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KALLMAN–ROTA TYPE INEQUALITY FOR DISCRETE EVOLUTION FAMILIES OF BOUNDED LINEAR OPERATORS

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Abstract. Let $\mathscr{X}$ be a complex Banach space and $\mathscr{Z}_+$ be the set of all nonnegative integers. Let $\mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{X})$ be the space of all $\mathscr{X}$-valued bounded sequences which decays to zero at 0 and at ∞. Using the space $\mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{X})$, we give Kallman-Rota type inequality for the discrete evolution family $\mathscr{U}=\{U(m,n): m,n\in\mathscr{Z}_+, m\geqslant n\}$ of bounded linear operators. We also present the same inequality for (r,q)-resolvent operators, which arises in the solution of fractional difference equation. In particular, if $\mathscr{A}$ is the algebraic generator of α-times family of bounded and linear operators, arising from the well posedness of fractional difference equations of order $\beta+1$, then we prove that the inequality

$$\|\mathscr{A}x\|^2\leqslant 8\eta^2\frac{\Gamma(\alpha+\beta+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2\beta+3)}\|x\|\|\mathscr{A}^2x\|,$$

holds for all $x\in D(\mathscr{A}^2)$.

1. Introduction

A well-known result established by Hardy, Littlewood and Pólya (see [9], p. 187) asserts that $\|f'\|_2^2\leqslant 2\|f\|_2\|f''\|_2$ for any function f on $\mathscr{R}_+$, where f, f', $f''\in\mathscr{L}^2(\mathscr{R}_+)$. In [12], Kallman and Rota proved that

$$\|\mathscr{A}x\|^2\leqslant 4\|x\|\|\mathscr{A}^2x\|,\ \text{ for all } x\in D(\mathscr{A}^2),\tag{1.1}$$

where $\mathscr{A}$ is the infinitesimal generator of a strongly continuous contraction semigroup on a Banach space $(\mathscr{X},\|.\|)$, and x, $\mathscr{A}x$ are in the domain of $\mathscr{A}$. In [14] Kraljević and Kurepa extended the above result for bounded and strongly continuous semigroups with the bound constant $\mathscr{W}>0$, as $\|\mathscr{A}x\|^2\leqslant 4\mathscr{W}^2\|x\|\|\mathscr{A}^2x\|$, for $x\in D(\mathscr{A}^2)$. In [12], it was shown that the Hardy-Littlewood-Pólya inequality implies the Kallman-Rota inequality for C_0-semigroups. A special interest has been taken to improve, and in some cases to obtain, the optimal constant 4 in the inequality (1.1). For Hilbert spaces, in [6] Goldstein showed that the optimal constant for a C_0-contraction semigroup is 2. In $\mathbb{C}$-Euclidean spaces for analytic semigroups, different optimal constants are obtained by different approaches, see [10, 19]. In [3] Buşe and Dragomir studied Kallman-Rota inequality by using the evolution semigroup approach in continuous case.

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On the other hand in the last decade, the study of existence and qualitative properties of fractional difference equations has drawn a great deal of interest, for instance, see [1, 8, 7, 11]. Miller and Ross [18] defined a fractional sum via the solution of a linear difference equation. More recently, Atici and Eloe [2] introduced the Riemann-Liouville like fractional difference equation by using the definition of fractional sum of Miller and Ross, and developed some of its properties that allow us to obtain solutions of certain fractional difference equations, such solutions lead to the idea of (r,q)-resolvent operators theory, see [15, 17]. Discrete (r,q)-resolvent operators treat different families of bounded linear operators in a unified way, where r and q represent sequences r_n and q_n such that $r_n \in l^1(\mathscr{Z}_+,\mathscr{X})$ and $q_n \in \mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{X})$, respectively.

Our main result of this paper is to give Kallman-Rota inequality with the help of discrete evolution family $\mathscr{U}$, over Banach space $\mathscr{X}$. Also we present the same inequality by using certain discrete (r,q)-resolvent families which arises from fractional difference equation. Our approach is based on "CP-condition". We also give applications of our results.

2. Preliminaries

In the following we introduce few sequence spaces which needed in next sections.

- $\mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{X})$, consisting of all $\mathscr{X}$-valued sequences f, such that $f(n)$ gives zero at 0 and at ∞.
- $l^p(\mathscr{Z}_+,\mathscr{X})$, $1 \leqslant p \leqslant \infty$ is the usual Lebesgue-Bochner space of all measurable sequences $N : \mathscr{Z}_+ \to \mathscr{X}$, which are equal almost everywhere, such that

$$\|N\|_p := \left(\sum_{s=0}^{n} \|N(s)\|^p\right)^{\frac{1}{p}} < \infty.$$

- $l^1(\mathscr{Z}_+,\mathscr{X})$ is the space of all sequences $Q : \mathscr{Z}_+ \to \mathscr{X}$, such that

$$\|Q\|_1 := \sup_{n\geqslant 0} \|Q(s)\| < \infty.$$

Let $\mathscr{X}$ be a real or complex Banach space and $\mathscr{L}(\mathscr{X})$ the Banach algebra of all linear and bounded operators acting on $\mathscr{X}$. The norm in $\mathscr{X}$ and in $\mathscr{L}(\mathscr{X})$ will be denoted by $\|\cdot\|$. Let $\mathscr{Z}_+$ be the set of all non-negative integers. The family $\mathscr{U} := \{U(n,m) : n,m \in \mathscr{Z}_+, n \geqslant m\}$ is called discrete evolution family of bounded linear operators on $\mathscr{X}$, if it satisfies the following properties:

