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QUASI–PERIODIC SOLUTIONS OF FRACTIONAL NABLA DIFFERENCE SYSTEMS

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Abstract. In this paper, we analyse periodic properties of fractional nabla difference systems. First, we prove that a system of fractional nabla difference equations with a periodic right hand side can not possess a periodic solution. Next, we establish sufficient conditions on the existence of a unique S-asymptotically T-periodic solution for this difference system. Finally, we provide an example illustrating the obtained results.

1. Introduction

Fractional nabla calculus is a new branch of mathematics that deals with the generalization of backward (nabla) sums and differences to arbitrary order. The combined efforts of a number of researchers during the past two decades produced a fairly strong basic theory of fractional nabla difference equations [1]. But a very little progress has been reported for fractional nabla difference systems. Atici and Eloe [4] studied linear systems of fractional nabla difference equations with constant coefficients and constructed the fundamental matrix for the homogeneous system and the causal Green's function for the nonhomogeneous system. Čermák et al. [13] derived stability regions for linear fractional nabla difference systems including a precise description of their asymptotics. Recently, the author [8, 9] established sufficient conditions on existence, uniqueness and stability of solutions of nonlinear fractional nabla difference systems.

Study of periodic solutions is one of the most interesting and important research directions in qualitative theory of fractional differential / difference equations, with applications ranging from celestial mechanics to biology and finance. Tavazoei et. al. [18, 19] have shown analytically that a time invariant Caputo type fractional order system contrary to its integer order counterpart cannot generate exactly periodic signals. Kaslik et. al. [3] have also shown the nonexistence of exact periodic solutions in a wide class of fractional order dynamical systems using the Mellin transform approach. Using the final value theorem of Laplace transform, Wang et. al. [14] have shown that nonhomogeneous fractional Cauchy problem does not have nonzero periodic solution and obtained two basic existence and uniqueness results for asymptotically periodic solution of semi linear fractional Cauchy problem in an asymptotically periodic functions

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space. Diblik et. al. [2] have shown that a fractional forward (delta) difference equation with a periodic right hand side cannot possess a periodic solution but it can have an S-asymptotically periodic solution and proved sufficient conditions for the existence of a unique S-asymptotically periodic solution. Recently, the author [10] has proved that any given fractional nabla difference system doesn't possess a nonconstant periodic solution and established sufficient conditions on the existence and uniqueness of S-asymptotically periodic solution for this difference system. Motivated by [2, 14], we present here the analogous results on quasi-periodic solutions in the field of fractional nabla calculus.

The present paper is organized as follows. Sections 2 contains preliminaries on fractional nabla calculus. In section 3, we obtain a few properties of N-transform. We discuss quasi-periodic properties of fractional nabla sums and differences in section 4. In section 5, we establish sufficient conditions on the existence and uniqueness of S-asymptotically periodic solutions of nonlinear fractional nabla difference systems.

2. Preliminaries

Throughout, we shall use the following notations, definitions and known results of fractional nabla calculus [1]. We denote the set of all integers, real numbers and complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. For $a \in \mathbb{R}$, define $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$. Assume that empty sums and products are taken to be 0 and 1, respectively.

DEFINITION 2.1. (Gamma Function) For any $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the gamma function is defined by

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} ds, \quad t > 0,$$
$$\Gamma(t+1) = t\Gamma(t).$$

DEFINITION 2.2. (Rising Factorial Function) For any $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$ and $\alpha \in \mathbb{R}$ such that $(t+\alpha) \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

We observe the following properties of rising factorial functions.

THEOREM 2.1. *Assume that the following factorial functions are well defined.*

1. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
2. *If* $t \leqslant r$ *then* $t^{\overline{\alpha}} \leqslant r^{\overline{\alpha}}$.
3. *If* $\alpha < t \leqslant r$ *then* $r^{\overline{-\alpha}} \leqslant t^{\overline{-\alpha}}$.
4. $(t+1)^{\alpha-1} \leqslant (t+1)^{\overline{\alpha-1}} \leqslant t^{\alpha-1}, \quad 0 \leqslant \alpha \leqslant 1$.

5. $(t+b)^{\overline{a-b}} = t^{a-b}\left[1+O\left(\frac{1}{t}\right)\right], \quad |t| \to \infty.$

DEFINITION 2.3. Let $u:\mathbb{N}_a \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The α^{th}-order nabla sum of u is given by

$$\left(\nabla_a^{-\alpha}u\right)(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t}(t-s+1)^{\overline{\alpha-1}}u(s), \quad t \in \mathbb{N}_a.$$

DEFINITION 2.4. Let $u:\mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$.

