FORCED OSCILLATION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH DAMPING TERM

HUANHUAN KONG AND RUN XU

(Communicated by N.-E. Tatar)

Abstract. In this paper, the forced oscillation of fractional partial differential equations of the form

\[ D_{+t}^{1+\alpha}u(x,t) + p(t)D_{+t}^{\alpha}u(x,t) = a(t)\Delta u(x,t) + \sum_{j=1}^{m} a_j(t)\Delta u(x,t - \tau_j) \]

\[-q(x,t) \int_0^t (t - \xi)^{-\alpha} u(x,\xi) d\xi + f(x,t), \quad (x,t) \in \Omega \times R_+ \equiv G \]

are investigated, and some examples are given to illustrate the useful of our results.

1. Introduction

Recently, the fractional differential equations have attracted much attention due to their applications being used widely in various areas of science, engineering or bioengineering, mechanics, finance, nonlinear control and so on. Based a series of fundamental theory [2–6], there are a lot of results in some aspects of fractional differential equation, such as the existence, uniqueness, boundedness, stability or oscillation of the solutions (see [5, 6, 16–19] and the references there in).

Recently, the research on the theory of fractional partial differential equations is becoming a hot topic, and some results are established, see [7–15]. In 2013, Jafari et al. [7] derived the exact and approximate analytical solutions of fractional partial differential equations by the method of iterative Laplace transform. In 2015, Chen and Jiang et al. [8] presented the techniques of Lie group analysis to solve n-order linear partial fractional differential equations.

At the same time, the oscillation of fractional partial differential equations has been studied. In [9, 10], Prakash et al. presented the oscillation of fractional partial differential equation

\[ \frac{\partial}{\partial t} \left( r(t)D_{+t}^{\alpha}u(x,t) \right) + q(x,t)f\left( \int_0^t (t - \nu)^{-\alpha} u(x,\nu) d\nu \right) = a(t)\Delta u(x,t), \quad (1.1) \]

Keywords and phrases: Forced, oscillation, fractional, partial differential equations.
This research is supported by National Science Foundation of China (11671227).
and the forced oscillation of a nonlinear fractional partial differential equation with damping term of the form
\[ D^{\alpha}_{+,t}(r(t)D^{\alpha}_{+,t}u(x,t)) + p(t)D^{\alpha}_{+,t}u(x,t) + q(x,t)f(u(x,t)) = a(t)\Delta u(x,t) + g(x,t), \quad (x,t) \in G. \] (1.2)

In [11], Harikrishnan et al. studied the oscillatory behavior of fractional partial differential equation of the form
\[ D^{\alpha}_{+,t}(r(t)D^{\alpha}_{+,t}u(x,t)) + q(x,t)f(u(x,t)) = a(t)\Delta u(x,t) + g(x,t), \quad (x,t) \in G. \] (1.3)

Li et al. and Sheng [12,13] established the oscillation of fractional partial differential equations of the form
\[ \frac{\partial}{\partial t} (D^{\alpha}_{+,t}u(x,t)) + p(t)D^{\alpha}_{+,t}u(x,t) = a(t)\Delta u(x,t) - q(x,t)u(x,t) + f(x,t), \quad (x,t) \in G, \] (1.4)
and
\[ D^{1+\alpha}_{+,t}u(x,t) + p(t)D^{\alpha}_{+,t}u(x,t) = a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,t - \tau_i) - q(x,t)\int_{0}^{t} (t - \xi)^{-\alpha}u(x,\xi)d\xi \] (1.5)
respectively. In this paper, We will study the oscillation for the fractional partial differential equation
\[ D^{1+\alpha}_{+,t}u(x,t) + p(t)D^{\alpha}_{+,t}u(x,t) = a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,t - \tau_i) - q(x,t)\int_{0}^{t} (t - \xi)^{-\alpha}u(x,\xi)d\xi + f(x,t), \]
\[ (x,t) \in \Omega \times R_+ \equiv G, \] (1.6)
with the boundary condition
\[ \frac{\partial u(x,t)}{\partial \mathbf{N}} + g(x,t)u(x,t) = 0, \quad (x,t) \in \partial \Omega \times R_+, \] (1.7)
or
\[ u(x,t) = 0, \quad (x,t) \in \partial \Omega \times R_+. \] (1.8)
where \( \Omega \) is a bounded domain in \( R^n \) with piecewise smooth boundary \( \partial \Omega, \alpha \in (0,1) \) is a constant, \( R_+ = (0,\infty) \). \( D^{\alpha}_{+,t}u(x,t) \) is the Riemann-Liouville fractional derivative of order \( \alpha \) of \( u \) with respect to \( t \), \( \Delta \) is the Laplacian in \( R^n \). \( N \) is the unit exterior normal vector to \( \partial \Omega \), and \( g(x,t) \) is a nonnegative continuous function on \( \partial \Omega \times R_+ \).

By a solution of (1.6), (1.7) (or (1.6), (1.8)) we mean a function \( u(x,t) \) satisfies (1.6) on \( \overline{G} \) and the boundary condition (1.7) (or (1.8)).