- $U(n,n) = I$, for all $n \in \mathscr{Z}_+$.
- $U(n,m)U(m,r) = U(n,r)$, for all $n \geqslant m \geqslant r$, $n,m,r \in \mathscr{Z}_+$.

It is well known that the evolution family $\mathscr{U}$ is exponentially bounded, if there exist $\sigma \in \mathscr{R}$ and $\mathscr{W}_\sigma \geqslant 0$ such that,

$$\|U(n,m)\| \leqslant \mathscr{W}_\sigma e^{\sigma(n-m)}, \text{ for all } n \geqslant m \in \mathscr{Z}_+, \tag{2.1}$$

and uniformly bounded if there exists $\mathscr{W} \geqslant 0$ such that,

$$\|U(n,m)\| \leqslant \mathscr{W} < \infty, \text{ for all } n \geqslant m \in \mathscr{Z}_+. \tag{2.2}$$

For more details about discrete evolution families we refer [4, 5, 13, 20, 21, 23, 22].

Here we define some conditions of (r,q)-resolvent operators. Let $r_n \in l^1(\mathscr{Z}_+, \mathscr{X})$ and $q_n \in \mathscr{K}_{00}(\mathscr{Z}_+, \mathscr{X})$, and let $\mathscr{A}$ be a bounded operator defined on a Banach space $\mathscr{X}$. Following [17], a bounded linear operator family $(K_n)_{n\geqslant 0} \subset \mathscr{L}(\mathscr{X})$ is called (r,q)-resolvent family, with algebraic generator $\mathscr{A}$ if the following holds:

(i) $K_n\mathscr{A}x = \mathscr{A}K_nx$ for all $x \in D(\mathscr{A})$; $n \geqslant 0$ and $K_0 = q_0 I$.

(ii) $K_nx = q_nx + \sum_{s=0}^{n} r_{(n-s)}\mathscr{A}K_sx$; $x \in D(\mathscr{A})$, $n \geqslant 0$.

In addition, if there is some constant $\eta > 1$, such that $\|K_n\| \leqslant \eta q_n$, for all $n \geqslant 0$, then the family $(K_n)_{n\geqslant 0}$ is called exponentially bounded. The usual convolution product $r * q$ is denoted by,

$$(r*q)_n := \sum_{s=0}^{n} r_{(n-s)}q_{(s)}, \quad n > 0.$$

In the case that r_n is a positive sequence a.e., then

$$\sum_{s=0}^{\infty}(r*r)_s = \left(\sum_{s=0}^{\infty} r_s\right)^2$$

and $\sum_{s=0}^{\infty} r*r_s = \infty$ if and only if $\sum_{s=0}^{\infty} r_s = \infty$.

DEFINITION 2.1. We say that the pair (r,q) satisfies CP-condition if for any $\mu > 0$ there exists $n_\mu > 0$ such that

$$\mu q(n_\mu) = (r*r*q)(n_\mu). \tag{2.3}$$

Let $g_\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha \geqslant 0$ and $e_{n_\mu} = e^{\mu n}$ for $\mu \in \mathscr{Z}_+$. Let $1_{\{1,\dots,n\}}$ be the characteristic sequence. It is easy to check that (g_θ, g_ϑ), with $\theta, \vartheta > 0$, (e_1, e_1), (e_{-1}, e_{-1}) and (e_1, e_{-1}) satisfies (2.3), however the pair (e_{-1}, e_1) does not satisfy it, because

$$\begin{aligned} e_{-1} * e_{-1} * e_1 &= \sum_{s=0}^{n} e^{-n+s}e^{-s}\sum_{s=0}^{n} e^{2s}, \quad s \geqslant 0,\\ &= ne^{-n}\sum_{s=0}^{n} e^{2s}, \quad s \geqslant 0. \end{aligned}$$

In the following lemma and example, we show some necessary conditions to get pairs (r,q), which satisfy the CP-condition.

LEMMA 2.2. *Let* $r_n \in l^1(\mathscr{Z}_+, \mathscr{X})$ *be a positive sequence and* $q_n \in \mathscr{K}_{00}(\mathscr{Z}_+, \mathscr{X})$.