1. (Nabla Difference) The first order backward (nabla) difference of u is defined by

$$\left(\nabla_a u\right)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\left(\nabla_a^N u\right)(t) = \left(\nabla_a\left(\nabla_a^{N-1}u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

2. (R-L Fractional Nabla Difference) The Riemann-Liouville type α^{th}-order nabla difference of u is given by

$$\begin{aligned}\left(\nabla_a^{\alpha}u\right)(t) &= \left(\nabla_a^N\left(\nabla_a^{-(N-\alpha)}u\right)\right)(t)\\ &= \frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{t}(t-s+1)^{\overline{-\alpha-1}}u(s), \quad t \in \mathbb{N}_{a+N}.\end{aligned}$$

3. (Caputo Fractional Nabla Difference) The Caputo type α^{th}-order nabla difference of u is given by

$$\begin{aligned}\left(\nabla_{a*}^{\alpha}u\right)(t) &= \left(\nabla_a^{-(N-\alpha)}\left(\nabla_a^N u\right)\right)(t)\\ &= \left(\nabla_a^{\alpha}u\right)(t) - \sum_{k=0}^{N-1}\frac{(t-a+1)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)}\left(\nabla_a^k u\right)(a-1), \quad t \in \mathbb{N}_{a+N}.\end{aligned}$$

THEOREM 2.2. (Power Rule) *Let α, $\mu \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$. Assume that the following factorial functions are well defined.*

1. $\nabla_a^N(t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-N+1)}(t-a+1)^{\overline{\mu-N}}, \quad t \in \mathbb{N}_{a+N}.$

2. $\nabla_a^{-\alpha}(t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a+1)^{\overline{\mu+\alpha}}, \quad t \in \mathbb{N}_a.$

3. $\nabla_a^{\alpha}(t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a+1)^{\overline{\mu-\alpha}}, \quad t \in \mathbb{N}_{a+N}.$

4. $\nabla_{a*}^{\alpha}(t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a+1)^{\overline{\mu-\alpha}}, \quad t \in \mathbb{N}_{a+N}.$

DEFINITION 2.5. Let $u : \mathbb{N}_a \to \mathbb{R}$. u is said to be T-periodic if there exists a least $T \in \mathbb{N}_1$ such that

$$u(t+T) = u(t), \quad t \in \mathbb{N}_a.$$

DEFINITION 2.6. Let $u : \mathbb{N}_a \to \mathbb{R}$. u is said to be S-asymptotically T-periodic if it is bounded and there exists $T \in \mathbb{N}_1$ such that

$$\lim_{t\to\infty} \big[u(t+T) - u(t)\big] = 0.$$

3. N-transform

In this section we recall the definition of nabla Laplace transform (N-transform) [4, 11] and obtain a few of its properties.

DEFINITION 3.1. [4] Let $u : \mathbb{N}_a \to \mathbb{R}$. The N-transform of u is defined by

$$N_a\big[u(t)\big] = \sum_{j=a}^{\infty} u(j)(1-z)^{j-1} = U(z),$$

for each $z \in \mathbb{C}$ for which the series converges.

DEFINITION 3.2. [11] Let $u : \mathbb{N}_a \to \mathbb{R}$. u is said to be of exponential order r, $r > 0$, if there exists a constant $A > 0$ such that

$$\big|u(t)\big| \leqslant Ar^{-t}, \tag{3.1}$$

for sufficiently large $t \in \mathbb{N}_a$.

THEOREM 3.1. [11] *Suppose u is of exponential order r, $r > 0$. Then, $U(z)$ exists for each z lies inside the open ball $B_1(r) = \{z \in \mathbb{C} : |1-z| < r\}$.*

REMARK 1. We note that every periodic sequence of real numbers is bounded and every bounded sequence of real numbers is of exponential order 1. Consequently, if $u : \mathbb{N}_a \to \mathbb{R}$ is T-periodic then $U(z)$ exists.

THEOREM 3.2. [11] *We observe the following properties of N-transform.*

1. *Let $\alpha \in \mathbb{R} \setminus \{\cdots, -3, -2, -1\}$ and $t \in \mathbb{N}_a$. The rising factorial function $(t-a+1)^{\overline{\alpha}}$ is of exponential order 1 and*

$$N_a\big[(t-a+1)^{\overline{\alpha}}\big] = (1-z)^{a-1}\frac{\Gamma(\alpha+1)}{z^{\alpha+1}}$$

for each $z \in B_1(1)$.

2. *Let u be of exponential order r, $0 < r \leqslant 1$ and $\alpha \in \mathbb{R}^+$. Then, $\left(\nabla_a^{-\alpha}u\right)$ is also of exponential order r, $0 < r \leqslant 1$ and*

$$N_a\left[\left(\nabla_a^{-\alpha}u\right)(t)\right] = z^{-\alpha}U(z)$$

for each $z \in B_1(r)$.

THEOREM 3.3. *Let u be of exponential order r, $r > 0$ and $T \in \mathbb{N}_1$. Then,*

1. $N_a\left[u(t-T)\right] = \sum_{j=a}^{a+T-1} u(j-T)(1-z)^{j-1} + (1-z)^T N_a\left[u(t)\right]$.

2. $N_a\left[u(t+T)\right] = (1-z)^{-T}\left[N_a\left[u(t)\right] - \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}\right]$.

