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A NONLOCAL FRACTIONAL HELMHOLTZ EQUATION

MOKHTAR KIRANE, BATIRKHAN KH. TURMETOV AND BERIKBOL T. TOREBEK

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Abstract. In this paper we study some boundary value problems for a fractional analogue of second order elliptic equation with an involution perturbation in a rectangular domain. Theorems on existence and uniqueness of a solution of the considered problems are proved by spectral method.

1. Introduction

The paper is concerned with four boundary value problems concerning the fractional analogue of Helmholtz equation with a perturbation term of involution type in the space variable. We obtain for them existence and uniqueness results based on the Fourier method.

To describe the problems, let $\Omega = \left\{(x,y) \in R^2 : 0 < x < 1, -\pi < y < \pi\right\}$. We consider the equation

$$D_x^\alpha D_x^\alpha u(x,y) + u_{yy}(x,y) - \varepsilon u_{yy}(x,-y) - c^2 u(x,y) = 0, \quad (x,y) \in \Omega, \tag{1}$$

where c, ε are real numbers and

$$D_x^\alpha u(x,y) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \frac{\partial u}{\partial s}(s,y)ds$$

is the Caputo derivative of order $\alpha \in (0,1]$ of u with respect to x [1].

Regular solution of Equation (1) is a function $u \in C\left(\bar{\Omega}\right)$, such that $D_x^\alpha u, D_x^{2\alpha} u, u_{yy} \in C(\Omega)$.

Since for $\alpha = 1$:

$$L_x^2 + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta.$$

Therefore, Equation (1) is a nonlocal generalization of the Helmholtz equation, which at $\varepsilon = 0$ coincides with the Helmholtz equation.

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PROBLEM D. *Find in the domain* Ω *a regular solution of Equation* (1)*, satisfying the following boundary value conditions:*

$$u(0,y) = \varphi(y), \quad u(1,y) = \psi(y), \quad -\pi \leqslant y \leqslant \pi, \tag{2}$$

$$u(x,-\pi) = u(x,\pi) = 0, \quad 0 \leqslant x \leqslant 1.$$

PROBLEM N. *Find in the domain* Ω *a regular solution of Equation* (1)*, such that* $u_y(x,y) \in C\left(\bar{\Omega}\right)$ *and satisfying conditions* (2) *and:*

$$u_y(x,-\pi) = u_y(x,\pi) = 0, \quad 0 \leqslant x \leqslant 1.$$

PROBLEM P. *Find in the domain* Ω *a regular solution of Equation* (1)*, such that* $u_y(x,y) \in C\left(\bar{\Omega}\right)$ *and satisfying conditions* (2) *and:*

$$u(x,-\pi) = u(x,\pi), \quad u_y(x,-\pi) = u_y(x,\pi), \quad 0 \leqslant x \leqslant 1.$$

PROBLEM AP. *Find in the domain* Ω *a regular solution of Equation* (1)*, such that* $u_y(x,y) \in C\left(\bar{\Omega}\right)$ *and satisfying conditions* (2) *and:*

$$u(x,-\pi) = -u(x,\pi), \quad u_y(x,-\pi) = -u_y(x,\pi), \quad 0 \leqslant x \leqslant 1.$$

Here $\varphi(y), \psi(y)$ *are given sufficiently smooth functions.*

Before we describe our results, let us dwell a while on the existing literature concerning equations with involution. Differential equations with modified arguments are equations in which the unknown function and its derivatives are evaluated with modifications of the time or space variables; such equations are called in general functional-differential equations. Among such equation, one can single, equations with involution; to describe them, let Γ be an open or a closed curve in the complex plane or the plane of real variables x and y.

The homeomorphism

$$a^2(t) = a(a(t)) = t, \ \ t \in \Gamma,$$

is called a Carleman shift (involution) [2].

Various problems for equations with involution were investigated in [3], [4].

Note that problems D, N and P for Equation (1) at $\varepsilon = 0$ were studied in [5], [6]. Some questions of solvability of boundary value problems with fractional analogues of the Laplace operator were studied in [7], [8].

Need to study boundary value problems for Equation (1) at $\varepsilon = 0$ is determined by using fractal Laplace equation to describe the production processes in mathematical modeling of socio-economic systems [9]. Note also that in [9] attention was drawn to the fact that the problem of finding a generalized two-factor Cobb-Douglas function is reduced to the classical boundary value problems for a generalized Laplace equation of fractional order.

