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IMPROVED MATHEMATICAL RESULTS AND SIMPLIFIED PEDAGOGICAL APPROACHES FOR GRONWALL'S INEQUALITY FOR FRACTIONAL CALCULUS

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Abstract. Gronwall's inequality plays an important role in producing new research and in the learning and teaching of differential and integral equations. The purpose of this work is to advance and simplify the current state of knowledge and pedagogical approaches regarding Gronwall's inequality. In particular: we extend known versions of Gronwall's inequality for fractional calculus; and we provide simpler and more accessible proofs that can be easily transferred to the classroom. Our work is also timely in the sense that it may be considered as a celebration of the upcoming centenary of the publication of Gronwall's original results. Thus, we believe this paper is important from mathematical research, pedagogical and historical viewpoints.

1. Introduction

For nearly 100 years, Gronwall's inequality [10] has played an important role in the research and pedagogy of differential and integral equations. For example, Gronwall's inequality is a central tool in the quantitative and qualitative analysis of solutions to initial value problems involving ordinary differential equations. It enables critical insight into error estimates, uniqueness of solutions, and *a priori* estimates in the Galerkin method [19], [2], [28, Ch. 3].

Gronwall's inequality is also found in undergraduate and postgraduate university courses, with the pedagogy supported by a range of textbooks and monographs in the literature, including: [16, Ch. 7, Sec. 3]; [18, Ch. 2, Sec. 3]; [4, Ch. 8, Sec. 2]; [5, Ch. 1, Sec 5]; [8, Sec. 12]; [6, Appendix 3]; [26, pp. 5–6]; [3, Appendix A3]; [20, Ch. 2, Sec. 9]; [11, Ch. 8, Sec. 10]; and [1, Lecture 7].

This work presents improved versions of Gronwall's inequality for fractional calculus. Fractional calculus involves generalisations of derivatives and integrals from integer order values to non–integer order values. Thus, for example, we can speak of "a derivative of the half–order" of a function. These ideas have a rich history dating back to L'Hôpital and the tautochrone problem. More recently, important applications have been identified and modelled through the analysis and use of fractional differential equations [17, 12].

The purpose of this work is to:

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- extend and advance the current state of knowledge and pedagogical approaches concerning Gronwall's inequality for fractional calculus and fractional integral inequalities;

- provide simpler and more accessible proofs than are currently available in the literature that have the potential to be easily transferred to the classroom.

Our work is also timely in the sense that it may be considered as a celebration of the upcoming centenary of Gronwall's original results. Thus, we believe this paper is important from research, pedagogical and historical viewpoints.

We organise this work as follows. In Section 2 we provide the necessary notation and definitions to keep the paper self–contained, while in Section 3 we present the main results.

For more recent developments in fractional calculus and nonlinear problems we refer the reader to [21, 22, 23, 24, 25, 13].

2. Preliminary notation

To keep this article self–contained, this section contains some preliminary definitions from fractional calculus and the associated notation. Define the Riemann–Liouville fractional derivative and integral of order $q > 0$ of a function $y : [0,a] \to \mathbb{R}$ at a point t, respectively, by:

$$D^q y(t) := \frac{d^{\lceil q \rceil}}{dt^{\lceil q \rceil}} \frac{1}{\Gamma(\lceil q \rceil - q)} \int_0^t (t-s)^{\lceil q \rceil - 1 - q} y(s)\, ds; \tag{1}$$

$$I^q y(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds$$

where $\lceil q \rceil$ being the integer that is the "ceiling value" of q and Γ is the gamma function. The Caputo derivative of $y : [0,a] \to \mathbb{R}$ at a point t is defined via

$$^{C}D^q y(t) := D^q \left(y - T_{\lceil q \rceil - 1}[y] \right)(t).$$

with $T_{\lceil q \rceil - 1}[y]$ denoting the Maclaurin polynomial of order (or degree) $\lceil q \rceil - 1$ of $y = y(t)$. Above, we make the assumption that y is a function such that all expressions are well defined.