Proof. (1) Consider

$$\begin{aligned} N_a\left[u(t-T)\right] &= \sum_{j=a}^{\infty} u(j-T)(1-z)^{j-1} \\ &= \sum_{j=a-T}^{\infty} u(j)(1-z)^{j+T-1} \\ &= \sum_{j=a-T}^{a-1} u(j)(1-z)^{j+T-1} + \sum_{j=a}^{\infty} u(j)(1-z)^{j+T-1} \\ &= \sum_{j=a}^{a+T-1} u(j-T)(1-z)^{j-1} + (1-z)^T N_a\left[u(t)\right]. \end{aligned}$$

(2) Consider

$$\begin{aligned} N_a\left[u(t+T)\right] &= \sum_{j=a}^{\infty} u(j+T)(1-z)^{j-1} \\ &= \sum_{j=a+T}^{\infty} u(j)(1-z)^{j-T-1} \\ &= \sum_{j=a}^{\infty} u(j)(1-z)^{j-T-1} - \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-T-1} \\ &= (1-z)^{-T}\left[N_a\left[u(t)\right] - \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}\right]. \quad \square \end{aligned}$$

THEOREM 3.4. *Let u be of exponential order r, $r > 0$ and T-periodic. Then,*

$$N_a\left[u(t)\right] = \frac{1}{\left[1-(1-z)^T\right]} \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}, \quad |1-z| < 1.$$

Proof. Using Theorem 3.3 (2), we have

$$N_a\left[u(t)\right] = (1-z)^{-T}\left[N_a\left[u(t)\right] - \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}\right]$$

implies

$$\left[(1-z)^{-T} - 1\right]N_a\left[u(t)\right] = (1-z)^{-T}\sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}.$$

Consequently, we get

$$N_a\left[u(t)\right] = \frac{1}{\left[1-(1-z)^T\right]}\sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}.$$

Hence the proof. □

THEOREM 3.5. *Let u be of exponential order r, $r > 0$ and $u(\infty)$ exists. Then,*

$$\lim_{z\to 0^+} zU(z) = \lim_{t\to\infty} u(t).$$

Proof. Consider

$$\begin{aligned} N_a\left[\left(\nabla_a u\right)(t)\right] &= \sum_{j=a}^{\infty}\left[\left(\nabla_a u\right)(j)\right](1-z)^{j-1} \\ &= \sum_{j=a}^{\infty}\left[u(j)-u(j-1)\right](1-z)^{j-1} \\ &= \lim_{t\to\infty}\sum_{j=a}^{t}\left[u(j)-u(j-1)\right](1-z)^{j-1}. \end{aligned} \tag{3.2}$$

On the other hand, using Theorem 3.3, we have

$$N_a\left[\left(\nabla_a u\right)(t)\right] = N_a\left[u(t)-u(t-1)\right] = zU(z) - u(a-1)(1-z)^{a-1}. \tag{3.3}$$

Letting $z \to 0^+$ in (3.2), we get

$$\lim_{z\to 0^+} N_a\left[\left(\nabla_a u\right)(t)\right] = \lim_{t\to\infty}\sum_{j=a}^{t}\left[u(j)-u(j-1)\right] = \lim_{t\to\infty}\left[u(t)-u(a-1)\right]. \tag{3.4}$$

Using (3.3) in (3.4), the proof is complete. □

THEOREM 3.6. *Let u be of exponential order r, $r > 0$ and T-periodic. Then,*

$$\lim_{t\to\infty} u(t) = \frac{1}{T}\sum_{j=a}^{a+T-1} u(j).$$

Proof. Clearly, from Remark 1, $u(\infty)$ exists. Using Theorems 3.4, 3.5 and L'Hôspital's rule, we have

$$\begin{aligned}\lim_{t\to\infty} u(t) = \lim_{z\to 0^+} zU(z) &= \lim_{z\to 0^+} \frac{z}{\left[1-(1-z)^T\right]} \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1} \\ &= \lim_{z\to 0^+} \frac{z}{\left[1-(1-z)^T\right]} \lim_{z\to 0^+} \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1} \\ &= \frac{1}{T} \sum_{j=a}^{a+T-1} u(j). \quad \square\end{aligned}$$

THEOREM 3.7. *Let u be of exponential order r, $r>0$ and T-periodic. If $\left(\nabla_a^{-\alpha} u\right)$ is bounded for $0<\alpha<1$ then*

$$\sum_{j=a}^{a+T-1} u(j) = 0. \tag{3.5}$$

Proof. Using Theorem 3.4 and L'Hôspital's rule, we have

$$\lim_{z\to 0^+} U(z) = \lim_{z\to 0^+} \sum_{j=a}^{a+T-1} u(j) \frac{(1-z)^{j-1}}{\left[1-(1-z)^T\right]} = -\frac{1}{T} \sum_{j=a}^{a+T-1} (j-1)u(j) < \infty.$$

Using Theorems 3.2, 3.5, 3.6 and L'Hôspital's rule, we have

$$\begin{aligned}\lim_{t\to\infty} \left(\nabla_a^{-\alpha} u\right)(t) = \lim_{z\to 0^+} zN_a\left[\left(\nabla_a^{-\alpha} u\right)(t)\right] &= \lim_{z\to 0^+} z^{1-\alpha} U(z) \\ &= \lim_{z\to 0^+} \frac{zU(z)}{z^\alpha} \\ &= \lim_{z\to 0^+} \frac{\frac{1}{T}\sum_{j=a}^{a+T-1} u(j)}{z^\alpha}.\end{aligned} \tag{3.6}$$

Hence it is clear from (3.6) that if $\left(\nabla_a^{-\alpha} u\right)$ is bounded, then (3.5) holds. $\square$

4. Periodic properties of fractional nabla differences

In [10], the author has proved that the fractional sum of a nonconstant periodic function is not periodic. However, since in this section we study quasi-periodic properties of fractional nabla sums and differences, we prove this fact in a similar way as in the cited case.