2. Solution of one-dimensional equation with fractional derivative

Let μ be a positive real number, $S = \{t : 0 < t < 1\}$, $\bar{S} = \{t : 0 \leqslant t \leqslant 1\}$. We consider the problem

$$D^{2\alpha} y(t) - \mu^2 y(t) = 0, \quad t \in S, \tag{3}$$

$$y(0) = a, \quad y(1) = b, \tag{4}$$

where a, b are real numbers.

A solution of problem (3)–(4) is the function $y \in C(\bar{S})$, such that $D^{\alpha} y \in C(\bar{S})$, $D^{2\alpha} y \in C(S)$.

LEMMA 1. *The solution of problem* (3)–(4) *exists, is unique and it can be written in the form*

$$y(t) = aC(\mu t) + bS(\mu t), \tag{5}$$

where

$$C(\mu t) = \frac{E_{\alpha,1}(\mu) E_{\alpha,1}(-\mu t^{\alpha}) - E_{\alpha,1}(-\mu) E_{\alpha,1}(\mu t^{\alpha})}{2\mu E_{2\alpha,\alpha+1}(\mu^2)}, \tag{6}$$

$$S(\mu t) = \frac{t^{\alpha} E_{2\alpha,\alpha+1}\left(\mu^2 t^{2\alpha}\right)}{E_{2\alpha,\alpha+1}(\mu^2)}. \tag{7}$$

Here

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

is the Mittag-Leffler type function [1].

Proof. From [5] it is known that the general solution of equation (3) has the form

$$y(t) = D_1 E_{\alpha,1}(-\mu t^{\alpha}) + D_2 E_{\alpha,1}(\mu t^{\alpha}), \tag{8}$$

where D_1, D_2 are arbitrary constants.

Substituting the function (8) into the boundary conditions (4) for unknown coefficients D_1 and D_2 we get

$$D_1 = \frac{aE_{\alpha,1}(\mu) - aE_{\alpha,1}(-\mu) + E_{\alpha,1}(-\mu) - b}{E_{\alpha,1}(\mu) - E_{\alpha,1}(-\mu)},$$

$$D_2 = \frac{b - aE_{\alpha,1}(-\mu)}{E_{\alpha,1}(\mu) - E_{\alpha,1}(-\mu)}.$$

Since $E_{\alpha,1}(\mu) - E_{\alpha,1}(-\mu) = 2\mu E_{2\alpha,\alpha+1}\left(\mu^2\right)$, after some transformations, the solution of the problem (3)–(4) can be reduced to the form (5). This proves the lemma. $\square$

Furthermore, for any $0 < \alpha < 1$, consider the equation

$$y''(t) - \mu D^{2-\alpha} y(t) = 0, \quad t \in S. \tag{9}$$

The following statement is known (see [10]):

LEMMA 2. *If the function* $y(t) \in C\left(\bar{S}\right) \cap C^2(S)$, $y(t) \neq Const$ *is a solution of equation* (9)*, then it can not attain its positive maximum (negative minimum) within the segment* $\bar{S}$.

LEMMA 3. [1] *For* $E_{\alpha,\beta}(z)$ *as* $|z| \to \infty$ *the following asymptotic estimation holds*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(\frac{1}{|z|^{p+1}}\right), \tag{10}$$

where $|\arg z| \leqslant \rho_1 \pi, \rho_1 \in \left(\frac{\alpha}{2}, \min\{1, \alpha\}\right), \alpha \in (0,2)$, *and for* $\arg z = \pi$

$$E_{\alpha,\beta}(z) = \frac{1}{1+|z|}, \quad |z| \to \infty. \tag{11}$$

From Lemmas 2 and 3 follows

LEMMA 4. *For any* $t \in [0,1]$ *the following inequalities hold:*

$$0 \leqslant S(\mu t), \quad C(\mu t) \leqslant 1.$$

3. Spectral properties of the perturbed Sturm-Liouville problem

Application of the Fourier method for solving problems D, N, P, AP leads to the spectral equation

$$Y''(y) - \varepsilon Y''(-y) + \lambda Y(x) = 0, \quad -\pi < y < \pi, \tag{12}$$

supplemented with one of the local

$$Y(-\pi) = Y(\pi) = 0, \tag{13}$$

$$Y'(-\pi) = Y'(\pi) = 0, \tag{14}$$

or nonlocal

$$Y(-\pi) = Y(\pi), \quad Y'(-\pi) = Y'(\pi), \tag{15}$$

$$Y(-\pi) = -Y(\pi), \quad Y'(-\pi) = -Y'(\pi) \tag{16}$$

boundary conditions.

