}{\partial\bar{n}(\omega)} - k\delta\frac{\partial w_2}{\partial\bar{n}(\omega)} = 0 \ \text{ on } \ \Upsilon_T, \quad k = \frac{k_2}{k_1}; \\
w_i = 0 \ \text{ on } \ \Gamma_{iT}; \quad \sigma(\omega,0) = \mathbf{D}_t^{\nu}\sigma(\omega,0) = 0, \quad \omega \in \Upsilon,
\end{gathered}
\tag{5.41}
$$

where $\delta \in [0,1]$.

If $\delta = 1$, problem (5.41) is just problem (5.29), when $\delta = 0$, problem (5.41) splits into two boundary value problems:

$$
\Delta_x w_1 = 0 \ \text{ in } \ \Omega_{1T}, \quad \frac{\partial w_1}{\partial\bar{n}(\omega)}|_{\Upsilon_T} = 0, \quad w_1|_{\Gamma_{1T}} = 0, \tag{5.42}
$$

so $w_1 \equiv 0$ in $\bar{\Omega}_{1T}$; and

$$
\begin{gathered}
\Delta_x w_2 = 0 \ \text{ in } \ \Omega_{2T}; \quad \sigma(\omega,0) = \mathbf{D}_t^{\nu}\sigma(\omega,0) = 0, \quad \omega \in \Upsilon; \\
w_2 + \sum_{lm=1}^{n-1} b_{lm}(\omega,t)\frac{\partial^2\sigma}{\partial\omega_l\partial\omega_m} - \sum_{l=1}^{n-1} b_l(\omega,t)\frac{\partial\sigma}{\partial\omega_l} = 0 \ \text{ on } \ \Upsilon_T; \\
\mathbf{D}_t^{\nu}\sigma + k_2 b_0(\omega)\frac{\partial w_2}{\partial\bar{n}(\omega)} = F_2 \ \text{ on } \ \Upsilon_T.
\end{gathered}
\tag{5.43}
$$

Problem (5.43) has been studied in Section 5 [29] (see Theorem 5.1) where one-to-one local solvability of (5.43) has been proved: $w_2 \in C^{2+\alpha,\frac{\nu\alpha}{3},\alpha}(\bar{\Omega}_{2T})$, $\sigma \in C^{4+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)$, $\mathbf{D}_t^\nu \sigma \in C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)$, and

$$\begin{aligned}&\|w_2\|_{C^{2+\alpha,\frac{\nu\alpha}{3},\alpha}(\bar{\Omega}_{2T})} + \|\sigma\|_{C^{4+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \|\mathbf{D}_t^\nu \sigma\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)}\\ &\leqslant C\|F_2\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)},\end{aligned} \tag{5.44}$$

where the positive constant $C$ depends only on the measure of $\Omega_i$, $\Gamma_i$, $\Upsilon$, and $\|b_0\|_{C^{1+\alpha}(\Upsilon)}$, $\|b_{lj}\|_{C^{2+\alpha,\nu,\alpha}(\Upsilon_T)}$, $\|b_i\|_{C^{2+\alpha,\nu,\alpha}(\Upsilon_T)}$.

To show the well-posedness of linear problem (5.29), we have to obtain apriori estimate uniformly with respect to $\delta$ for the solution $(w_1, w_2, \sigma)$ of problem (5.41).

Adapting the standard Schauder technique to the case of a fractional derivative and applying the results of Theorem 2.1 and Remark 4.1, we deduce:

$$\begin{aligned}&\sum_{i=1}^{2}\|w_i\|_{C^{2+\alpha,\frac{\nu\alpha}{3},\alpha}(\bar{\Omega}_{iT})} + \|\sigma\|_{C^{4+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \|\mathbf{D}_t^\nu \sigma\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)}\\ &\leqslant C_1\|F_2\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \frac{C_1}{\varepsilon^{2+\alpha}}\sum_{i=1}^{2}\langle w_i\rangle_{t,\Omega_{iT}}^{(\nu\alpha/3)},\end{aligned} \tag{5.45}$$

where the positive constant $C_1$ is independent of $\delta$; $\varepsilon$ is a positive constant which will be chosen below.

Let us denote

$$V_i := w_i(\cdot,t_1) - w_i(\cdot,t_2), \quad i = 1,2, \quad \mathscr{S} := \sigma(\cdot,t_1) - \sigma(\cdot,t_2),$$

$$B_{lj} := b_{lj}(\cdot,t_1) - b_{lj}(\cdot,t_2), \qquad B_j := b_j(\cdot,t_1) - b_j(\cdot,t_2), \quad l,j = \overline{1,n-1},$$

and apply the results from [27] and embedding theorem to the following transmission problem

$$\begin{aligned}&\Delta_x V_i = 0 \text{ in } \Omega_{iT}, \ V_i = 0 \text{ on } \Gamma_{iT};\\ &V_1 - V_2 = \sum_{lj=1}^{n-1}\left[B_{lj}(\omega,t)\frac{\partial^2\sigma}{\partial\omega_l\partial\omega_j} + b_{lj}(\omega,t_2)\frac{\partial^2\mathscr{S}}{\partial\omega_l\partial\omega_j}\right]\\ &-\sum_{l=1}^{n-1}\left[B_l(\omega,t)\frac{\partial\sigma}{\partial\omega_l} + b_l(\omega,t_2)\frac{\partial\mathscr{S}}{\partial\omega_l}\right] \equiv g_0 \text{ on } \Upsilon_T;\\ &\frac{\partial V_1}{\partial\bar{n}(\omega)} - k\delta\frac{\partial V_2}{\partial\bar{n}(\omega)} = 0 \text{ on } \Upsilon_T.\end{aligned} \tag{5.46}$$