In fractional calculus the Mittag-Leffler function plays a similar role to that of the exponential function in classical calculus [17, Ch.1, Podlubny], [14, 15, Mittag-Leffler], [9, Ch.16, Erdélyi et al], [27, Wiman]. The Mittag-Leffler function of order $q > 0$ is defined and denoted by

$$E_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, \quad z \in \mathbb{C}.$$

In particular, we shall be interested in the function

$$E_q(\beta t^q) = \sum_{k=0}^{\infty} \frac{(\beta t^q)^k}{\Gamma(qk+1)}, \quad t \in \mathbb{R} \tag{2}$$

where $q>0$ and $\beta>0$ is a constant. An important property within the context of this work is that $E_q(\beta t^q)$ is the unique solution to the fractional initial value problem

$$\begin{gathered} {}^{C}D^{q}x := \beta x \\ x(0)=1,\ x'(0)=0,\ldots,\ x^{(\lceil q\rceil-1)}(0)=0 \end{gathered}$$

for $t \geqslant 0$.

3. Main results

In this section, the main results and new, simplified proofs are presented.

The following result generalises [7, Lemma 4.3] and presents a significantly simpler proof.

THEOREM 1. *Let A and B be non–negative constants and let $\rho:[0,a]\to[0,\infty)$ be uniformly bounded on $[0,a]$. If*

$$\rho(t) \leqslant A + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} B\rho(s)\,ds, \quad \text{for all } t\in[0,a] \tag{3}$$

with the right hand side of inequality (3) well defined, then

$$\rho(t) \leqslant AE_q(Bt^q), \quad \text{for all } t\in[0,a]. \tag{4}$$

Proof. Our style of proof is similar to that of [28, p82–83] and adapted to the fractional setting.

Since ρ is non–negative and uniformly bounded above on $[0,a]$, there is a constant $M>0$ such that

$$0 \leqslant \rho(t) \leqslant M, \quad \text{for all } t\in[0,a]. \tag{5}$$

Inserting (5) into the right–hand side of (3) we obtain, for all $t\in[0,a]$:

$$\begin{aligned} \rho(t) &\leqslant A + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} BM\,ds \\ &= A + \frac{MBt^q}{\Gamma(q+1)}. \end{aligned} \tag{6}$$

Now, in a similar fashion, inserting (6) into (3) we obtain, for all $t\in[0,a]$:

$$\begin{aligned} \rho(t) &\leqslant A + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} B\left[A + \frac{MBs^q}{\Gamma(q+1)}\right]\,ds \\ &= A + \frac{ABt^q}{\Gamma(q+1)} + \frac{MB^2t^{2q}}{\Gamma(2q+1)}. \end{aligned}$$

Continuing with this process, we see that the n-th interation is

$$\rho(t) \leqslant A\sum_{k=0}^{n-1}\frac{(Bt^q)^k}{\Gamma(qk+1)} + \frac{M(Bt^q)^n}{\Gamma(qn+1)}, \quad \text{for all } t\in[0,a]. \tag{7}$$

Taking limits as $n\to\infty$ in (7) and in light of (2) we obtain (4). $\square$

REMARK 1. The right hand side of (3) will be well defined, for example, when: ρ is continuous; or when ρ is an integrable, discontinuous function, for example, when ρ is piecewise continuous.

REMARK 2. If $q = 1$ in Theorem 1 then (3) becomes

$$\rho(t) \leqslant A + \int_0^t B\rho(s)\, ds, \quad \text{for all } t \in [0,a]$$

and (4) becomes

$$\rho(t) \leqslant Ae^{Bt}, \quad \text{for all } t \in [0,a]$$

which is the classical version of Gronwall's inequality.

REMARK 3. In the notation of fractional calculus, (3) is

$$\rho \leqslant A + BI^q\rho, \quad \text{on } [0,a].$$

REMARK 4. Theorem 1 generalizes [7, Lemma 4.3] but we note that (4) appears in both results. There are two significant differences to communicate. Firstly, [7] make the assumption that ρ is continuous and furthermore, their assumption is necessary due to the style of their proof, which invokes the intermediate value theorem. We make no such assumption of continuity. Secondly, our proof is very simple and straightforward and does not require a knowledge of fractional inequalities or fractional calculus, rather it involves simple integration and a particular convergent series.

THEOREM 2. *Let A, B and C be non–negative constants and let $\rho : [0,a] \to [0,\infty)$ be continuous. If*

$$\rho(t) \leqslant A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[B\rho(s) + C]\, ds, \quad \textit{for all } t \in [0,a] \tag{8}$$

then

$$\rho(t) \leqslant \left[A + \frac{Ct^q}{\Gamma(q+1)}\right] E_q(Bt^q), \quad \textit{for all } t \in [0,a]. \tag{9}$$

Proof. *Case* $B = 0$: Inequality (8) may be integrated directly to obtain (9).