The Sturm-Liouville problem for Equation (12) with one of the boundary conditions (13), (14), (15), (16) is self-adjoint so they have real eigenvalues, and their corresponding eigenfunctions form a complete orthonormal basis in $L^2(-\pi,\pi)$ [11].

For further investigation of the problems under consideration, we need to calculate the explicit form of the eigenvalues and eigenfunctions.

For $|\varepsilon| < 1$ problem (12), (13) has the following eigenvalues:

$$\lambda_{2k-1,1} = (1+\varepsilon)k^2, \quad \lambda_{2k,1} = (1-\varepsilon)\left(k - \frac{1}{2}\right)^2, \quad k = 1,2,\ldots,$$

with the corresponding eigenfunctions

$$Y_{2k-1,1} = \sin ky, \quad Y_{2k,1} = \cos\left(k - \frac{1}{2}\right) y, \quad k = 1, 2, \ldots. \tag{17}$$

Similarly, problem (12), (14) has the eigenvalues

$$\lambda_{2k+1,2} = (1+\varepsilon)\left(k + \frac{1}{2}\right)^2, \quad \lambda_{2k,2} = (1-\varepsilon)k^2, \quad k = 0, 1, \ldots,$$

with the corresponding eigenfunctions

$$Y_{2k+1,2} = \sin\left(k + \frac{1}{2}\right) y, \quad Y_{2k,2} = \cos ky, \quad k = 0, 1, \ldots. \tag{18}$$

The eigenvalues of problem (12), (15) are

$$\lambda_{2k-1,3} = (1+\varepsilon)k^2, \quad k = 1, 2, \ldots, \quad \lambda_{2k,3} = (1-\varepsilon)k^2, \quad k = 0, 1, \ldots,$$

with the corresponding eigenfunctions

$$Y_{2k-1,3} = \sin ky, \quad k = 1, 2, \ldots, \quad Y_{2k,3} = \cos ky, \quad k = 0, 1, \ldots. \tag{19}$$

Problem (12), (16) has the following eigenvalues

$$\lambda_{2k+1,4} = (1+\varepsilon)\left(k + \frac{1}{2}\right)^2, \quad \lambda_{2k,4} = (1-\varepsilon)\left(k + \frac{1}{2}\right)^2, \quad k = 0, 1, \ldots,$$

and corresponding eigenfunctions

$$Y_{2k+1,4} = \sin\left(k + \frac{1}{2}\right) y, \quad Y_{2k,4} = \cos\left(k + \frac{1}{2}\right) y, \quad k = 0, 1, \ldots. \tag{20}$$

LEMMA 5. *The systems of functions* (17)*,* (18)*,* (19)*,* (20) *are complete and orthonormal in* $L^2(-\pi, \pi)$.

Proof. We prove only the completeness of system (17) in $L^2(-\pi, \pi)$. We will prove that from the equalities

$$\int_{-\pi}^{\pi} f(y) \sin ky dy = 0, \quad k = 1, 2, \ldots,$$

$$\int_{-\pi}^{\pi} f(y) \cos\left(k - \frac{1}{2}\right) y dy = 0, \quad k = 1, 2, \ldots,$$

for $f \in L_2(-\pi, \pi)$, we should obtain $f(y) = 0$ in $L^2(-\pi, \pi)$.

Suppose that the second equation holds. We transform it as follows

$$0=\int_{-\pi}^{\pi} f(y)\cos\left(k-\frac{1}{2}\right)ydy=\int_{0}^{\pi}(f(y)+f(-y))\cos\left(k-\frac{1}{2}\right)ydy.$$

Then by the completeness of the system [12] $\left\{\cos\left(k-\frac{1}{2}\right)y\right\}_{k=1}^{\infty}$ in $L^2(-\pi,\pi)$ we have $f(y)=-f(-y)$, $0<y<\pi$.

Similarly

$$0=\int_{-\pi}^{\pi} f(y)\sin kydy=\int_{0}^{\pi}(f(y)-f(-y))\sin kydy.$$

Then by the completeness of the system [12] $\{\sin ky\}_{k=1}^{\infty}$ in $L_2(-\pi,\pi)$ we have $f(y)=f(-y)$, $0<y<\pi$. Then we obtain $f(y)=0$ in $L^2(0,\pi)$, and consequently $f(y)=0$ in $L^2(-\pi,\pi)$.