We assume throughout this paper that
\[ (A_1) \quad a \in C(R_+;R_+), p \in C(R_+;R), a_i \in C(R_+;R_+) \text{ and } \tau_i \geq 0 \text{ are constants, } \]
\[ i = 1, 2, 3, \ldots, m; \]
\[ (A_2) \quad q \in C(\overline{G};R_+) \text{ and } q(t) = \min_{x \in \Omega} q(x,t); \]
\[ (A_3) \quad f \in C(\overline{G};R). \]
2. Preliminaries and Lemmas

In this part, we will present some useful preliminaries and lemmas, which will be used in the following proof for our results.

**Definition 1.** (see [3]) The Riemann-Liouville fractional integral \( I_\alpha^a \) of order \( \alpha > 0 \) of a function \( y : R_+ \rightarrow R \) on the half-axis \( R_+ \) is given by

\[
(I_\alpha^a y)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \quad t > a,
\]

provided that the right-hand side is pointwise defined on \( R_+ \), where \( \Gamma \) is the gamma function.

**Definition 2.** (see [3]) The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) of a function \( y : R_+ \rightarrow R \) on the half-axis \( R_+ \) is given by

\[
(D_+^\alpha y)(t) := \frac{d^{[\alpha]}}{dt^{[\alpha]}}(I_+^{[\alpha]-\alpha} y)(t)
\]

\[
= \frac{1}{\Gamma([\alpha] - \alpha)} \frac{d^{[\alpha]}}{dt^{[\alpha]}} \int_0^t (t-s)^{[\alpha]-\alpha-1} y(s) ds, \quad t > 0,
\]

provided that the right-hand side is pointwise defined on \( R_+ \), where \( [\alpha] \) is the ceiling function of \( \alpha \).

**Definition 3.** (see [3]) The Riemann-Liouville fractional partial derivative of order \( 0 < \alpha < 1 \) with respect to \( t \) of a function \( u(x,t) \) is given by

\[
D_+^{\alpha,x} u(x,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(x,s) ds,
\]

provided the right-hand side is pointwise defined on \( R_+ \).

There are some lemmas which are very useful in the proof of our main results.

**Lemma 2.4.** (see [3]) Let \( \alpha \geq 0, m \in N \) and \( D = \frac{d}{dt} \). If the fractional derivatives \( D_+^\alpha y(t) \) and \( D_+^{\alpha+m} y(t) \) exist, then

\[
D^m(D_+^\alpha y(t)) = D_+^{\alpha+m} y(t).
\]

**Lemma 2.5.** (see [9]) Let

\[
\tilde{E}(t) =: \int_0^t (t-s)^{-\alpha} y(s) ds \text{ for } \alpha \in (0,1) \text{ and } t \geq 0,
\]

Then \( \tilde{E}'(t) = \Gamma(1-\alpha)D_+^\alpha y(t) \).

**Lemma 2.6.** (see [3]) Let \( \alpha \in (0,1) \) and \( I_{a+}^{1-\alpha} y(t) \) is the fractional integral of order \( 1-\alpha \), then

\[
(I_\alpha^a D_+^\alpha y(t)) = y(t) - \frac{I_{a+}^{1-\alpha} y(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}.
\]
**Remark** that where \( C \) is positive and the corresponding eigenfunction \( \varphi(x) \) is positive in \( \Omega \).

### 3. The oscillation of the problem (1.6), (1.7)

**Theorem 3.1.** Suppose that \((A_1) - (A_3)\) hold, \(\lim_{t \to \infty} I_t^{1-\alpha} U(t_0) = C_0\). If

\[
\lim\inf_{t \to \infty} \int_0^t \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^\xi F(s) V(s) ds \right) d\xi < 0, \tag{3.1}
\]

and

\[
\lim\sup_{t \to \infty} \int_0^t \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^\xi F(s) V(s) ds \right) d\xi > 0, \tag{3.2}
\]

where \( C_0, C \) are two constants, \( V(t) = e^{\int_0^t p(s) ds} \), \( F(t) = \int_\Omega f(x, t) dx \), then each solution of problem (1.6), (1.7) is oscillatory.

**Proof.** Suppose to the contrary that there is a non-oscillatory solution \( u(x, t) \) of (1.6), (1.7), then there exists a \( t_0 \geq 0 \), such that \( u(x, t) > 0 \) (or \( u(x, t) < 0 \)), \( t \geq t_0 \).