(i) If $q_n > 0$, q *is decreasing sequence and* $\sum_{s=0}^{\infty} r_s = \infty$, *then the pair* (r,q) *satisfies the CP-condition.*

(ii) The pair $(r_n, 1_{1,\dots,n})$ *satisfies the CP-condition if and only if* $\sum_{s=0}^{\infty} r_s = \infty$.

Proof. (i) Fixed $\mu > 0$. We apply Bolzano's theorem to the sequence $g := \mu q - r*r*q$. Note that $g(0) > 0$ and

$$\lim_{n\to\infty} \frac{r*r*q_n}{q_n} \geqslant \lim_{n\to\infty} \sum_{s=0}^{n} r*r_n = +\infty,$$

and then $\lim_{n\to\infty} g_n = -\infty$. We conclude that there exists $n_\mu > 0$ such that $\mu q(n_\mu) = (r*r*q)(n_\mu)$.

(ii) If the pair $(r_n, 1_{1,\dots,n})$ satisfies the CP-condition, then there exists n_μ such that

$$\sum_{s=0}^{n_\mu} r*r_s = \mu$$

for all $\mu > 0$. We may conclude that $\sum_{s=0}^{\infty} r*r_s = \infty$ and then $\sum_{s=0}^{\infty} r_s = \infty$. The converse statement is proven in a similar way. □

EXAMPLE 2.3. The pairs $(g_\alpha.e_\mu, g_\beta)$ and $(g_\alpha.e_\mu, 1_{1,\dots,n})$ with $\alpha \geqslant 0$, $\mu > 0$ and $0 < \beta \leqslant 1$ satisfy the CP-condition.

3. Evolution semigroups

Let $\mathscr{X}$ be either space $\mathscr{K}_{00}(\mathscr{Z}_+, \mathscr{X})$ or $l^p(\mathscr{Z}_+, \mathscr{X})$ and $\mathscr{U} = \{U(n,m);\ n,m \in \mathscr{Z}_+,\ n \geqslant m\}$ be an exponentially bounded discrete evolution family of bounded linear operators on Banach space $\mathscr{X}$. For every $n \in \mathscr{Z}_+$ and each $f \in \mathscr{X}$, the sequence

$$n \to (\mathscr{T}(s)f)(n) := U(n,n-s)f(n-s) : \mathscr{Z}_+ \to \mathscr{X} \tag{3.1}$$

belongs to $\mathscr{X}$ and the family $\mathbb{T} = \{\mathscr{T}(n) : n \in \mathscr{Z}_+\}$ is a discrete semigroup on $\mathscr{X}$, [16].

LEMMA 3.1. *The discrete semigroup* $\mathbb{T} = \{\mathscr{T}(n) : n \in \mathscr{Z}_+\}$ *described by* (3.1) *acts on* $l^p(\mathscr{Z}_+, \mathscr{X})$.

Proof. Let f_n be a sequence on the space $\mathscr{X}$ such that $\|f(0)\|_p = 0 = f(0)$. It can be seen that for all $n \in \mathscr{Z}_+$, we have

$$(\mathscr{T}(s)f)(n) := U(n,n-s)f(n-s) : \mathscr{Z}_+ \to \mathscr{X}.$$

Taking norm on both sides we get

$$\begin{aligned}
\|(\mathscr{T}(s)f)(n)\|_p &= \|U(n,n-s)f(n-s)\|_p \\
&= \Big(\sum_{s=0}^{n} \|U(n,n-s)f(n-s)\|^p\Big)^{\frac{1}{p}} \\
&\leqslant \Big(\sum_{s=0}^{n} \|U(n,n-s)\|^p \times \|f(n-s)\|^p\Big)^{\frac{1}{p}} \\
&\leqslant \Big(\sum_{s=0}^{n} \mathscr{W}^p \|f(n-s)\|^p\Big)^{\frac{1}{p}} \\
&= \mathscr{W}\Big(\sum_{s=0}^{n} \|f(n-s)\|^p\Big)^{\frac{1}{p}} \\
&= \mathscr{W}\|f(n-s)\|_p < \infty,
\end{aligned}$$

belong to $l^p(\mathscr{Z}_+,\mathscr{X})$. Hence, the lemma is proved. □

The discrete semigroup $\mathbb{T}_n$ defined by (3.1) is called evolution semigroup associated to $\mathscr{U}$ on the space $\mathscr{X}$. The "infinitesimal generator" of the discrete semigroup is denoted by $\mathscr{A}$ is define as $\mathscr{A} := \mathscr{T}(1) - I$. It is clear that

$$\mathscr{T}(n)x - x = \sum_{s=0}^{n-1} \mathscr{T}(s)\mathscr{A}x, \quad \text{for all } n \in \mathscr{Z}_+,\ x \in \mathscr{X}. \tag{3.2}$$

LEMMA 3.2. *Let $\mathbb{T} = \{\mathscr{T}(n)\}_{n\in\mathscr{Z}_+}$ be the evolution semigroup associated to the discrete evolution family $\mathscr{U}$ on the space $\mathscr{X}$ and let $x, f \in \mathscr{X}$. The following two statements are equivalent:*

(i) $\mathscr{A}x = -f$.