The completeness of the systems (18), (19) and (20) can be proved similarly. □

4. Main results

For the considered problems D, N, P, AP, the following theorems hold.

Suppose $\varphi_{2k-j,i}=\left(\varphi,Y_{2k-j,i}\right)$, $\psi_{2k-j,i}=\left(\psi,Y_{2k-j,i}\right)$, $j=0,1$, $i=1,2,3,4$ are the Fourier coefficients of functions φ,ψ, by system $Y_{2k-j,i}$, $\lambda_{2k-j,i}$ are the corresponding eigenvalues, $\mu^2_{2k-j,i}=\lambda_{2k-j,i}+c^2$, and functions $C\left(\mu_{2k-j,i}x\right)$ and $S\left(\mu_{2k-j,i}x\right)$ are defined by (6) and (7).

THEOREM 1. *Let* $|\varepsilon|<1$, $0<\delta<1$, $\varphi(y)\in C^{2+\delta}[-\pi,\pi]$, $\psi(y)\in C^{1+\delta}[-\pi,\pi]$ *and* $\varphi(-\pi)=\varphi(\pi)=0$, $\psi(-\pi)=\psi(\pi)=0$. *Then the solution of the problem D exists, is unique and it can be written in the form*

$$\begin{aligned}u(x,y)=&\sum_{k=1}^{\infty}\left[\varphi_{2k-1,1}C\left(\mu_{\mu_{2k-1,1}}x\right)+\psi_{2k-1,1}S\left(\mu_{\mu_{2k-1,1}}x\right)\right]Y_{2k-1,1}(y)\\&+\sum_{k=1}^{\infty}\left[\varphi_{2k,1}C\left(\mu_{\mu_{2k,1}}x\right)+\psi_{2k,1}S\left(\mu_{\mu_{2k,1}}x\right)\right]Y_{2k,1}(y).\end{aligned}$$

THEOREM 2. *Let* $|\varepsilon|<1$, $0<\delta<1$, $\varphi(y)\in C^{3+\delta}[-\pi,\pi]$, $\psi(y)\in C^{2+\delta}[-\pi,\pi]$ *and* $\varphi'(-\pi)=\varphi'(\pi)=0$, $\psi'(-\pi)=\psi'(\pi)=0$. *Then the solution of the problem N exists, is unique and it can be written in the form*

$$\begin{aligned}u(x,y)=&\left(1-x^{\alpha}\right)\varphi_{0,2}+x^{\alpha}\psi_{0,2}\\&+\sum_{k=0}^{\infty}\left[\varphi_{2k-1,2}C\left(\mu_{2k-1,2}x\right)+\psi_{2k-1,2}S\left(\mu_{2k-1,2}x\right)\right]Y_{2k-1,2}\\&+\sum_{k=1}^{\infty}\left[\varphi_{2k,2}C\left(\mu_{2k,2}x\right)+\psi_{2k,2}S_{2k,2}\left(\mu_{2k,2}x\right)\right]Y_{2k,2}.\end{aligned}$$

THEOREM 3. *Let* $|\varepsilon|<1,\ 0<\delta<1,\ \varphi(y)\in C^{3+\delta}[-\pi,\pi],\ \psi(y)\in C^{2+\delta}[-\pi,\pi]$ *and* $\varphi(-\pi)=\varphi(\pi),\ \varphi'(-\pi)=\varphi'(\pi),\ \psi(-\pi)=\psi(\pi),\ \psi'(-\pi)=\psi'(\pi)$. *Then the solution of the problem P exists, is unique and it can be written in the form*

$$\begin{aligned}u(x,y)=&(1-x^{\alpha})\varphi_{0,3}+x^{\alpha}\psi_{0,3}\\&+\sum_{k=1}^{\infty}\left[\varphi_{2k,3}C\left(\mu_{2k,3}x\right)+\psi_{2k,3}S\left(\mu_{2k,3}x\right)\right]Y_{2k,3}(y)\\&+\sum_{k=1}^{\infty}\left[\varphi_{2k-1,3}C\left(\mu_{2k-1,3}x\right)+\psi_{2k-1,3}S\left(\mu_{2k-1,3}x\right)\right]Y_{2k-1,3}(y).\end{aligned}$$