**Case 1.** Suppose that \( u(x, t) > 0, t \geq t_0 \).

Integrating both sides of (1.6) with respect to \( x \) over the domain \( \Omega \), we have

\[
\int_\Omega D_{+t}^{1+\alpha} u(x, t) dx + p(t) \int_\Omega D_{+t}^{\alpha} u(x, t) dx
\]

\[
= a(t) \int_\Omega \Delta u(x, t) dx + \sum_{i=1}^m a_i(t) \int_\Omega \Delta u(x, t - \tau_i) dx
\]

\[
- \int_\Omega q(x, t) \int_0^t (t - \xi)^{-\alpha} u(x, \xi) d\xi dx + \int_\Omega f(x, t) dx, \tag{3.3}
\]

\[
D_{+t}^{1+\alpha} \int_\Omega u(x, t) dx + p(t) D_{+t}^{\alpha} \int_\Omega u(x, t) dx
\]

\[
= a(t) \int_\Omega \Delta u(x, t) dx + \sum_{i=1}^m a_i(t) \int_\Omega \Delta u(x, t - \tau_i) dx
\]

\[
- \int_\Omega q(x, t) \int_0^t (t - \xi)^{-\alpha} u(x, \xi) d\xi dx + \int_\Omega f(x, t) dx. \tag{3.4}
\]

Remark that \( \int_\Omega u(x, t) dx = U(t) \), then \( U(t) > 0 \).

By the Green’s formula and (1.7), we can get

\[
\int_\Omega \Delta u(x, t) dx = \int_{\partial \Omega} \frac{\partial u}{\partial N} ds = - \int_{\partial \Omega} g(x, t) u(x, t) ds < 0, \tag{3.5}
\]
\[
\int_{\Omega} \Delta u(x, t - \tau_i) dx < 0, \tag{3.6}
\]
and
\[
\int_{\Omega} q(x, t) \int_{0}^{t} (t - \xi)^{-\alpha} u(x, \xi) d\xi dx \geq q(t) \int_{0}^{t} (t - \xi)^{-\alpha} \int_{\Omega} u(x, \xi) dx d\xi
\]
\[
= q(t) \int_{0}^{t} (t - \xi)^{-\alpha} U(\xi) d\xi = q(t) E(t), \tag{3.7}
\]
where
\[
E(t) = \int_{0}^{t} (t - \xi)^{-\alpha} U(\xi) d\xi. \tag{3.8}
\]
By (3.3)–(3.8), we can get
\[
D_{+}^{1+\alpha} U(t) + p(t) D_{+}^{\alpha} U(t) \leq -q(t) E(t) + F(t) \leq F(t). \tag{3.9}
\]
By lemma 2.4 and (3.9), we have
\[
[D_{+}^{\alpha} U(t)V(t)]' = D_{+}^{1+\alpha} U(t)V(t) + p(t) D_{+}^{\alpha} U(t)V(t) \leq F(t)V(t). \tag{3.10}
\]
Integrating both sides of the above inequality from \(t_0\) to \(t\), we get
\[
D_{+}^{\alpha} U(t)V(t) \leq D_{+}^{\alpha} U(t_0)V(t_0) + \int_{t_0}^{t} F(s)V(s) ds.
\]
Setting \(D_{+}^{\alpha} U(t_0)V(t_0) = C\), we can get
\[
D_{+}^{\alpha} U(t) \leq \frac{C}{V(t)} + \frac{\int_{t_0}^{t} F(s)V(s) ds}{V(t)}. \tag{3.11}
\]
Taking the Riemann-Liouville fractional integral of order \(\alpha\) on both sides of (3.11), by lemma 2.6 and the above inequality, we obtain
\[
I_{+}^{\alpha} D_{+}^{\alpha} U(t) = U(t) - \frac{I_{0}^{1-\alpha} U(0)}{\Gamma(\alpha)} t^{\alpha-1} \leq I_{+}^{\alpha} \left[ \frac{C}{V(t)} + \frac{\int_{t_0}^{t} F(s)V(s) ds}{V(t)} \right].
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F(s)V(s) ds \right) d\xi. \tag{3.12}
\]
In (3.12), set \(t \to \infty\), we have
\[
\liminf_{t \to \infty} U(t) \leq \liminf_{t \to \infty} \frac{C_0}{\Gamma(\alpha)} t^{\alpha-1} + \liminf_{t \to \infty} \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F(s)V(s) ds \right) d\xi.
\]
From (3.1), we can get \(\liminf_{t \to \infty} U(t) \leq 0\). Which contradicts the assumption that \(U(t) > 0\).

Case 2. Suppose that \(u(x, t) < 0\), \(t \geq t_0\).
Just as the case 1, we can obtain (3.3) holds and \( U(t) < 0 \). Then,

\[
\int_\Omega \Delta u(x,t)dx = \int_{\partial \Omega} \frac{\partial u}{\partial N}ds = - \int_{\partial \Omega} g(x,t)u(x,t)ds > 0,
\]

\[
\int_\Omega \Delta u(x,t - \tau_i) > 0,
\]

and

\[
\int_\Omega q(x,t) \int_0^t (t - \xi)^{-\alpha} u(x,\xi)d\xi dx \leq q(t) \int_0^t (t - \xi)^{-\alpha} \int_\Omega u(x,\xi)d\xi d\xi = q(t)E(t).
\]

From the conditions (3.13)–(3.15), we get

\[
D_+^{1+\alpha} U(t) + p(t)D_+^\alpha U(t) \geq -q(t)E(t) + F(t) \geq F(t).
\]