(ii) $x(n) = \sum_{s=0}^{n} U(n,s)f(s)$ *for all* $n \in \mathscr{Z}_+$.

Proof. $(i) \Rightarrow (ii)$ For $n = 0$ the assertion is obvious. Let $n \in \mathscr{Z}_+$, $n \geqslant 1$. From (3.2) follows:

$$\begin{aligned}
\mathscr{T}(n)x - x = \sum_{s=0}^{n-1} \mathscr{T}(s)\mathscr{A}x &= -\sum_{s=0}^{n-1} \mathscr{T}(s)f \\
(\mathscr{T}(n)x - x)(n) &= -\sum_{s=0}^{n-1} (\mathscr{T}(s)f)(n) \\
x(n) &= \big(\mathscr{T}(n)x\big)(n) + \Big(\sum_{s=0}^{n-1} \mathscr{T}(s)f\Big)(n) \\
&= U(n,0)x(0) + \sum_{s=0}^{n-1} U(n,n-s)f(n-s) \\
&= \sum_{r=0}^{n} U(n,r)f(r).
\end{aligned}$$

$(ii) \Rightarrow (i)$ Let $n \geqslant 1$, successively one has:

$$\begin{aligned}(\mathscr{A}x)(n) &= [(\mathscr{T}(1)-I)x](n)\\ &= U(n,n-1)x(n-1)-x(n), \text{ using (3.1)}\\ &= \sum_{s=0}^{n-1} U(n,s)f(s)-x(n)\\ &= \sum_{s=0}^{n} U(n,s)f(s)-U(n,n)f(n)-x(n)\\ &= -f(n).\end{aligned}$$

Hence proved that statements (i) and (ii) are equivalent. □

4. Main results

Now we are in the position to prove our main results about Kallman-Rota type inequality.

LEMMA 4.1. *Let $\mathbb{T} = \{\mathscr{T}(n) : n \in \mathscr{Z}_+\}$ be a discrete semigroup. If $\mathbb{T}$ is uniformly bounded, that is, there is a positive constant $\mathscr{W}$ such that $\sup_{n\in\mathscr{Z}_+}\|\mathscr{T}(n)\| \leqslant \mathscr{W}$, then*

$$\|\mathscr{A}x\|^2 \leqslant 4\mathscr{W}^2\|x\|\|\mathscr{A}^2x\|, \; for\; all\; x \in D(\mathscr{A}^2). \tag{4.1}$$

For proof see [12].

Now we state the same inequality for evolution semigroups on the space $\mathscr{X}$ as given in section 2.

THEOREM 4.2. *Let $\mathscr{U}$ be a uniformly stable evolution family of bounded linear operators acting on $\mathscr{X}$, and let $f_n \in \mathscr{X}$. Suppose that the following conditions are fulfilled:*

(i) $\sum_{s=0}^{n} U(n,s)f(s)$ belong to $\mathscr{X}$.

(ii) $\sum_{s=0}^{n} 1_{\{0,\dots,r\}}(s)(n-s)U(n,s)f(s)$ belong to $\mathscr{X}$.

Then the following inequality holds

$$\Big\|\sum_{s=0}^{n} U(n,s)f(s)\Big\|_{\mathscr{X}}^2 \leqslant 4\mathscr{W}^2\|f\|_{\mathscr{X}} \times \Big\|\sum_{s=0}^{n} 1_{\{0,\dots,r\}}(s)(n-s)U(n,s)f(s)\Big\|_{\mathscr{X}}, \tag{4.2}$$

where $\mathscr{W}$ is a constant from the estimation of (2.2) and $1_{\{0,\dots,r\}}$ is the characteristic function.

Proof. Let $\mathbb{T}$ be the evolution semigroup associated to $\mathscr{U}$ on the space $\mathscr{X}$ and let $\mathscr{A}$ be its algebraic generator. Let f be any arbitrary element of the space $\mathscr{X}$, From condition (i) we know that $\sum_{s=0}^{n} U(n,s)f(s)$ is also an element of $\mathscr{X}$, so let us denote $\sum_{s=0}^{n} U(n,s)f(s)$ by ϑ_n i.e.

