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SOME k–FRACTIONAL ASSOCIATES OF HERMITE–HADAMARD'S INEQUALITY FOR QUASI–CONVEX FUNCTIONS AND APPLICATIONS TO SPECIAL MEANS

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Abstract. This article brings together some inequalities associated with Hermite-Hadamard's inequality for quasi-convex functions by way of k-Riemann-Liouville fractional integrals of order α. The inequalities thus obtained are applied to some special means of real numbers.

1. Introduction

A function $f : I \subset R \longrightarrow R$ is said to be convex on I if for every $a,b \in I$ and $t \in [0,1]$, we have

$$f(ta+(1-t)b) \leqslant tf(a)+(1-t)f(b).$$

An immediate consequence of convexity is the integrability of the function in the Riemann's sense. Subsequently the lower and upper estimations for the integral average of a convex function defined on the compact interval $[a,b]$, involving the midpoint and the endpoints of the domain, are given by the following inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}.$$

This is the celebrated Hermite-Hadamard's inequality. Almost since the outset of this inequality in 1881 by Hermite (see [4]), it has been worked on. As a result many generalizations, refinements, extensions and counter part of this inequality are available in literature. So persisting the tradition of generalization, this inequality is generalized here for quasi-convex functions by means of remarkable k-Riemann-Liouville fractional integrals.

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DEFINITION 1. The function $f : I \subset R \longrightarrow R$ is said to be quasi-convex if for every $a,b \in I$ and $t \in [0,1]$, we have

$$f(ta+(1-t)b) \leqslant \max\{f(a), f(b)\},$$

(*see* [4]). Quasi-convexity is a weaker convexity, that is it generalizes the notion of convexity. Therefore every convex function is quasi-convex whereas there are quasi-convex functions which are not convex (see [7]).

In [3] Dragomir and Pearce proved the following result for quasi-convex function, connected with Hermite-Hadamard inequality:

LEMMA 1. *Let $f : [a,b] \longrightarrow R$ be a quasi-convex function and $f \in L_1[a,b]$, we have the inequality*

$$\frac{1}{b-a}\int_a^b f(t)dt \leqslant \max\{f(a), f(b)\}. \tag{1}$$

In [7] Ion proved the following two results connected with quasi-convex function:

THEOREM 1. *Let $f : [a,b] \longrightarrow R$ be differentiable function on (a,b). If $|f'|$ is quasi-convex on $[a,b]$, the subsequent inequality is valid*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{b-a}{4}\max\{|f'(a)|, |f'(b)|\}. \tag{2}$$

THEOREM 2. *Let $f : [a,b] \longrightarrow R$ be differentiable function on (a,b). If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a,b]$ with $p>1$, the subsequent inequality is valid*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}. \tag{3}$$

In [1] the following result connected with quasi-convex function is proved:

THEOREM 3. *Let $f : I^o \subset R \longrightarrow R$ be differentiable function on $I^o, a,b \in I^o$ with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q \geqslant 1$, the subsequent inequality is valid*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{b-a}{4}\left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}. \tag{4}$$

In [8] Mubeen and Habibullah introduced the following class k-fractional integrals:

DEFINITION 2. Let $f \in L_1[a,b]$, the k-Riemann-Lioville fractional integrals ${}_kJ^{\alpha}_{a^+}f(u)$ and ${}_kJ^{\alpha}_{b^-}f(u)$ of order $\alpha > 0$ with $a \geqslant 0$, $k > 0$, are defined by

$$ {}_kJ^{\alpha}_{a^+}f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_a^u (u-t)^{\frac{\alpha}{k}-1} f(t)dt, \qquad 0 \leqslant a < u < b $$

and

$$ {}_kJ^{\alpha}_{b^-}f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_u^b (t-u)^{\frac{\alpha}{k}-1} f(t)dt, \qquad 0 \leqslant a < u < b $$

respectively, where $\Gamma_k(\alpha)$ is the k-gamma function given as $\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt$, see [2].

In fact k-Riemann-Liouville fractional integrals of order α are generalization of Riemann-Liouville fractional integrals of order α. If we take $k \to 1$, the k-Riemann-Liouville fractional integrals of order α turn out to be Riemann-Liouville fractional integrals of order α which are described in [5].

In [6] following useful identity related to k-fractional integrals is proved:

LEMMA 2. *Let $f : [a,b] \longrightarrow R$ be differentiable function on (a,b). If $f' \in L[a,b]$, the following equality for k-fractional integrals is valid*

$$ \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} [{}_kJ^{\alpha}_{a^+}f(b) + {}_kJ^{\alpha}_{b^-}f(a)] $$
$$ = \frac{b-a}{2} \int_0^1 [(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}] f'(ta+(1-t)b)dt. $$

2. Main results

The main aim of the present paper is to establish new inequalities for quasi-convex functions via k-Riemann-Liouville fractional integrals. Starting with the following lemma.