Similarly, by lemma 2.4 and (3.16), we can obtain

\[
[D_+^\alpha U(t)V(t)]' = D_+^{1+\alpha} U(t)V(t) + p(t)D_+^\alpha U(t)V(t) \geq F(t)V(t).
\]

Then integrating both sides of the above inequality from \( t_0 \) to \( t \), we have

\[
D_+^\alpha U(t)V(t) \geq D_+^\alpha U(t_0)V(t_0) + \int_{t_0}^t F(s)V(s)ds = C + \int_{t_0}^t F(s)V(s)ds,
\]

and

\[
D_+^\alpha U(t) \geq \frac{C}{V(t)} + \frac{\int_{t_0}^t F(s)V(s)ds}{V(t)}.
\]

Taking the Riemann-Liouville fractional integral of order \( \alpha \) on both sides of (3.18), by lemma 2.6 and the above inequality, we obtain

\[
I_+^\alpha D_+^\alpha U(t) = U(t) - \frac{I_+^{1-\alpha}U(0)}{\Gamma(\alpha)} t^{-\alpha} \geq I_+^\alpha \left[ \frac{C}{V(t)} + \frac{\int_{t_0}^t F(s)V(s)ds}{V(t)} \right]
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} V(\xi) \left( C + \int_{t_0}^\xi F(s)V(s)ds \right) d\xi.
\]

In (3.19), setting \( t \to \infty \), we have

\[
\limsup_{t \to \infty} U(t) \geq \limsup_{t \to \infty} \frac{C_0}{\Gamma(\alpha)} t^{-\alpha} + \limsup_{t \to \infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} V(\xi) \left( C + \int_{t_0}^\xi F(s)V(s)ds \right) d\xi.
\]

By the condition (3.2), we can get \( \limsup_{t \to \infty} U(t) \geq 0 \). Which contradicts the assumption that \( U(t) < 0 \). This proof is completed. \( \Box \)
4. The oscillation of the problem (1.6), (1.8)

**Theorem 4.1.** Suppose that \((A_1)-(A_3)\) hold, \(\lim_{t \to +\infty} I_{+}^{1-\alpha} U_1(t_0) = C_1\). If

\[
\liminf_{t \to +\infty} \int_0^t \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s) V(s) \, ds \right) d\xi < 0, \tag{4.1}
\]

and

\[
\limsup_{t \to +\infty} \int_0^t \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s) V(s) \, ds \right) d\xi > 0, \tag{4.2}
\]

where \(C_1, C\) are constants, \(V(t) = e^{\int_0^t p(s) \, ds}\), \(F_1(s) = \int_\Omega f(x,t) \phi(x) \, dx\), \(U_1(t) = \int_\Omega u(x,t) \phi(x) \, dx\), then each solution \(u(x,t)\) of problem (1.6), (1.8) is oscillatory.

**Proof.** Suppose to the contrary that there is a non-oscillatory solution \(u(x,t)\) of (1.6), (1.8), then there exists a \(t_0 \geq 0\), such that \(u(x,t) > 0\) (or \(u(x,t) < 0\), \(t \geq t_0\).

**Case 1.** Suppose that \(u(x,t) > 0\), \(t \geq t_0\).

Multiplying both sides of (1.6) by \(\phi(x)\) and integrating with respect to \(x\) over the domain \(\Omega\), we have

\[
D_{+}^{1-\alpha} \int_\Omega u(x,t) \phi(x) \, dx + p(t) D_{+}^{\alpha} \int_\Omega u(x,t) \phi(x) \, dx = a(t) \int_\Omega \Delta u(x,t) \phi(x) \, dx + \sum_{i=1}^{m} a_i(t) \int_\Omega \Delta u(x,t - \tau_i) \phi(x) \, dx
\]

\[
- \int_\Omega q(x,t) \int_0^t (t - \xi)^{-\alpha} u(x,\xi) \phi(x) \, d\xi \, dx + \int_\Omega f(x,t) \phi(x) \, dx. \tag{4.3}
\]

By the Green’s formula and Lemma 2.7, we can get

\[
\int_\Omega \Delta u(x,t) \phi(x) \, dx = \int_\Omega u(x,t) \Delta \phi(x) \, dx = - \int_\Omega u(x,t) \beta_0 \phi(x) \, dx < 0, \tag{4.4}
\]

\[
\int_\Omega \Delta u(x,t - \tau_i) \phi(x) \, dx < 0, \tag{4.5}
\]

and

\[
\int_\Omega q(x,t) \int_0^t (t - \xi)^{-\alpha} u(x,\xi) \phi(x) \, d\xi \, dx \geq q(t) \int_0^t (t - \xi)^{-\alpha} \int_\Omega u(x,\xi) \phi(x) \, dx \, d\xi = q(t) E_1(t). \tag{4.6}
\]

Therefore, \(U_1(t) > 0\), \(E_1(t) = \int_0^t (t - \xi)^{-\alpha} U_1(\xi) \, d\xi > 0\).