$$\vartheta_n := \sum_{s=0}^{n} U(n,s)f(s).$$

Then by using Lemma 3.2 we have

$$\mathscr{A}\vartheta_n = -f_n. \tag{4.3}$$

Also we claimed that $\sum_{s=0}^{n} U(n,s)\vartheta(s) \in \mathscr{X}$, because,

$$\begin{aligned}
\sum_{s=0}^{n} U(n,s)\vartheta(s) &= \sum_{r=0}^{n} U(n,r)\sum_{s=0}^{r} U(r,s)f(s) \quad \text{where } n \geqslant r \geqslant s \geqslant 0\\
&= \sum_{r=0}^{n}\sum_{s=0}^{n} 1_{\{0,\dots,r\}}(s)U(n,s)f(s)\\
&= \sum_{s=0}^{n}\sum_{r=0}^{n} 1_{\{0,\dots,r\}}(s)U(n,s)f(s)\\
&= \sum_{r=0}^{n} U(n,0)f(0) + \sum_{r=0}^{n} U(n,1)f(1) + \dots\\
&\quad + \sum_{r=0}^{n} U(n,n)f(n), \text{ where } n \geqslant r \geqslant s \geqslant 0\\
&= nU(n,0)f(0) + (n-1)U(n,1)f(1) + \dots + (n-r)U(n,r)f(r)\\
&= \sum_{s=0}^{n} 1_{\{0,\dots,r\}}(s)(n-s)U(n,s)f(s), \text{ where } n \geqslant r \geqslant s \geqslant 0
\end{aligned}$$

i.e. $\sum_{s=0}^{n} U(n,s)\vartheta(s) = \sum_{s=0}^{n} 1_{\{0,\dots,r\}}(s)(n-s)U(n,s)f(s)$.

The condition (ii) $\sum_{s=0}^{n} 1_{\{0,\dots,r\}}(s)(n-s)U(n,s)f(s) \in \mathscr{X}$ implies that $\sum_{s=0}^{n} U(n,s)\vartheta(s) \in \mathscr{X}$. Let $\hat{b}_n = \sum_{s=0}^{n} U(n,s)\vartheta(s)$ i.e. $\hat{b}_n, \vartheta_n \in \mathscr{X}$. Then again by Lemma 3.2, we get

$$\mathscr{A}\hat{b}_n = -\vartheta_n.$$

Applying $\mathscr{A}$ on both sides we get

$$\begin{aligned}
\mathscr{A}(\mathscr{A}\hat{b}_n) &= \mathscr{A}(-\vartheta_n)\\
&= -\mathscr{A}\vartheta_n, \quad \text{(using (4.3))}\\
&= -(-f_n),
\end{aligned}$$

$$\mathscr{A}^2\hat{b}_n = f_n \in \mathscr{X}. \tag{4.4}$$

As f_n is an element from $\mathscr{X}$, so $\hat{b}_n$ belong to $\mathscr{X}$, i.e. $\hat{b}_n \in D(\mathscr{A}^2)$. If we replace x by $\hat{b}_n$ and $\mathscr{A}x$, $\mathscr{A}^2x$ from (4.3) and (4.4) respectively, from (4.1), we get

$$\Big\|\sum_{s=0}^{n} U(n,s)f(s)\Big\|_{\mathscr{X}}^2 \leqslant 4\mathscr{W}^2\|f\|_{\mathscr{X}} \times \Big\|\sum_{s=0}^{n} 1_{\{0,\ldots,r\}}(n-s)U(n,s)f(s)\Big\|_{\mathscr{X}}.$$

Hence this completes the proof. $\square$

Now we prove some applications of inequality (4.2) on the spaces $\mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{Z})$ and $l_p(\mathscr{Z}_+,\mathscr{Z})$, as corollaries.

COROLLARY 4.3. *Let $\{v\}_{\mathscr{Z}_+}$ or $v:\mathscr{Z}_+\to\mathscr{Z}$ be a sequence such that it decays to zero at* 0 *and* ∞, *so* $\lim_{n\to\infty} v_n = 0$. *Suppose that the sequences*

$$n \mapsto y(n) := \sum_{s=0}^{n} v(s)$$

and

$$n \mapsto J(n) := \sum_{s=0}^{n} (n-s)v(s)$$

verifies the condition $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} J(n) = 0$.

Then the following inequality holds:

$$\sup_{n\in\mathscr{Z}_+} |\sum_{s=0}^{n} v(s)|^2 \leqslant 4 \sup_{n\in\mathscr{Z}_+} |v(s)| \times |\sum_{s=0}^{n}(n-s)v(s)|.$$

Proof. We apply Theorem 4.2 for $\mathscr{X} = \mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{Z})$ and for $U(n,m)x = x$ where $n \geqslant m \geqslant 0$ and $x \in \mathscr{Z}$. $\square$

COROLLARY 4.4. *Let v, y, J be as in Corollary 4.3 and λ_n be a positive non-decreasing sequence on $\mathscr{Z}_+$. The following inequality holds:*

$$\sup_{n\in\mathscr{Z}_+}\left[\frac{\left|\sum_{s=0}^{n}\lambda(s)v(s)\right|^2}{\lambda(n)^2}\right] \leqslant 4 \sup_{n\in\mathscr{Z}_+}|v(s)| \times \sup_{n\in\mathscr{Z}_+}\left[\frac{\left|\sum_{s=0}^{n}(n-s)\lambda(s)v(s)\right|}{\lambda(n)}\right].$$

Proof. Follows by Theorem 4.2 for $\mathscr{X} = \mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{Z})$ and $U(n,m) = \frac{\lambda(m)}{\lambda(n)}$. $\square$