THEOREM 4. *Let* $|\varepsilon|<1,\ 0<\delta<1,\ \varphi(y)\in C^{3+\delta}[-\pi,\pi],\ \psi(y)\in C^{2+\delta}[-\pi,\pi]$ *and* $\varphi(-\pi)=-\varphi(\pi),\ \varphi'(-\pi)=-\varphi'(\pi),\ \psi(-\pi)=-\psi(\pi),\ \psi'(-\pi)=-\psi'(\pi)$. *Then the solution of the problem AP exists, is unique and it can be written in the form*

$$\begin{aligned}u(x,y)=&\sum_{k=0}^{\infty}\left[\varphi_{2k,4}C\left(\mu_{2k,4}x\right)+\psi_{2k,4}S\left(\mu_{2k,4}x\right)\right]Y_{2k,4}(y)\\&+\sum_{k=0}^{\infty}\left[\varphi_{2k-1,4}C\left(\mu_{2k-1,4}x\right)+\psi_{2k-1,4}S\left(\mu_{2k-1,4}x\right)\right]Y_{2k-1,4}(y).\end{aligned}$$

5. Proofs of the main results

As the proofs for the uniqueness of the solutions of each problems are similar, we will present only the proof for problem D.

As the system of eigenfunctions (17) of problem D forms an orthonormal basis in $L_2(-\pi,\pi)$, the function can be represented as follows

$$u(x,y)=\sum_{k=1}^{\infty}u_{2k-1,1}(x)Y_{2k-1,1}(y)+\sum_{k=1}^{\infty}u_{2k,1}(x)Y_{2k,1}(y), \tag{21}$$

where $u_{2k-1,1}(x),\ u_{2k,1}(x)$ are unknown coefficients. It is well known that if $\varphi(y)$, $\psi(y)$ satisfy the conditions of Theorem 1, then they can be uniquely represented in the form of a uniformly and absolutely convergent Fourier series by the systems $\{Y_{2k-1,1}(y),$ $Y_{2k,1}(y)\}$:

$$\varphi(y)=\sum_{k=1}^{\infty}\varphi_{2k-1,1}Y_{2k-1,1}(y)+\sum_{k=1}^{\infty}\varphi_{2k,1}Y_{2k,1}(y),$$

$$\psi(y)=\sum_{k=1}^{\infty}\psi_{2k-1,1}Y_{2k-1,1}(y)+\sum_{k=1}^{\infty}\psi_{2k,1}Y_{2k,1}(y),$$

where $\varphi_{2k-j,1}=\left(\varphi,Y_{2k-j,1}\right),\ \psi_{2k-j,1}=\left(\psi,Y_{2k-j,1}\right),\ j=0,1.$

Putting (21) into Equation (1) and boundary conditions (2), for finding unknown functions $u_k(x)$, we obtain the following problem

$$D_y^{2\alpha}u_{2k-j,1}(x)-\mu_{2k-j,1}^2u_{2k-j,1}(x)=0,\quad 0<x<1, \tag{22}$$

$$u_{2k-j,1}(0) = \varphi_{2k-j,1}, \quad u_{2k-j,1}(1) = \psi_{2k-j,1}, \tag{23}$$

where $\mu^2_{2k-j,1} = \lambda_{2k-j,1} + c^2,\ j = 0,1.$

Due to Lemma 1 the solution of problem (22)–(23) exists, is unique and it can be written in the form

$$u_{2k-j,1}(x) = \varphi_{2k-j,1}C\left(\mu_{2k-j,1}x\right) + \psi_{2k-j,1}S\left(\mu_{2k-j,1}x\right),$$

where $C\left(\mu_{2k-j,1}x\right)$ and $S\left(\mu_{2k-j,1}x\right)$ are defined by (6) and (7), respectively. Furthermore, according to Lemma 4 inequalities

$$0 \leqslant S\left(\mu_{2k-j,1}x\right), \quad C\left(\mu_{2k-j,1}x\right) \leqslant 1, \quad x \in [0,1]$$

are true.