Consider (4.3)–(4.6), we can get

\[
D_{+}^{1-\alpha} U_1(t) + p(t) D_{+}^{\alpha} U_1(t) \leq - q(t) E_1(t) + F_1(t) \leq F_1(t), \tag{4.7}
\]

By lemma 2.4 and (4.7), we have

\[
[D_{+}^{\alpha} U_1(t) V(t)]' = D_{+}^{1-\alpha} U_1(t) V(t) + p(t) D_{+}^{\alpha} U_1(t) V(t) \leq F_1(t) V(t). \tag{4.8}
\]
Then integrating both sides of the above inequality from \( t_0 \) to \( t \), we have

\[
D^\alpha_u U_1(t)V(t) \leq D^\alpha_u U_1(t_0)V(t_0) + \int_{t_0}^{t} F_1(s)V(s)ds.
\]

Setting \( D^\alpha_u U_1(t_0)V(t_0) = C \), We can get

\[
D^\alpha_u U_1(t) \leq \frac{C}{V(t)} + \frac{\int_{t_0}^{t} F_1(s)V(s)ds}{V(t)}.
\] (4.9)

Taking the Riemann-Liouville fractional integral of order \( \alpha \) on both sides of (4.9), by lemma 2.6 and the above inequality, we obtain

\[
I^\alpha_+ D^\alpha_+ U_1(t) = U_1(t) - \frac{I^{1-\alpha}_+ U_1(0)}{\Gamma(\alpha)} t^{\alpha-1} \leq I^\alpha_+ \left[ \frac{C}{V(t)} + \frac{\int_{t_0}^{t} F(s)V(s)ds}{V(t)} \right] = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi,
\]

and

\[
U_1(t) \leq \frac{I^{1-\alpha}_+ U_1(0)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi.
\] (4.10)

In (4.10), setting \( t \to \infty \), we have

\[
\liminf_{t \to \infty} U_1(t) \leq \liminf_{t \to \infty} \frac{C_1}{\Gamma(\alpha)} t^{\alpha-1} + \liminf_{t \to \infty} \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi.
\]

From (4.1), we can get \( \liminf_{t \to \infty} U_1(t) \leq 0 \). Which contradicts the assumption that \( U_1(t) > 0 \).

Case 2. Suppose that \( u(x,t) < 0 \), \( t \geq t_0 \).

Multiplying both sides of (1.6) by \( \varphi(x) \) and integrating with respect to \( x \) over the domain \( \Omega \), we have (4.3) holds.

By the Green’s formula, we can get

\[
\int_{\Omega} \Delta u(x,t) \varphi(x) dx = \int_{\Omega} u(x,t) \Delta \varphi(x) dx = - \int_{\Omega} u(x,t) \beta_0 \varphi(x) dx > 0,
\] (4.11)

\[
\int_{\Omega} \Delta u(x,t - \tau_1) \varphi(x) dx > 0,
\] (4.12)

and

\[
\int_{\Omega} q(x,t) \int_{0}^{t} (t - \xi)^{-\alpha} u(x,\xi) \varphi(\xi) d\xi d\xi dx \leq q(t) \int_{0}^{t} (t - \xi)^{-\alpha} \int_{\Omega} u(x,\xi) \varphi(x) dx d\xi = q(t) E_1(t).
\] (4.13)

Therefore, \( U_1(t) < 0 \), \( E_1(t) = \int_{0}^{t} (t - \xi)^{-\alpha} U_1(\xi) d\xi < 0 \).
From (4.11)–(4.13), we can get
\[ D_{+}^{1+\alpha}U_1(t) + p(t)D_{+}^{\alpha}U_1(t) \geq -q(t)E_1(t) + F_1(t) \geq F_1(t). \quad (4.14) \]

By lemma 2.4 and (4.14)
\[ [D_{+}^{\alpha}U_1(t)V(t)]' = D_{+}^{1+\alpha}U_1(t)V(t) + p(t)D_{+}^{\alpha}U_1(t)V(t) \geq F_1(t)V(t). \]

Integrating both sides of the above inequality from \( t_0 \) to \( t \), we have
\[ D_{+}^{\alpha}U_1(t)V(t) \geq D_{+}^{\alpha}U_1(t_0)V(t_0) + \int_{t_0}^{t} F_1(s)V(s)ds. \]

Setting \( D_{+}^{\alpha}U_1(t_0)V(t_0) = C \), We can get
\[ D_{+}^{\alpha}U_1(t) \geq \frac{C}{V(t)} + \frac{\int_{t_0}^{t} F_1(s)V(s)ds}{V(t)}. \quad (4.15) \]

Taking the Riemann-Liouville fractional integral of order \( \alpha \) on both sides of (4.15), by lemma 2.6 and the above inequality, we obtain
\[ I_{+}^{\alpha}D_{+}^{\alpha}U_1(t) = U_1(t) - \frac{I_{+}^{1-\alpha}U_1(0)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{F_1(s)V(s)ds}{V(t)} \geq \frac{C}{V(t)} \]
\[ = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi, \]
then
\[ U_1(t) \geq \frac{I_{+}^{1-\alpha}U_1(0)}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi. \quad (4.16) \]

Therefore, we have
\[ \limsup_{t \to \infty} U_1(t) \geq \limsup_{t \to \infty} \frac{C}{\Gamma(\alpha)} \frac{1}{t^{\alpha-1}} + \limsup_{t \to \infty} \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left( C + \int_{t_0}^{\xi} F_1(s)V(s)ds \right) d\xi. \]

From (4.2), we can get \( \liminf_{t \to \infty} U_1(t) \geq 0 \). Which contradicts to \( U_1(t) < 0 \). The proof is completed. \( \Box \)

5. Examples

In this part, we will give two examples to illustrate the usefulness of our conclusions.