LEMMA 3. *Let $f : [a,b] \longrightarrow R$ be positive function and $f \in L_1[a,b]$. If f is quasi-convex on $[a,b]$, the subsequent inequality for k-fractional integrals is valid*

$$ \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} [{}_kJ^{\alpha}_{a^+}f(b) + {}_kJ^{\alpha}_{b^-}f(a)] \leqslant \max\{f(a), f(b)\} $$

with $\frac{\alpha}{k} > 0$.

Proof. Since f is quasi-convex on $[a,b]$, we have

$$ f(ta+(1-t)b) \leqslant \max\{f(a), f(b)\} $$

and

$$f((1-t)a+tb) \leqslant \max\{f(a), f(b)\}$$

by adding these inequalities we get

$$\frac{1}{2}[f(ta+(1-t)b)+f((1-t)a+tb)] \leqslant \max\{f(a), f(b)\}$$

now multiplying both sides by $t^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1} f(ta+(1-t)b)dt + \int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)a+tb)dt \\
&\quad = \int_b^a \left(\frac{b-u}{b-a}\right)^{\frac{\alpha}{k}-1} f(u)\frac{du}{a-b} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\frac{\alpha}{k}-1} f(v)\frac{dv}{b-a} \\
&\quad \leqslant \frac{2k}{\alpha}\max\{f(a), f(b)\}
\end{aligned}$$

by using the definition of k-Riemann-Liouville fractional integrals, we get

$$\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a^+}f(b) + {}_kJ^{\alpha}_{b^-}f(a)] \leqslant \max\{f(a), f(b)\},$$

hence the proof is complete. □

REMARK 1. If we choose $\alpha = k$ in Lemma 3, with properties of gamma function we have the inequality (1).

THEOREM 4. *Let $f : [a,b] \longrightarrow R$ be a differentiable function on (a,b). If $|f'|$ is quasi-convex on $[a,b], \alpha > 0$, the subsequent inequality for k-fractional integrals is valid*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a^+}f(b) + {}_kJ^{\alpha}_{b^-}f(a)]\right| \\
&\qquad \leqslant \frac{b-a}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\max\left\{|f'(a)|, |f'(b)|\right\}.
\end{aligned}$$

Proof. Using Lemma 2, the fact that $|f'|$ is quasi-convex and properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b) + {}_kJ^{\alpha}_{b^-}f(a)\right]\right| \\
&\leqslant \frac{b-a}{2}\int_0^1 |(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}||f'(ta+(1-t)b)|dt \\
&\leqslant \frac{b-a}{2}\int_0^1 |(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}|\max\left\{|f'(a)|, |f'(b)|\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right]dt\right\}\max\left\{|f'(a)|,|f'(b)|\right\}\\
&= \frac{b-a}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\max\left\{|f'(a)|,|f'(b)|\right\}.
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}|dt &= \int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right]dt\\
&= \frac{2}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)
\end{aligned}
$$

which completes the proof. □

REMARK 2. If we choose $\alpha = k$ in Theorem 4, we have the inequality (2).

THEOREM 5. *Let $f : [a,b] \longrightarrow R$ be a differentiable function on (a,b) such that $f' \in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q > 1$, the subsequent inequality for k-fractional integrals is valid*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ_{a^+}^{\alpha}f(b)+{}_kJ_{b^-}^{\alpha}f(a)\right]\right|\\
&\qquad\leqslant \frac{b-a}{2\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{\alpha}{k}\in[0,1]$.

Proof. From Lemma 2 and using Hölder's inequality with properties of modulus, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ_{a^+}^{\alpha}f(b)+{}_kJ_{b^-}^{\alpha}f(a)\right]\right|\\
&\leqslant \frac{b-a}{2}\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}||f'(ta+(1-t)b)|dt\\
&\leqslant \frac{b-a}{2}\left(\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

We know that for $\frac{\alpha}{k}\in[0,1]$ and for all $t_1,t_2\in[0,1]$, $\left|t_1^{\frac{\alpha}{k}}-t_2^{\frac{\alpha}{k}}\right|\leqslant|t_1-t_2|^{\frac{\alpha}{k}}$, therefore

$$
\begin{aligned}
\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}|^p dt &\leqslant \int_0^1 |1-2t|^{\frac{\alpha}{k}p}dt\\
&= \int_0^{\frac{1}{2}} |1-2t|^{\frac{\alpha}{k}p}dt+\int_{\frac{1}{2}}^1 |2t-1|^{\frac{\alpha}{k}p}dt\\
&= \frac{1}{\frac{\alpha}{k}p+1}.
\end{aligned}
$$

Since $|f'|^q$ is quasi-convex on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)]\right| \leqslant \frac{b-a}{2\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}. \quad \square$$

REMARK 3. If in Theorem 5, we choose $\alpha=k$, we have the inequality (3).

THEOREM 6. *Let $f:[a,b]\longrightarrow R$ be a differentiable function on (a,b) such that $f'\in L_1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q\geqslant 1$, the subsequent inequality for k-fractional integrals is valid*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)]\right| \leqslant \frac{b-a}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}$$

with $\frac{\alpha}{k}>0$.