THEOREM 4.1. *Let u be of exponential order r, $0<r\leqslant 1$ and nonconstant T-periodic. Let $\alpha\in\mathbb{R}$ and choose $N\in\mathbb{N}_1$ such that $N-1<\alpha<N$. Then,*

1. *$\left(\nabla_a^N u\right)$ is also nonconstant T-periodic.*

2. $\left(\nabla_a^{-\alpha}u\right)$ *cannot be* T*-periodic.*
3. $\left(\nabla_a^{\alpha}u\right)$ *cannot be* T*-periodic.*
4. $\left(\nabla_{a*}^{\alpha}u\right)$ *cannot be* T*-periodic.*

Proof. The proof of (1) is trivial. Now, we prove (2). Take $0<\alpha<1$. Suppose there exists a nonconstant T-periodic function u such that $\left(\nabla_a^{-\alpha}u\right)$ is also T-periodic. Let

$$\left(\nabla_a^{-\alpha}u\right)(t)=v(t), \quad t\in\mathbb{N}_a.$$

Applying N-transform on both sides, we get

$$z^{-\alpha}N_a\big[u(t)\big]=N_a\big[v(t)\big].$$

Using Theorem 3.2, v is also of exponential order r, $0<r\leqslant 1$. Since u and v are T-periodic, using Theorem 3.4, we have

$$\frac{1}{\left[1-(1-z)^T\right]}\sum_{j=a}^{a+T-1}u(j)(1-z)^{j-1}=\frac{z^\alpha}{\left[1-(1-z)^T\right]}\sum_{j=a}^{a+T-1}v(j)(1-z)^{j-1}.$$

So,

$$\sum_{j=a}^{a+T-1}u(j)(1-z)^{j-1}=z^\alpha\sum_{j=a}^{a+T-1}v(j)(1-z)^{j-1}.$$

Consequently,

$$\sum_{j=0}^{T-1}u(j+a)(1-z)^j=z^\alpha\sum_{j=0}^{T-1}v(j+a)(1-z)^j. \tag{4.1}$$

Putting

$$P(z)=\sum_{j=0}^{T-1}u(j+a)(1-z)^j \text{ and } Q(z)=\sum_{j=0}^{T-1}v(j+a)(1-z)^j$$

in (4.1), we have

$$P(z)=z^\alpha Q(z). \tag{4.2}$$

Since $P(z)$ and $Q(z)$ are polynomials of degree $T-1$ over $\mathbb{C}$, we write

$$P(z)=\sum_{k=0}^{T-1}p_kz^k \text{ and } Q(z)=\sum_{k=0}^{T-1}q_kz^k.$$

Then, (4.2) gives

$$\sum_{k=0}^{T-1}p_kz^k=z^\alpha\sum_{k=0}^{T-1}q_kz^k, \quad |1-z|<1. \tag{4.3}$$

Letting $z\to 0^+$, we get $p_0=0$. Then, (4.3) implies

$$\sum_{k=1}^{T-1}p_kz^{k-\alpha}=\sum_{k=0}^{T-1}q_kz^k, \quad |1-z|<1. \tag{4.4}$$

Letting $z \to 0^+$, we get $q_0 = 0$. Following this argument step by step, we derive

$$p_k = q_k = 0, \quad k = 0, 1, 2, \cdots, (T-1).$$

Hence

$$P(z) = Q(z) = 0, \quad |1-z| < 1,$$

or equivalently,

$$\sum_{j=0}^{T-1} u(j+a)(1-z)^j = \sum_{j=0}^{T-1} v(j+a)(1-z)^j = 0.$$

Then, we have

$$u(j+a) = v(j+a) = 0, \quad j = 0, 1, 2, \cdots, (T-1).$$

This is a contradiction. Hence, our assumption is false, and therefore, $\left(\nabla_a^{-\alpha} u\right)$ cannot be T-periodic. The proof for an arbitrary $\alpha > 0$ follows easily.

Replacing α by $-\alpha$ in (2), we get (3).

From Definition 2.4, the Caputo type α^{th}-order nabla difference of u is given by

$$\left(\nabla_{a*}^{\alpha} u\right)(t) = \left(\nabla_a^{-(N-\alpha)}\left(\nabla_a^N u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Given that u is a nonconstant T-periodic function. Then, from (1), $\left(\nabla_a^N u\right)$ is also a nonconstant T-periodic function and hence from (2), $\left(\nabla_a^{-(N-\alpha)}\left(\nabla_a^N u\right)\right)$ cannot be T-periodic. Thus, we have (4). $\square$

THEOREM 4.2. *Let $0 < \alpha < 1$, u be of exponential order r, $0 < r \leqslant 1$ and T-periodic. Assume that $\left(\nabla_a^{-\alpha} u\right)$ is bounded. Then,*

1. *$\left(\nabla_a u\right)$ is S-asymptotically T-periodic.*
2. *$\left(\nabla_a^{-\alpha} u\right)$ is S-asymptotically T-periodic.*
3. *$\left(\nabla_a^{\alpha} u\right)$ is S-asymptotically T-periodic.*
4. *$\left(\nabla_{a*}^{\alpha} u\right)$ is S-asymptotically T-periodic.*