Thus, we can deduce

$$\sup_{\bar{\Omega}_{iT}}|V_i| \leqslant C\|g_0\|_{C([0,T],C^1(\Upsilon))}, \quad i = 1,2. \tag{5.47}$$

Finally, interpolation inequality (2.1) together with (5.47) allow us to conclude

$$\sum_{i=1}^{2}\langle w_i\rangle_{t,\Omega_{iT}}^{(\frac{1+\alpha}{3}\nu)} \leqslant CT^{\nu\alpha/3}[\|\sigma\|_{C^{4+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \|\mathbf{D}_t^{\nu}\sigma\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)}]. \tag{5.48}$$

Then we choose $\varepsilon$ such that $C_1CT^{\frac{1+\alpha}{3}\nu}\varepsilon^{-2-\alpha} < 1/2$ and get from (5.48) and (5.45) the uniformly estimate

$$\begin{aligned}&\sum_{i=1}^{2}\|w_i\|_{C^{2+\alpha,\frac{\nu\alpha}{3},\alpha}(\bar{\Omega}_{iT})} + \|\sigma\|_{C^{4+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)} + \|\mathbf{D}_t^{\nu}\sigma\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)}\\ &\leqslant C\|F_2\|_{C^{1+\alpha,\frac{\nu\alpha}{3},\alpha}(\Upsilon_T)},\end{aligned} \tag{5.49}$$

where the positive constant $C$ is independent of $\delta$. Thus, we have proved Theorem 5.2 under condition (5.40). To avoid this restriction it is enough to use statement $(ii)$ from Proposition 2.3 [31]. $\square$

Next we prove solvability of nonlinear problem (5.29), and hence, (5.15)–(5.19). We introduce the functional spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, such that $\mathbf{z}\in\mathscr{H}_1$ and $\mathscr{F}\mathbf{z}\in\mathscr{H}_2$,

$$\begin{aligned}\mathscr{H}_1 &= \underset{0}{C}^{2+\alpha,\nu\alpha/3,\alpha}(\bar{\Omega}_{1T})\times \underset{0}{C}^{2+\alpha,\nu\alpha/3,\alpha}(\bar{\Omega}_{2T})\times \underset{0}{C}^{4+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)\\ &\quad\times \underset{0}{C}^{1+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T);\\ \mathscr{H}_2 &= \underset{0}{C}^{\alpha,\nu\alpha/3,\alpha}(\bar{\Omega}_{1T})\times \underset{0}{C}^{\alpha,\nu\alpha/3,\alpha}(\bar{\Omega}_{2T})\times \underset{0}{C}^{2+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)\times \underset{0}{C}^{1+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)\\ &\quad\times \underset{0}{C}^{1+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)\times C^{5+\alpha}(\Gamma_1)\times C^{5+\alpha}(\Gamma_2);\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{z}\|_{\mathscr{H}_1} &= \|(w_1,w_2,\sigma)\|_{\mathscr{H}_1}\\ &= \sum_{i=1}^{2}\|w_i\|_{C^{2+\alpha,\nu\alpha/3,\alpha}(\bar{\Omega}_{iT})} + \|\sigma\|_{C^{4+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)} + \|\mathbf{D}_t^{\nu}\sigma\|_{C^{1+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)};\\ \|\mathscr{F}\mathbf{z}\|_{\mathscr{H}_2} &= \|(\mathscr{F}_{10}(\mathbf{z}),\mathscr{F}_{20}(\mathbf{z}),\mathscr{F}_1(\mathbf{z}),\mathscr{F}_2(\mathbf{z}),\mathscr{F}_3(\mathbf{z}),0,0)\|_{\mathscr{H}_2}\\ &= \sum_{i=1}^{2}\|\mathscr{F}_{i0}(\mathbf{z})\|_{C^{\alpha,\nu\alpha/3,\alpha}(\bar{\Omega}_{iT})} + \|\mathscr{F}_1(\mathbf{z})\|_{C^{2+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)}\\ &\quad+\sum_{j=2}^{3}\|\mathscr{F}_j(\mathbf{z})\|_{C^{1+\alpha,\nu\alpha/3,\alpha}(\Upsilon_T)}.\end{aligned}$$

By Theorem 5.2 we can rewrite equation (5.30) as

$$\mathbf{z} = \mathscr{A}^{-1}\mathscr{F}(\mathbf{z}) \equiv \mathfrak{R}(\mathbf{z}),$$

where $\mathfrak{R}(\mathbf{z})$ is a nonlinear operator, $\mathfrak{R}:\mathscr{H}_1\to\mathscr{H}_2$. To show that $\mathfrak{R}$ is a contraction operator we apply standard arguments from Subsection 5.2 [31] together with results of Theorem 5.2. That finishes the proof of Theorem 5.1.

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