COROLLARY 4.5. *Let $1 \leqslant p \leqslant \infty$ and $\hat{f} \in l^p(\mathscr{Z}_+,\mathscr{Z})$. If the sequences*

$$n \mapsto \hat{g}(n) := \sum_{s=0}^{n} \hat{f}(s) \quad \textit{and} \quad n \mapsto \hat{h}(n) := \sum_{s=0}^{n}(n-s)\hat{f}(s)$$

belong to $l^p(\mathscr{Z}_+,\mathscr{Z})$, then the following inequality holds:

$$\|\sum_{s=0}^{n}\hat{f}(s)\|_p^2 \leqslant 4\|\hat{f}\|_p \times \|\sum_{s=0}^{n}(n-s)\hat{f}(s)\|_p\,.$$

Proof. We apply Theorem 4.2 for $\mathscr{X} = l^p(\mathscr{Z}_+, \mathscr{Z})$ and for $U(n,m)x = x$ where $n \geqslant m \geqslant 0$ and $x \in \mathscr{Z}$. $\square$

Now we state the same inequality for (r,q)-resolvent, when both sequences $r_n \in l^1(\mathscr{Z}_+, \mathscr{X})$ and $q_n \in \mathscr{K}_{00}(\mathscr{Z}_+, \mathscr{X})$ are positive.

THEOREM 4.6. *Let (r,q) be the pair satisfying the CP-condition and*

$$C_{r,q} := \sup_{n>0} \frac{(r*r*q)_n q_n}{(q*r)_n^2} < \infty. \tag{4.5}$$

Assume that $\mathscr{A}$ is the algebraic generator of the (r,q)-resolvent $\{K_n\}_{n\geqslant 0}$, such that

$$\|K_n\| \leqslant \eta q_n,\ n \geqslant 0, \tag{4.6}$$

with $\eta \geqslant 1$. Then the Kallman-Rota inequality,

$$\|\mathscr{A}x\|^2 \leqslant 8\eta^2 C_{r,q}\|x\|\|\mathscr{A}^2x\|, \tag{4.7}$$

holds for all $x \in D(\mathscr{A}^2)$.

Proof. For all $x \in D(\mathscr{A}^2)$ and $n \in \mathscr{Z}_+$ we have $K_n x \in D(\mathscr{A})$ and $\mathscr{A}K_n x \in D(\mathscr{A})$, hence

$$\begin{aligned} K_n x &= (r*\mathscr{A}K)_n x + q_n x \\ &= r*\mathscr{A}[(r*\mathscr{A}K)_n x + q_n x] + q_n x \\ &= (r*r*\mathscr{A}^2K)_n x + (r*q)_n \mathscr{A}x + q_n x. \end{aligned}$$

Therefore,

$$\begin{aligned} \|(r*q)_n\mathscr{A}x\| &\leqslant \|K_n x\| + \|(r*r*\mathscr{A}^2K)_n x\| + \|q_n x\| \\ &= \|K_n x\| + \|\sum_{s=0}^{n}(r*r)_{(n-s)}\mathscr{A}^2K_s x\| + \|q_n x\| \\ &\leqslant \|K_n x\| + \sum_{s=0}^{n}(r*r)_{(n-s)}\|K_s\mathscr{A}^2x\| + \|q_n x\| \\ &\leqslant \|K_n x\| + \sum_{s=0}^{n}(r*r)_{(n-s)}\|K_s\|\|\mathscr{A}^2x\| + \|q_n x\| \\ &\leqslant \eta q_n\|x\| + \eta\sum_{s=0}^{n}(r*r)_{(n-s)}q_s\|\mathscr{A}^2x\| + q_n\|x\|, \text{ using (4.6)} \\ \|(r*q)_n\mathscr{A}x\| &\leqslant \eta q_n\|x\| + \eta(r*r*q)_n\|\mathscr{A}^2x\| + q_n\|x\|, \end{aligned}$$

or, equivalently,

$$\|\mathscr{A}x\| \leqslant 2\eta\frac{q_n}{(r*q)_n}\|x\| + \eta\frac{(r*r*q)_n}{(r*q)_n}\|\mathscr{A}^2x\|,\ n > 0. \tag{4.8}$$

Let $d = 2\eta\|x\|$ and $e = \eta\|\mathscr{A}^2x\|$, we define

$$y_n = d\frac{q_n}{(r*q)_n} + e\frac{(r*r*q)_n}{(r*q)_n},$$

$$y_n = \frac{(\sqrt{e}\sqrt{(r*r*q)_n} - \sqrt{d}\sqrt{q_n})^2 + 2\sqrt{de}\sqrt{(r*r*q)_n q_n}}{(r*q)_n}. \tag{4.9}$$