Proof. The proof of (1) is trivial. Now, we prove (2). Consider

$$\begin{aligned}
\lim_{t\to\infty}\left[\left(\nabla_a^{-\alpha} u\right)(t+T) - \left(\nabla_a^{-\alpha} u\right)(t)\right] &= \frac{1}{\Gamma(\alpha)} \lim_{t\to\infty}\left[\sum_{s=a}^{a+T-1} (t+T-s+1)^{\overline{\alpha-1}} u(s)\right] \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{a+T-1} \lim_{t\to\infty}\left[(t+T-s+1)^{\overline{\alpha-1}}\right] u(s) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{a+T-1} \lim_{t\to\infty}\left[t^{\alpha-1} + O\left(\frac{1}{t}\right)\right] u(s) \\
&= 0,
\end{aligned}$$

implies $\left(\nabla_a^{-\alpha} u\right)$ is S-asymptotically T-periodic. Thus, we have (2). Proofs of (3) and (4) are similar to the proofs of (3) and (4) of Theorem 4.1. So, we omit it. $\square$

THEOREM 4.3. *Let* $0 < \alpha < 1$*,* u *be of exponential order* r*,* $0 < r \leqslant 1$ *and* S*-asymptotically* T*-periodic. Assume that* $\left(\nabla_a^{-\alpha} u\right)$ *is bounded. Then,*

1. $\left(\nabla_a u\right)$ *is* S*-asymptotically* T*-periodic.*
2. $\left(\nabla_a^{-\alpha} u\right)$ *is* S*-asymptotically* T*-periodic.*
3. $\left(\nabla_a^{\alpha} u\right)$ *is* S*-asymptotically* T*-periodic.*
4. $\left(\nabla_{a*}^{\alpha} u\right)$ *is* S*-asymptotically* T*-periodic.*

Proof. Let $v(t) = u(t+T) - u(t)$. Since u is S-asymptotically T-periodic,

$$\lim_{t\to\infty} \left[u(t+T) - u(t)\right] = 0 \Rightarrow \lim_{t\to\infty} v(t) = 0.$$

Consequently, from Theorem 3.5, we have

$$\lim_{z\to 0^+} zV(z) = 0,$$

where $N_a\left[v(t)\right] = V(z)$. We know that $\left(\nabla_a u\right)$ is bounded and

$$\lim_{t\to\infty} \left[\left(\nabla_a u\right)(t+T) - \left(\nabla_a u\right)(t)\right] = \lim_{t\to\infty} \left(\nabla_a v\right)(t) = \lim_{t\to\infty} \left[v(t) - v(t-1)\right] = 0,$$

implies $\left(\nabla_a u\right)$ is S-asymptotically T-periodic. Thus, we have (1). Next, we prove (2). Consider

$$\begin{aligned} V(z) = N_a\left[v(t)\right] &= \sum_{j=a}^{\infty} v(j)(1-z)^{j-1} \\ &= \sum_{j=a}^{\infty} u(j+T)(1-z)^{j-1} - \sum_{j=a}^{\infty} u(j)(1-z)^{j-1} \\ &= (1-z)^{-T} \sum_{j=a+T}^{\infty} u(j)(1-z)^{j-1} - \sum_{j=a}^{\infty} u(j)(1-z)^{j-1} \\ &= \left[(1-z)^{-T} - 1\right]U(z) - (1-z)^{-T} \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1}. \end{aligned}$$

Clearly, $u(\infty)$ exists. From Theorems 3.5–3.7 and L'Hôspital's rule, we have

$$\begin{aligned} \lim_{z\to 0^+} V(z) &= \lim_{z\to 0^+} \frac{\left[1-(1-z)^T\right]}{z(1-z)^T} \lim_{z\to 0^+} zU(z) - \lim_{z\to 0^+} (1-z)^{-T} \sum_{j=a}^{a+T-1} u(j)(1-z)^{j-1} \\ &= \lim_{z\to 0^+} \frac{T(1-z)^{T-1}}{Tz(1-z)^{T-1} + (1-z)^T} \lim_{t\to\infty} u(t) - \sum_{j=a}^{a+T-1} u(j) = 0. \end{aligned} \tag{4.5}$$

Using (4.5) and Theorems 2.1, 3.2 and 3.5, we have

$$\begin{aligned}
&\lim_{t\to\infty}\left[\left(\nabla_a^{-\alpha}u\right)(t+T)-\left(\nabla_a^{-\alpha}u\right)(t)\right]\\
&=\frac{1}{\Gamma(\alpha)}\lim_{t\to\infty}\left[\sum_{s=a}^{t+T}(t+T-s+1)^{\overline{\alpha-1}}u(s)-\sum_{s=a}^{t}(t-s+1)^{\overline{\alpha-1}}u(s)\right]\\
&=\frac{1}{\Gamma(\alpha)}\lim_{t\to\infty}\left[\sum_{s=a}^{t}(t-s+1)^{\overline{\alpha-1}}v(s)+\sum_{s=a}^{a+T-1}(t+T-s+1)^{\overline{\alpha-1}}u(s)\right]\\
&=\lim_{t\to\infty}\left(\nabla_a^{-\alpha}v\right)(t)+\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}\left[\lim_{t\to\infty}(t+T-s+1)^{\overline{\alpha-1}}\right]u(s)\\
&=\lim_{z\to0^+}zN_a\left[\left(\nabla_a^{-\alpha}v\right)(t)\right]+\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{a+T-1}\lim_{t\to\infty}\left[t^{\alpha-1}+O\left(\frac{1}{t}\right)\right]u(s)\\
&=\lim_{z\to0^+}z^{1-\alpha}V(z)\\
&=\lim_{z\to0^+}z^{1-\alpha}\lim_{z\to0^+}V(z)=0,
\end{aligned}$$

implies $\left(\nabla_a^{-\alpha}u\right)$ is S-asymptotically T-periodic. Thus, we have (2). Proofs of (3) and (4) are similar to the proofs of (3) and (4) of Theorem 4.1. So, we omit it. □