EXAMPLE 1. Consider the oscillation of the fractional partial differential equation
\[ D_{+,t}^{1+\frac{1}{2}}u(x,t) - D_{+,t}^{\frac{1}{2}}u(x,t) = \frac{1}{\pi} \Delta u(x,t) + 2t \Delta u(x,t - 1) \]
\[ - \left( x^2 + \frac{1}{t^2} \right) \int_{0}^{t} \left( t - \xi \right)^{-\frac{1}{2}} u(x, \xi) d\xi + e^{2t} \sin t \sin x, \quad (x,t) \in (0, \pi) \times \mathbb{R}_+ \quad (5.1) \]
with the boundary condition,

$$u_x(0,t) = u_x(\pi,t) = 0. \quad (5.2)$$

Proof. Compare with (1.6) and (1.7), where $$\alpha = \frac{1}{2}, \ \Omega = (0, \pi), \ n = 1, \ m = 1, \ \rho(t) = -1, \ a(t) = \frac{1}{\pi}, \ a_1(t) = 2t, \ \tau_1 = 1, \ q(x,t) = x^2 + \frac{1}{2}, \ q(t) = \min q(x,t) = \frac{1}{r^2}, \ f(x,t) = e^{2t} \sin t \sin x.$$  

Then

$$F(t) = \int_0^\pi f(x,t)dx = 2e^{2t} \sin t,$$

$$V(t) = e^{\int_0^t p(s)ds} = e^{1-t},$$

and

$$\int_{t_0}^\xi F(s)V(s)ds = \int_{t_0}^\xi 2e^{2s} \sin e^{t_0-s}ds$$

$$= e^{t_0}(e^{\xi} \sin \xi - e^{\xi} \cos \xi + e^{t_0} \cos t_0 - e^{t_0} \sin t_0).$$

Letting $$t_0 = \xi,$$ we obtain

$$\int_0^t \left( \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \right) \left( C + \int_0^\xi F(s)V(s)ds \right) d\xi$$

$$= \int_0^t (t - \xi)^{-\frac{1}{2}}e^{\xi} \left[ C + e^{t_0}(e^{\xi} \sin \xi - e^{\xi} \cos \xi + e^{t_0} \cos t_0 - e^{t_0} \sin t_0) \right] d\xi$$

$$= \int_0^t (t - \xi)^{-\frac{1}{2}}e^{\xi} \left[ C + \xi e^{\xi} \sin \xi - e^{\xi} e^{\xi} \cos \xi - \xi \right] d\xi. \quad (5.3)$$

Let $$t - s^2 = \xi,$$ then

$$\int_0^t \left( \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \right) \left( C + \int_0^\xi F(s)V(s)ds \right) d\xi$$

$$= \int_0^{\sqrt{t}} e^{t-s^2} \left[ C + e^{\frac{\pi}{2}} e^{t-s^2} \sin(t-s^2) - e^{\frac{\pi}{2}} e^{t-s^2} \cos(t-s^2) - \xi \right] (-2s)ds$$

$$= e^{t} \left\{ 2(C - e^{\pi})e^{-\frac{\pi}{4}} \right\} \left[ \sin \left(t - \frac{\pi}{4}\right) \int_0^{\sqrt{t}} e^{-2s^2} \cos s^2 ds + 2\sqrt{2}e^{t} \int_0^{\sqrt{t}} e^{-2s^2} \cos s^2 ds \right.$$

$$- \cos \left(t - \frac{\pi}{4}\right) \int_0^{\sqrt{t}} e^{-2s^2} \sin s^2 ds \right\}.$$

Noting that

$$|e^{-2s^2} \cos s^2| \leq e^{-2s^2}, \quad |e^{-2s^2} \sin s^2| \leq e^{-2s^2},$$
and

\[ \lim_{t \to -\infty} \int_0^{\sqrt{t}} e^{-2s^2} \, ds = \frac{\sqrt{2\pi}}{4}, \]

we can get that \( \lim_{t \to -\infty} \int_0^{\sqrt{t}} e^{-2s^2} \cos^2 s^2 \, ds \) and \( \lim_{t \to -\infty} \int_0^{\sqrt{t}} e^{-2s^2} \sin^2 s^2 \, ds \) are convergent, then

\[ \lim_{t \to -\infty} \left( \sin \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{t}} e^{-2s^2} \cos^2 s^2 \, ds - \cos \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{t}} e^{-2s^2} \sin^2 s^2 \, ds \right) \]

is convergent. Setting

\[ \lim_{t \to -\infty} \int_0^{\sqrt{t}} e^{-2s^2} \cos^2 s^2 \, ds = A, \]
\[ \lim_{t \to -\infty} \int_0^{\sqrt{t}} e^{-2s^2} \sin^2 s^2 \, ds = B, \]
and

\[ t_k = \frac{3\pi}{2} + \frac{\pi}{4} + 2k\pi + \arccos \frac{A}{\sqrt{A^2 + B^2}}. \]