Proof. From Lemma 2, using power mean inequality with properties of modulus and using the fact that $|f'|^q$ is quasi-convex, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)]\right| \\
&\leqslant \frac{b-a}{2}\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}||f'(ta+(1-t)b)|dt \\
&\leqslant \frac{b-a}{2}\left(\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leqslant \frac{b-a}{2}\left(\int_0^1 |(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}|dt\right)\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\int_0^{\frac{1}{2}}[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}]dt+\int_{\frac{1}{2}}^1[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}]dt\right)\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}} \\
&= \frac{b-a}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. $\square$

REMARK 4. If in theorem 6, we choose $\alpha=k$, we have the inequality (4).

3. Applications to special means

We now consider special means of positive real numbers $\xi,\eta \ (\xi \neq \eta)$, as follows

Arithmetic mean

$$A(\xi,\eta) = \frac{\xi+\eta}{2}.$$

Geometric Mean

$$G(\xi,\eta) = \sqrt{\xi\eta}.$$

Harmonic Mean

$$H(\xi,\eta) = \frac{2\,\xi\eta}{\xi+\eta}.$$

Logarithmic mean

$$L(\xi,\eta) = \frac{\eta-\xi}{ln|\eta| - ln|\xi|}.$$

Genralised Log-mean

$$L_n(\xi,\eta) = \left[\frac{\eta^{n+1}-\xi^{n+1}}{(n+1)(\eta-\xi)}\right]^{\frac{1}{n}}, \quad n \in Z\setminus\{-1,0\}.$$

Indentric Mean

$$I(\xi,\eta) = \frac{1}{e}\left(\frac{\eta^\eta}{\xi^\xi}\right)^{\frac{1}{\eta-\xi}}.$$

PROPOSITION 1. *Let* $\xi,\eta \in R^+, \xi < \eta$, *we have an obvious result*

$$A(\xi,\eta) \leqslant \max\{\xi,\eta\}, \tag{5}$$

$$L_n^n(\xi,\eta) \leqslant \max\{\xi^n,\eta^n\}. \tag{6}$$

Proof. The assertion (5) follows from Lemma 3 applied to the quasi-convex function $f(x) = x$, where $x \in R$ and taking $\alpha = k = 1$. The second assertion (6) also follows from Lemma 3 applied to the quasi-convex function $f(x) = x^n$ and $\alpha = k = 1$. Note that $f(x) = x^n$ is quasi-convex for ; $x \in R$ when $n \in Z$ is even and $x \in R^+$ when $n \in Z$ is odd. □

PROPOSITION 2. *Let* $\xi,\eta \in R^+, \xi < \eta$, *and* $n \in Z\setminus\{-1,0\}$, *we have*

$$|A(\xi^n,\eta^n) - L_n^n(\xi,\eta)| \leqslant \frac{\eta-\xi}{4} \max\{|n\xi^{n-1}|,|n\eta^{n-1}|\}.$$

Proof. The assertions follow from Theorem 4, applied to function $f(x)=x^n$ $\alpha=k=1$. Note that if $f(x)=x^n$, then $|f'(x)|$ is quasi-convex for; $x\in R$ with $n\in Z$ is $0,2$ or odd, and $x\in R^+$ with $n\in Z\setminus\{0,2\}$ is even. □

PROPOSITION 3. *Let $\xi,\eta\in R^+,\xi<\eta$, and $n\in Z$, we have*

$$|A(\xi,\eta)-L(\xi,\eta)|\leqslant\frac{ln\,\eta-ln\,\xi}{4}\ \max\{\xi,\eta\}.$$

Proof. The assertions follow from Theorem 4, applied to the function $f(x)=e^x$ and $\alpha=k=1$. Note that if $f(x)=e^x$ then $|f'(x)|$ is convex for all $x\in R$. □

PROPOSITION 4. *Let $\xi,\eta\in R^+,\xi<\eta,\ q>1$ and $n\in Z$, we have*

$$|G(\xi,\eta)-I(\xi,\eta)|\leqslant e^{\frac{\eta-\xi}{2(p+1)^{\frac{1}{p}}}\left(\max\left\{\left|\frac{1}{\xi}\right|^q,\left|\frac{1}{\eta}\right|^q\right\}\right)^{\frac{1}{q}}}.$$

Proof. The assertions follow from Theorem 5, applied the function $f(x)=lnx$ and $\alpha=k=1$. Note that if $f(x)=lnx$ then $|f'(x)|^q$ is quasi-convex for $x\in R\setminus\{0\}$. □

PROPOSITION 5. *Let $\xi,\eta\in R^+,\xi<\eta,\ q\geqslant1$ and $n\in Z$, we have*

$$|H^{-1}(\xi,\eta)-L(\xi,\eta)|\leqslant\frac{\eta-\xi}{4}\ \left(\max\left\{\left|\frac{1}{\xi}\right|^q,\left|\frac{1}{\eta}\right|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertions follow from Theorem 6, applied to the function $f(x)=\frac{1}{x}$ and $\alpha=k=1$. Note that if $f(x)=\frac{1}{x}$ the function $|f'(x)|^q$ is convex for all $x\in R$. □

4. Conclusion

In the present paper we have presented genralization of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals of order α, given in [9], to the corresponding inequalities for k-Riemann-Liouville fractional integrals of order α.

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