5. Periodic solutions of fractional nabla difference systems

In this section, we establish sufficient conditions on the existence and uniqueness of S-asymptotically T-periodic solutions of the following initial value problems using fixed point theory.

$$\begin{aligned}\left(\nabla_0^{\alpha}\mathbf{u}\right)(t)&=\mathbf{f}(t,\mathbf{u}(t)),\quad t\in\mathbb{N}_1,\\ \mathbf{u}(0)&=\mathbf{c},\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}\left(\nabla_{0*}^{\alpha}\mathbf{u}\right)(t)&=\mathbf{f}(t,\mathbf{u}(t)),\quad t\in\mathbb{N}_1,\\ \mathbf{u}(0)&=\mathbf{c},\end{aligned}\tag{5.2}$$

where $\alpha\in\mathbb{R}$ such that $0<\alpha<1$, $\mathbf{u}:\mathbb{N}_0\to\mathbb{R}^n$ and $\mathbf{f}:\mathbb{N}_0\times\mathbb{R}^n\to\mathbb{R}^n$.

The main result (Theorem 4.1) presented in the previous section leads us to the following consequence.

THEOREM 5.1. *Assume that the function* $\mathbf{f}$ *is* T*-periodic with respect to its first argument. Then there are no nonconstant* T*-periodic solutions of (5.1) and (5.2).*

Proof. Assume that there exists a nonconstant T-periodic solution $\mathbf{u}$ of (5.1) (or (5.2)). From the T-periodicity of the function $\mathbf{f}$ with respect to its first argument, it follows that $\left(\nabla_0^{\alpha}\mathbf{u}\right)$ is also T-periodic, which contradicts Theorem 4.1. □

We know that, $\mathbf{u}$ is a solution of (5.1) if and only if

$$\mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{5.3}$$

Similarly, $\mathbf{u}$ is a solution of (5.2) if and only if

$$\mathbf{u}(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{5.4}$$

Define the operators

$$\big(F\mathbf{u}\big)(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0, \tag{5.5}$$

$$\big(F'\mathbf{u}\big)(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{5.6}$$

It is evident from (5.3)–(5.6) that $\mathbf{u}$ is a fixed point of F if and only if $\mathbf{u}$ is a solution of (5.1) and $\mathbf{u}$ is a fixed point of F' if and only if $\mathbf{u}$ is a solution of (5.2).

DEFINITION 5.1. Let X be a locally convex space of functions $\mathbf{u} : \mathbb{N}_0 \to \mathbb{R}^n$ with the topology of pointwise convergence on finite subsets. Let $p \in [0,\infty)$ and $\phi : \mathbb{N}_0 \to [0,\infty)$ such that

$$\lim_{t\to\infty}\phi(t) = 0.$$

Define

$$B_{p,\phi} = \big\{\mathbf{u} \in X : \|\mathbf{u}(t)\| \leqslant p,\ \|\mathbf{u}(t+T) - \mathbf{u}(t)\| \leqslant \phi(t),\ t \in \mathbb{N}_0\big\}.$$

Clearly, $B_{p,\phi}$ is a nonempty convex subset of X. Since $B_{p,\phi}$ is a countable product of compact sets, by Tychonoff product theorem [17], it is also compact.

Now we make the following assumptions to establish main results of this section:

(H1) $\mathbf{f}$ is continuous on $\mathbb{N}_0 \times \mathbb{R}^n$.

(H2) There exists a function $a : \mathbb{N}_0 \to [0,\infty)$ such that, for all $(t,\mathbf{u}(t))$, $(t,\mathbf{v}(t)) \in \mathbb{N}_0 \times \mathbb{R}^n$,

$$\|\mathbf{f}(t,\mathbf{u}(t)) - \mathbf{f}(t,\mathbf{v}(t))\| \leqslant a(t)\|\mathbf{u}(t) - \mathbf{v}(t)\|, \quad t \in \mathbb{N}_0,$$

and

$$\sup_{t\in\mathbb{N}_0}\big(\nabla_1^{-\alpha}a\big)(t) = L < 1.$$

(H3) Assume that

$$\sup_{t\in\mathbb{N}_0}\big\|\nabla_1^{-\alpha}\mathbf{f}(s,\mathbf{0})\big\| = M < \infty.$$

(H4) There exists a function $b : \mathbb{N}_0 \to [0,\infty)$ such that, for all $(t,\mathbf{u}(t)) \in \mathbb{N}_0 \times \mathbb{R}^n$,

$$\|\mathbf{f}(t+T,\mathbf{u}(t)) - \mathbf{f}(t,\mathbf{u}(t))\| \leqslant b(t)\|\mathbf{u}(t)\|, \quad t \in \mathbb{N}_0,$$

and

$$\lim_{t\to\infty}\left(\nabla_1^{-\alpha}b\right)(t)=0.$$

(H5) Let

$$\max_{t\in\{1,2,\cdots,T\}}\|\mathbf{f}(t,\mathbf{0})\|=q \text{ and } \max_{t\in\{1,2,\cdots,T\}}a(t)=r.$$