Further, if the function $f(x)$ belongs to the class $C^{m+\delta}[a,b],\ m = 0,1,\ldots,\ 0 < \delta < 1,$ then for Fourier coefficients of this function the following estimation holds (see [13]):

$$|f_k| = O\left(\frac{1}{k^{m+\delta}}\right), \quad k \to \infty.$$

If $\varphi''(y) \in C^{\delta}[-\pi,\pi],\ \psi'(y) \in C^{\delta}[-\pi,\pi]$ and conditions $\varphi(-\pi) = \varphi(\pi) = \psi(-\pi) = \psi(\pi) = 0$ hold, then

$$\left|\varphi_{2k-1,1}\right| \leqslant \frac{C}{k^{2+\delta}}, \quad \left|\varphi_{2k,1}\right| \leqslant \frac{C}{\left(k-\frac{1}{2}\right)^{2+\delta}},$$

$$\left|\psi_{2k-1,1}\right| \leqslant \frac{C}{k^{1+\delta}}, \quad \left|\psi_{2k,1}\right| \leqslant \frac{C}{\left(k-\frac{1}{2}\right)^{1+\delta}}, \quad C = const.$$

For such functions, we obtain

$$\begin{aligned} \left|u_{2k-1,1}(x)\right| &\leqslant C\left(\frac{1}{k^{2+\delta}} + \frac{1}{k^{1+\delta}}\right), \\ \left|u_{2k-1,1}(x)\right| &\leqslant C\left(\frac{1}{\left(k-\frac{1}{2}\right)^{2+\delta}} + \frac{1}{\left(k-\frac{1}{2}\right)^{1+\delta}}\right). \end{aligned} \tag{24}$$

Then the series (21) converges uniformly in the domain $\bar{\Omega}$ and therefore $u(x,y) \in C\left(\bar{\Omega}\right)$. Further, using estimations (10) and (11), we get

$$S_{2k-j,1}\left(\mu_{2k-j,1}x\right) = O\left(e^{\mu^{\frac{1}{\alpha}}_{2k-j,1}(x-1)}\right),$$

$$C\left(\mu_{2k-j,1}x\right) = O\left(\frac{1}{\mu_{2k-j,1}}\right).$$

Taking derivative term by term from the series (21) twice by y, we have

$$u_{yy}(x,y) = -\sum_{k=1}^{\infty} \lambda_{2k-1,1} u_{2k-1,1}(x) Y_{2k-1,1}(y) - \sum_{k=1}^{\infty} \lambda_{2k,1} u_{2k,1}(x) Y_{2k,1}(y).$$

Then for all $x \geqslant x_0 > 0,\ 0 \leqslant y \leqslant 1,$ taking into account inequalities (24), we have

$$\begin{aligned}
\left|u_{yy}(x,y)\right| &\leqslant C \sum_{k=1}^{\infty} \left(\left|\lambda_{2k-1,1}\right| \left|u_{2k-1,1}(x)\right| + \left|\lambda_{2k,1}\right| \left|u_{2k,1}(x)\right|\right) \\
&\leqslant C \sum_{k=1}^{\infty} \left(\left|\lambda_{2k-1,1}\right| \left|u_{2k-1,1}(x)\right| + \left|\lambda_{2k,1}\right| \left|u_{2k,1}(x)\right|\right) \\
&\leqslant C \sum_{k=1}^{\infty} k^{-1-\delta} + k^{1-\delta} e^{-\mu_{2k-1,1}(1-x)} + \left(k - \frac{1}{2}\right)^{-1-\delta} \\
&\quad + C \sum_{k=1}^{\infty} \left(k - \frac{1}{2}\right)^{1-\delta} e^{-\mu_{2k,1}(1-x)} < \infty.
\end{aligned}$$

Similarly, estimate the series

$$D_x^{2\alpha} u(x,y) = \sum_{k=1}^{\infty} \left(\mu_{2k-1,1}^2 u_{2k-1,1}(x) Y_{2k-1,1}(y) + \mu_{2k,1}^2 u_{2k,1}(x) Y_{2k,1}(y)\right).$$

Then $u_{yy}(x,y), D_x^{2\alpha} u(x,y) \in C(\Omega)$.

The uniqueness of the solution of problem D follows from the uniqueness of the solution of problem (22)–(23). The theorem is proved.

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Mokhtar Kirane
LaSIE, Faculté des Sciences
Pôle Sciences et Technologies
Université de La Rochelle
Avenue M. Crepeau, 17042 La Rochelle Cedex, France
e-mail: mkirane@@univ-lr.fr

Batirkhan Kh. Turmetov
Department of Mathematics
Akhmet Yasawi University
161200, Sattarkhanov st., 29, Turkestan, Kazakhstan
e-mail: batirkhan.turmetov@@ayu.edu.kz

Berikbol T. Torebek
Department of Differential Equations
Institute of Mathematics and Mathematical Modeling
050010, Pushkin st., 125, Almaty, Kazakhstan
e-mail: torebek@@math.kz