As $(\sqrt{e}\sqrt{(r*r*q)_n} - \sqrt{d}\sqrt{q_n})^2 \geqslant 0$, the equation (4.9) can be written as

$$y_n \geqslant 2\sqrt{de}\sqrt{\frac{(r*r*q)_n q_n}{(r*q)_n^2}}, \quad \text{for all } n > 0, \tag{4.10}$$

and

$$y_n = 2\sqrt{de}\sqrt{\frac{(r*r*q)_n q_n}{(r*q)_n^2}}, \tag{4.11}$$

for those $n > 0$ such that $\sqrt{e}\sqrt{(r*r*q)_n} - \sqrt{d}\sqrt{q_n} = 0$. Since the pair (r,q) satisfies the CP-condition, we conclude that there exists $n_0 > 0$, depending on d and e, such that

$$\frac{d}{e}q(n_0) = (r*r*q)(n_0). \tag{4.12}$$

Hence,

$$y_{n_0} = 2\sqrt{de}\sqrt{\frac{(r*r*q)_{n_0} q_{n_0}}{(r*q)_{n_0}^2}} = 2d\frac{q_{n_0}}{(r*q)_{n_0}}. \tag{4.13}$$

From (4.8) we deduce that for all $x \in D(\mathscr{A}^2)$,

$$\|\mathscr{A}x\| \leqslant \min_{n>0} y(n) \leqslant y_{n_0} \leqslant 2\eta\sqrt{2\|x\|\|\mathscr{A}^2x\|}\sqrt{\frac{(r*r*q)_n q_n}{(r*q)_n^2}}. \tag{4.14}$$

Putting (4.5) in (4.14), we get

$$\|\mathscr{A}x\|^2 \leqslant 8\eta^2 C_{r,q}\|x\|\|\mathscr{A}^2x\|.$$

Hence the proof is complete. □

In the next main result, we used the sequence h_α, defined by

$$h_\alpha(b) := \frac{\Gamma(\alpha+b+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2b+3)}, \quad b > -1, \tag{4.15}$$

for all $\alpha > -1$, and it will play an important role in several estimates, see Theorem 4.8 below. In the following proposition, we collect some interesting properties of h_α.

PROPOSITION 4.7. *Let* $\alpha > -1$ *and* h_α *be defined by* (4.15). *Then* $h_\alpha(-1) = 1$, $0 < h_\alpha(b) \leqslant 1$, h_α *is a decreasing sequence in* $(-1,+\infty)$ *for any* $\alpha > -1$ *and* $\lim_{\alpha\to\infty} h_\alpha(b) = 1$ *for any* $b \in (-1,+\infty)$.

Proof. We directly check that $h_\alpha(-1) = 1$. To show that h_α is decreasing, we prove that $\Delta h_\alpha(b) < 0$ for $b > -1$. Note that

$$\Delta h_\alpha(b) := \left(\frac{\Gamma(\alpha+b+3)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2b+5)} - \frac{\Gamma(\alpha+b+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2b+3)}\right) < 0,$$

if and only if

$$\frac{\Gamma(\alpha+2b+3)}{\Gamma(\alpha+2b+5)} < \frac{\Gamma(\alpha+b+2)^2}{\Gamma(\alpha+b+3)^2}, \text{ for all } b > -1.$$

The above inequality is obvious, so we conclude that h_α is decreasing for any $\alpha > -1$. Then $1 = h_\alpha(-1) \geqslant h_\alpha(b)$ for any $b > -1$.

It is known that

$$\begin{aligned}\lim_{\alpha\to\infty} h_\alpha(b) &= \lim_{\alpha\to\infty} \frac{\Gamma(\alpha+b+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2b+3)} \\ &= \lim_{\alpha\to\infty} \frac{\Big((\alpha+b+1)!\Big)^2}{\alpha!(\alpha+2b+2)!} \\ &= 1\end{aligned}$$

and we conclude the proof. $\square$

In what follows we will deduce from our main result examples concerning different types of algebraic generators families arising in applications to abstract evolution equations. We begin with norm inequalities for generators of α-times β-resolvents $(S_{\alpha,\beta}(n))_{n\in\mathscr{Z}_+}$ families. According to the definition given in preliminaries, they satisfy:

$$S_{\alpha,\beta}(n)x = \frac{n^\alpha}{\Gamma(\alpha+1)}x + \sum_{s=1}^{n-1} \frac{(n-s)^\beta}{\Gamma(\beta+1)} \mathscr{A} S_{\alpha,\beta}(s)x, \quad n \in \mathscr{Z}_+, \quad x \in \mathscr{X},$$

i.e. a $(g_{\beta+1}, g_{\alpha+1})$-resolvent for some $\alpha, \beta > -1$. Recall that for $\alpha = 0$, the existence of $(S_{0,\beta}(n))_{n\in\mathscr{Z}_+}$ is equivalent to the well-posedness of the abstract fractional difference equation

$$\Delta_c^{\beta+1} v(n) = \mathscr{A} v(n), \;\; n \in \mathscr{Z}_+, \;\; \beta > -1, \tag{4.16}$$

with some initial conditions, where $\Delta_c^{\beta+1}$ denotes the Caputo's fractional difference, see [1]. In case $\alpha > 0$, there families corresponds to α-times solutions of the above equation.