THEOREM 5.2. *Let (H1)–(H5) hold. Then there exists a unique S-asymptotically T-periodic solution of (5.1) (or (5.2)) in $B_{p,\phi}$ where*

$$p=\frac{\left(\|\mathbf{c}\|+M\right)}{(1-L)} \tag{5.7}$$

and

$$\phi(t)=\frac{1}{(1-L)}\left[p\left(\nabla_1^{-\alpha}b\right)(t)+\frac{T(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(pr+q)\right],\quad t\in\mathbb{N}_0. \tag{5.8}$$

Proof. We use Tychonoff fixed point theorem [17] to establish the existence of S-asymptotically T-periodic solutions of (5.1) and (5.2) in $B_{p,\phi}$. Since $\mathbf{f}$ is continuous, F and F' are also continuous. Next we show that F' maps $B_{p,\phi}$ into $B_{p,\phi}$. Let $\mathbf{u}\in B_{p,\phi}$. Consider

$$\begin{aligned}\left\|\left(F'\mathbf{u}\right)(t)\right\| &\leqslant \|\mathbf{c}\|+\frac{1}{\Gamma(\alpha)}\left\|\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))\right\|\\
&\leqslant \|\mathbf{c}\|+\left\|\nabla_1^{-\alpha}\mathbf{f}(t,\mathbf{0})\right\|+\frac{1}{\Gamma(\alpha)}\left\|\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\left[\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{0})\right]\right\|\\
&\leqslant \|\mathbf{c}\|+M+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\left\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{0})\right\|\\
&\leqslant \|\mathbf{c}\|+M+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}a(s)\|\mathbf{u}(s)\|\\
&\leqslant \|\mathbf{c}\|+M+p\left(\nabla_a^{-\alpha}a\right)(t)\\
&\leqslant \|\mathbf{c}\|+M+Lp\\
&= p.\end{aligned}$$

Now, consider

$$\begin{aligned}&\left\|\left(F'\mathbf{u}\right)(t+T)-\left(F'\mathbf{u}\right)(t)\right\|\\
&=\frac{1}{\Gamma(\alpha)}\left\|\sum_{s=1}^{t+T}(t+T-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))-\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))\right\|\\
&=\frac{1}{\Gamma(\alpha)}\left\|\sum_{s=1-T}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s+T,\mathbf{u}(s+T))-\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))\right\|\\
&\leqslant\frac{1}{\Gamma(\alpha)}\left\|\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\left[\mathbf{f}(s+T,\mathbf{u}(s+T))-\mathbf{f}(s,\mathbf{u}(s+T))\right]\right\|\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\Big\|\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\big[\mathbf{f}(s,\mathbf{u}(s+T))-\mathbf{f}(s,\mathbf{u}(s))\big]\Big\| \\
&+\frac{1}{\Gamma(\alpha)}\Big\|\sum_{s=1}^{T}(t+T-s+1)^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s))\Big\| \\
&= S_1+S_2+S_3. \qquad (5.9)
\end{aligned}
$$

We have

$$
S_1 \leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}b(s)\|\mathbf{u}(s+T)\| \leqslant p\big(\nabla_1^{-\alpha}b\big)(t), \qquad (5.10)
$$

$$
\begin{aligned}
S_2 &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}a(s)\|\mathbf{u}(s+T)-\mathbf{u}(s)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}a(s)\phi(s) \\
&\leqslant \frac{1}{(1-L)}\Big[p\big(\nabla_1^{-\alpha}b\big)(t)+\frac{T(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(pr+q)\Big]\big(\nabla_1^{-\alpha}a\big)(t) \\
&\leqslant \frac{L}{(1-L)}\Big[p\big(\nabla_1^{-\alpha}b\big)(t)+\frac{T(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(pr+q)\Big] \qquad (5.11)
\end{aligned}
$$

and

$$
\begin{aligned}
S_3 &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{T}(t+T-s+1)^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))\| \\
&\leqslant \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\sum_{s=1}^{T}\|\mathbf{f}(s,\mathbf{u}(s))\| \\
&\leqslant \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\sum_{s=1}^{T}\big\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{0})\big\|+\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\sum_{s=1}^{T}\big\|\mathbf{f}(s,\mathbf{0})\big\| \\
&\leqslant \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\sum_{s=1}^{T}a(s)\|\mathbf{u}(s)\|+\frac{Tq(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \\
&\leqslant \frac{T(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(pr+q). \qquad (5.12)
\end{aligned}
$$

Using (5.10)–(5.12) in (5.9), we get

$$
\big\|\big(F'\mathbf{u}\big)(t+T)-\big(F'\mathbf{u}\big)(t)\big\| \leqslant \frac{1}{(1-L)}\Big[p\big(\nabla_1^{-\alpha}b\big)(t)+\frac{T(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(pr+q)\Big]. \qquad (5.13)
$$

Thus, $F'\mathbf{u}\in B_{p,\phi}$ and hence $F':B_{p,\phi}\to B_{p,\phi}$. Similarly, we prove that $F:B_{p,\phi}\to B_{p,\phi}$. Clearly, $\phi(t)\to 0$ as $t\to\infty$. The existence follows from Tychonoff fixed point theorem.