Then

\[ \lim_{k \to -\infty} \left[ \sin \left( t_k - \frac{\pi}{4} \right) \int_0^{\sqrt{t_k}} e^{-2s^2} \cos^2 s^2 \, ds - \cos \left( t_k - \frac{\pi}{4} \right) \int_0^{\sqrt{t_k}} e^{-2s^2} \sin^2 s^2 \, ds \right] \]

\[ = \lim_{k \to -\infty} \left[ \sin \left( t_k - \frac{\pi}{4} \right) \int_0^{\sqrt{t_k}} e^{-2s^2} \cos^2 s^2 \, ds - \cos \left( t_k - \frac{\pi}{4} \right) \int_0^{\sqrt{t_k}} e^{-2s^2} \sin^2 s^2 \, ds \right] \]

\[ = \sqrt{A^2 + B^2} \sin \left( \frac{3\pi}{2} + \frac{\pi}{4} + 2k\pi + \arccos \frac{A}{\sqrt{A^2 + B^2}} - \frac{\pi}{4} - \arccos \frac{A}{\sqrt{A^2 + B^2}} \right) \]

\[ = \sqrt{A^2 + B^2} \sin \left( \frac{3\pi}{2} + 2k\pi \right) \]

\[ = -\sqrt{A^2 + B^2}. \]

We have

\[ \lim_{t \to -\infty} \int_0^t \frac{(t - \xi)^{\alpha-1}}{V(\xi)} \left( C + \int_0^\xi F(s)V(s) \, ds \right) \, d\xi \]

\[ = \lim_{t \to -\infty} \left\{ 2(C - e^{\pi}) e^{-\frac{\pi}{4}} \int_0^{\sqrt{t}} e^{-2s^2} \, ds + 2\sqrt{2} e^{\pi} \left[ \sin \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{t}} e^{-2s^2} \cos^2 s^2 \, ds \right. \right. \]

\[ - \cos \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{t}} e^{-2s^2} \sin^2 s^2 \, ds \left. \right] \}

\[ = \lim_{k \to -\infty} \left\{ 2(C - e^{\pi}) e^{-\frac{\pi}{4}} \int_0^{\sqrt{t_k}} e^{-2s^2} \, ds + 2\sqrt{2} e^{\pi} \left[ \sin \left( t_k - \frac{\pi}{4} \right) \int_0^{\sqrt{t_k}} e^{-2s^2} \cos^2 s^2 \, ds \right. \right. \]

\[ - \cos \left( t_k - \frac{\pi}{4} \right) \int_0^{\sqrt{t_k}} e^{-2s^2} \sin^2 s^2 \, ds \left. \right] \}

\[ = -\infty, \]
then
\[
\liminf_{t \to \infty} \int_0^t \frac{(t - \xi)^{\alpha - 1}}{V(\xi)} \left( C + \int_{I_0}^\xi F(s)V(s)ds \right) d\xi = -\infty < 0. \tag{5.4}
\]

Similarly, taking \( t_j = \frac{\pi}{2} + \frac{\pi}{4} + 2j\pi + \arccos \frac{A}{\sqrt{A^2 + B^2}} \), we can get that
\[
\lim_{j \to \infty} \left[ \sin \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{j}} e^{-s^2} \cos^2 s^2 ds - \cos \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{j}} e^{-s^2} \sin^2 s^2 ds \right] = \lim_{j \to \infty} \left[ \sin \left( t_j - \frac{\pi}{4} \right) \int_0^{\sqrt{j}} e^{-s^2} \cos^2 s^2 ds - \cos \left( t_j - \frac{\pi}{4} \right) \int_0^{\sqrt{j}} e^{-s^2} \sin^2 s^2 ds \right] = \sqrt{A^2 + B^2} \sin \left( \frac{\pi}{2} + \frac{\pi}{4} + 2j\pi + \arccos \frac{A}{\sqrt{A^2 + B^2}} - \frac{\pi}{4} - \arccos \frac{A}{\sqrt{A^2 + B^2}} \right) = \sqrt{A^2 + B^2} \sin \left( \frac{\pi}{2} + 2j\pi \right) = \sqrt{A^2 + B^2}.
\]