THEOREM 4.8. *Let $\mathscr{A}$ be the algebraic generator of α-times β-resolvent $(S_{\alpha,\beta}(n))_{n\in\mathscr{Z}_+}$ for some $\alpha,\beta > -1$ and suppose that there is $\eta \geqslant 1$ such that*

$$\|S_{\alpha,\beta}(n)\| \leqslant \eta \frac{n^{\alpha}}{\Gamma(\alpha+1)}, \ \ n \in \mathscr{Z}_+.$$

Then for all $x \in D(\mathscr{A}^2)$ we have

$$\|\mathscr{A}x\|^2 \leqslant 8\eta^2 \frac{\Gamma(\alpha+\beta+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2\beta+3)} \|x\|\|\mathscr{A}^2 x\|. \tag{4.17}$$

Proof. The pair $(g_{\beta+1}, g_{\alpha+1})$ satisfies the CP-condition and the following inequality

$$\frac{(r*r*q)_n q_n}{(r*q)_n^2} \leqslant \frac{\Gamma(\alpha+\beta+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2\beta+3)}$$

holds for any $n \in \mathscr{Z}_+$. Hence, the conclusion follows from Theorem 4.6. □

REMARK 4.9. In the case of the well-posed fractional difference equation (4.16), for different values of α, we find out the qualitative behavior of $h_\alpha(\beta)$, given in Proposition 4.7. Then we apply it in the study of the Kallman-Rota type inequality (4.17).

(i) When $\alpha \to -1$ then $h_\alpha(\beta) \to 0$ for some $\beta > -1$. As a consequence, we can choose constant $\omega_{\alpha,\beta}$ as smaller as we want, such that

$$\|\mathscr{A}x\|^2 \leqslant \omega_{\alpha,\beta}\|x\|\|\mathscr{A}^2 x\|, \ \ x \in D(\mathscr{A}^2). \tag{4.18}$$

(ii) When $\alpha = 0$ then $h_0(\beta) = \frac{\Gamma(\beta+2)^2}{\Gamma(2\beta+3)}$ for $\beta > -1$. Note that such sequence is decreasing so that the constant in case of the second order abstract difference equation $(h_0(1) = \frac{1}{6})$, will be always smaller than the constant in case of the first order equation $(h_0(0) = \frac{1}{2})$.

(iii) When $\alpha \to \infty$ then $h_\alpha(\beta) \to 1$. The situation is different as in case (i) and (ii): If $\alpha \to \infty$, then the constant $\omega_{\alpha,\beta}$ in (4.18) tends to $8\eta^2$. Moreover, again the constant near to the abstract difference equation of order 2 is smaller than the constant near to abstract difference equation of order 1, for the same value of α.

The cases $\beta = 0$ and $\beta = 1$ in Theorem 4.8 give, respectively, the following corollaries.

COROLLARY 4.10. *Let $\mathscr{A}$ be the algebraic generator of the α-times semigroup $(S_\alpha(n))_{n\in\mathscr{Z}_+}$ for some $\alpha \geqslant 0$ and suppose that there is $\eta \geqslant 1$ such that*

$$\|S_\alpha(n)\| \leqslant \eta \frac{n^{\alpha}}{\Gamma(\alpha+1)}, \ \ n \in \mathscr{Z}_+.$$

Then for all $x \in D(\mathscr{A}^2)$ we have

$$\|\mathscr{A}x\|^2 \leqslant \eta^2 \frac{(\alpha+1)}{(\alpha+2)} \|x\|\|\mathscr{A}^2 x\|. \tag{4.19}$$

COROLLARY 4.11. *Let $\mathscr{A}$ be the algebraic generator of the α-times sums cosine sequence $(C_\alpha(n))_{n\in\mathscr{Z}_+}$ for some $\alpha \geqslant 0$ and suppose that there is $\eta \geqslant 1$ such that*

$$\|C_\alpha(n)\| \leqslant \eta \frac{n^\alpha}{\Gamma(\alpha+1)}, \ \ n \in \mathscr{Z}_+.$$

Then for all $x \in D(\mathscr{A}^2)$ we have

$$\|\mathscr{A}x\|^2 \leqslant \eta^2 \frac{(\alpha+1)(\alpha+2)}{(\alpha+3)(\alpha+4)} \|x\| \|\mathscr{A}^2 x\|. \tag{4.20}$$

5. Conclusion

We have proved that Kallman-Rota inequality holds for a discrete evolution family using the space $\mathscr{K}_{00}(\mathscr{Z}_+,\mathscr{X})$ and we have also proved the same inequality for (r,q)-resolvent operators which arises in the solution of fractional difference equation of order $\beta+1$. Finally, some applications of the obtained inequality is given.

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