We use mathematical induction to prove uniqueness. Let $\mathbf{u}$, $\mathbf{v}\in B_{p,\phi}$ be two solutions of (5.2). Consider $w(t)=\|\mathbf{u}(t)-\mathbf{v}(t)\|$. Clearly, $w(0)=0$. Assume that

$w(s)=0$ for $s=1,2,\cdots,(t-1)$. Then,

$$\begin{aligned} w(t) &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\big\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{v}(s))\big\| \\ &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}a(s)\|\mathbf{u}(s)-\mathbf{v}(s)\| \\ &= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}a(s)w(s)=a(t)w(t), \end{aligned}$$

implies $w(t)=0$. Hence, by principle of mathematical induction, $w(t)=0$ for all $t\in\mathbb{N}_0$. This completes the proof. □

The following example demonstrates the applicability of Theorem 5.2.

EXAMPLE 1. Consider the scalar initial value problem

$$\begin{gathered} \left(\nabla_0^{0.5}u\right)(t)=\frac{1}{(t+1)^{0.6}}\left[(0.1)\sin u+(0.2)u\sin\left(\frac{\pi t}{2}\right)\right], \quad t\in\mathbb{N}_1, \\ u(0)=1. \end{gathered} \tag{5.14}$$

SOLUTION. We have
(H1) $f(t,u)=(t+1)^{-0.6}\left[(0.1)\sin u+(0.2)u\sin\left(\frac{\pi t}{2}\right)\right]$ is continuous on $\mathbb{N}_0\times\mathbb{R}$.
(H2) For any (t,u), $(t,v)\in\mathbb{N}_0\times\mathbb{R}$, using Theorem 2.1,

$$\begin{aligned} |f(t,u)-f(t,v)| &= (t+1)^{-0.6}\left[(0.1)|\sin u-\sin v|+(0.2)|u-v|\left|\sin\left(\frac{\pi t}{2}\right)\right|\right] \\ &\leqslant (0.3)(t+1)^{\overline{-0.6}}|u-v|. \end{aligned}$$

Here $a(t)=(0.3)(t+1)^{\overline{-0.6}}$ and

$$\begin{aligned} L &= \sup_{t\in\mathbb{N}_0}\left[(0.3)\nabla_1^{-0.5}(t+1)^{\overline{-0.6}}\right] \\ &= \sup_{t\in\mathbb{N}_0}\left[\frac{(0.3)\Gamma(0.4)}{\Gamma(0.9)}(t+1)^{\overline{-0.1}}\right] \\ &= \frac{(0.3)\Gamma(0.4)}{\Gamma(0.9)}=0.6227<1. \end{aligned}$$

(H3) Here $f(t,0)=0$. So, $M=\nabla_1^{-0.5}f(t,0)=0$.

(H4) Take $T=4$. For any $(t,u)\in\mathbb{N}_0\times\mathbb{R}$, using Theorem 2.1,

$$\begin{aligned}
&|f(t+T,u)-f(t,u)|\\
&\leqslant (0.1)|\sin u|\left|(t+5)^{-0.6}-(t+1)^{-0.6}\right|\\
&\quad +(0.2)|u|\left|(t+5)^{-0.6}\sin\left(\frac{\pi(t+4)}{2}\right)-(t+1)^{-0.6}\sin\left(\frac{\pi t}{2}\right)\right|\\
&\leqslant \left[(0.1)|u|+(0.2)|u|\left|\sin\left(\frac{\pi t}{2}\right)\right|\right]\left|(t+5)^{-0.6}-(t+1)^{-0.6}\right|\\
&\leqslant (0.3)(t+1)^{-0.6}\left|\left(\frac{t+1}{t+5}\right)^{0.6}-1\right||u|\\
&\leqslant (0.6)(t+1)^{\overline{-0.6}}|u|.
\end{aligned}$$

Here $b(t)=(0.6)(t+1)^{\overline{-0.6}}\geqslant 0$ and

$$\begin{aligned}
\left(\nabla_0^{-0.5}b\right)(t) &= (0.6)\nabla_0^{-0.5}(t+1)^{\overline{-0.6}}\\
&= \frac{(0.6)\Gamma(0.4)}{\Gamma(0.9)}(t+1)^{\overline{-0.1}}\\
&= \frac{(0.3)\Gamma(0.4)}{\Gamma(0.9)}(t+1)^{\overline{-0.1}}=(1.2454)(t+1)^{\overline{-0.1}}.
\end{aligned}$$

Consequently,

$$\lim_{t\to\infty}\left(\nabla_1^{-0.5}b\right)(t)=0.$$

(H5) We have

$$\max_{t\in\{1,2,3,4\}}|f(t,0)|=q=0 \text{ and } \max_{t\in\{1,2,3,4\}}a(t)=r=0.2662.$$

Thus, all the assumptions of Theorem 5.2 hold and hence the initial value problem (5.14) has a unique S-asymptotically 4-periodic solution in $B_{p,\phi}$, where $p=2.6504$ and

$$\phi(t)=(8.7485)(t+1)^{\overline{-0.1}}-(1.0527)(t+1)^{\overline{-0.5}},\quad t\in\mathbb{N}_0.$$

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