Then
\[
\liminf_{t \to \infty} \int_0^t \frac{(t - \xi)^{\alpha - 1}}{V(\xi)} \left( C + \int_{I_0}^\xi F(s)V(s)ds \right) d\xi = \liminf_{t \to \infty} e^t \left\{ 2(C - e^\pi)e^{-\frac{\pi}{4}} \int_0^{\sqrt{t}} e^{-s^2} ds + 2\sqrt{2}e^{\frac{\pi}{2}} \left[ \sin \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{t}} e^{-2s^2} \cos^2 s^2 ds 
  - \cos \left( t - \frac{\pi}{4} \right) \int_0^{\sqrt{t}} e^{-2s^2} \sin^2 s^2 ds \right] \right\} = \limsup_{j \to \infty} e^j \left\{ 2(C - e^\pi)e^{-\frac{\pi}{4}} \int_0^{\sqrt{j}} e^{-s^2} ds + 2\sqrt{2}e^{\frac{\pi}{2}} \left[ \sin \left( t_j - \frac{\pi}{4} \right) \int_0^{\sqrt{j}} e^{-2s^2} \cos^2 s^2 ds 
  - \cos \left( t_j - \frac{\pi}{4} \right) \int_0^{\sqrt{j}} e^{-2s^2} \sin^2 s^2 ds \right] \right\} = \infty,
\]
\[
\limsup_{t \to \infty} \int_0^t \frac{(t - \xi)^{\alpha - 1}}{V(\xi)} \left( C + \int_{I_0}^\xi F(s)V(s)ds \right) d\xi = \infty > 0. \tag{5.5}
\]

Combine (5.4) with (5.5), by Theorem 3.1, all solutions of (5.1), (5.2) are oscillatory.

**Example 2.** Consider the oscillation of the fractional partial differential equation
\[
D_{+,t}^{\frac{1}{2}}u(x,t) - D_{+,\xi}^{\frac{1}{2}}u(x,t) = e^{-t}\Delta u(x,t) + 2t\Delta u(x,t - 1) - \left( x^2 + \frac{1}{t^2} \right) \int_0^t (t - \xi)^{-\frac{1}{2}} u(x,\xi) d\xi + e^{2t} \sin t \sin x,
\]
with the boundary condition,

\[ u(0, t) = u(\pi, t) = 0. \]  \hspace{1cm} (5.7)

**Proof.** Compare with (1.6), (1.8), where \( \alpha = \frac{1}{2}, \Omega = (0, \pi), n = 1, m = 1, p(t) = -1, a(t) = e^{-t}, a_1(t) = 2t, \tau_1 = 1, q(x,t) = x^2 + \frac{1}{t^2}, q(t) = \min q(x,t) = \frac{1}{t^2}, f(x,t) = e^{2t}\sin t\sin x. \)

So we can obtain the smallest eigenvalue \( \beta_0 \) of the Dirichlet problem above and the corresponding eigenfunction \( \phi(x) \), where \( \beta_0 = 1 \) and \( \phi(x) = \sin x. \) As a result,

\[ F_1(t) = \int_\Omega f(x,t)\phi(x)dx = \int_0^\pi e^{2t}\sin t\sin xdx = \frac{\pi}{2}e^{2t}\sin t, \]

\[ V(t) = e^{\int_0^t-1dx} = e^{0-t}. \]

We can get

\[ \int_0^\xi F_1(s)V(s)ds = \int_0^\xi \frac{\pi}{2}e^{2s}\sin se^{0-s}ds = \frac{\pi}{4}e^{0}(e^\xi \sin \xi - e^\xi \cos \xi + e^0 \cos t_0 - e^0 \sin t_0) \]

and

\[ \int_0^t \left( t - \xi \right)^{\alpha-1} \left( C + \int_0^\xi F_1(s)V(s)ds \right) d\xi \]

\[ = \int_0^t \left( t - \xi \right)^{-\frac{1}{2}}e^{\xi-t_0} \left[ C + \frac{\pi}{4}e^{0}(e^\xi \sin \xi - e^\xi \cos \xi + e^0 \cos t_0 - e^0 \sin t_0) \right] d\xi. \]

Using a similar way in example 1, we have

\[ \liminf_{t \to \infty} \int_0^t \left( t - \xi \right)^{\alpha-1} \frac{1}{V(\xi)} \left( C + \int_0^\xi F_1(s)V(s)ds \right) d\xi = -\infty < 0, \]  \hspace{1cm} (5.8)

\[ \limsup_{t \to \infty} \int_0^t \left( t - \xi \right)^{\alpha-1} \frac{1}{V(\xi)} \left( C + \int_0^\xi F_1(s)V(s)ds \right) d\xi = \infty > 0. \]  \hspace{1cm} (5.9)

Therefore, by theorem 4.1, it is easy to see that every solution of (5.6), (5.7) is oscillatory in \((0, \pi) \times R_+\). \( \Box \)

**Remark 3.** We note that our results obtained here can give sufficient conditions to guarantee the oscillatory of Eq. (5.1), (5.2) and Eq. (5.6), (5.7). So we can get the forced oscillation easily. However, the results in [13] can’t solve problem which with forced term.
REFERENCES


(Received December 13, 2016)

Huanhuan Kong
Department of Mathematics
Qufu Normal University
Qufu, 273165, Shandong
People’s Republic of China

Run Xu
Department of Mathematics
Qufu Normal University
Qufu, 273165, Shandong
People’s Republic of China
e-mail: xurun2005@163.com