Case $B > 0$: In the interest of diversity, we present a different style of proof from that used in Theorem 1. Once again, it is a very simple approach and only requires a basic understanding of functions and fractional calculus.

For $0 \leqslant t \leqslant t_2 \leqslant a$, define

$$g(t) := \frac{\rho(t)}{E_q(Bt^q)}. \tag{10}$$

Since g is continuous on a compact interval, it must attain its maximum value at some point $t_1 \in [0,t_2]$. Let

$$m := \max_{t\in[0,t_2]} g(t) = g(t_1).$$

Thus, from (10) we see that

$$\rho(t_1) = mE_q(Bt_1^q). \tag{11}$$

Using (11) and (9) we have

$$\begin{aligned} mE_q(Bt_1^q) &= \rho(t_1) \\ &\leqslant A + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} [B\rho(s) + C]\, ds \\ &= A + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} [BE_q(Bs^q)g(s) + C]\, ds \\ &\leqslant A + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} [BE_q(Bs^q)m + C]\, ds \\ &= A + \frac{Ct_1^q}{\Gamma(q+1)} + m[E_q(Bt_1^q) - 1] \end{aligned}$$

where, in the previous line we applied a fundamental identity from [12, (2.4.42), p. 96], namely

$$I^q[{}^C D^q y(t)] = y(t) - \sum_{i=0}^{\lceil q \rceil - 1} y^{(i)}(0) t^i / i!. \tag{12}$$

Thus, we have

$$mE_q(Bt_1^q) \leqslant A + \frac{Ct_1^q}{\Gamma(q+1)} + m[E_q(Bt_1^q) - 1]$$

from which we can eliminate the Mittag-Leffler functions and simplify to

$$\begin{aligned} m &\leqslant A + \frac{Ct_1^q}{\Gamma(q+1)} \\ &\leqslant A + \frac{Ct_2^q}{\Gamma(q+1)}. \end{aligned} \tag{13}$$

Thus, from (10) and (13), for each $t_2 \in [0,a]$ we have

$$\begin{aligned} \rho(t_2) &= g(t_2)E_q(Bt_2^q) \\ &\leqslant mE_q(Bt_2^q) \\ &\leqslant \left[A + \frac{Ct_2^q}{\Gamma(q+1)} \right] E_q(Bt_2^q). \end{aligned} \tag{14}$$

Thus, (9) holds by replacing t_2 with t in both sides of (14). $\square$

The following generalisation of Theorem 2 is now presented.

THEOREM 3. *Let B and C be non–negative constants; let $A : [0,a] \to [0,\infty)$ be continuous and nondecreasing; and let $\rho : [0,a] \to [0,\infty)$ be continuous. If*

$$\rho(t) \leqslant A(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [B\rho(s) + C]\, ds, \quad \textit{for all } t \in [0,a] \tag{15}$$

then

$$\rho(t) \leqslant \left[A(t) + \frac{Ct^q}{\Gamma(q+1)}\right] E_q(Bt^q), \quad \textit{for all } t \in [0,a]. \tag{16}$$

Proof. *Case* $B = 0$: Inequality (15) may be integrated directly to obtain (16).

Case $B > 0$: If (15) holds then, for $0 \leqslant t \leqslant t_3 \leqslant a$ we have

$$\rho(t) \leqslant A(t_3) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[B\rho(s) + C]\, ds \tag{17}$$

where t_3 is now regarded as a constant. The conditions of Theorem 2 hold and the conclusion (9) can then be applied, so that we have

$$\rho(t) \leqslant \left[A(t_3) + \frac{Ct^q}{\Gamma(q+1)}\right] E_q(Bt^q). \tag{18}$$

Thus replacing t with t_3 in (18) we obtain

$$\rho(t_3) \leqslant \left[A(t_3) + \frac{Ct_3^q}{\Gamma(q+1)}\right] E_q(Bt_3^q), \quad \text{for all } t_3 \in [0,a]. \tag{19}$$

so that (16) holds. $\square$

We now present our final result.

THEOREM 4. *Let* A, B *and* C *be non–negative constants and let* $\rho : [0,a] \to [0,\infty)$ *satisfy*

$$\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)} < \infty. \tag{20}$$

If

$$\rho(t) \leqslant A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[B\rho(s) + C]\, ds, \quad \textit{for all } t \in [0,a] \tag{21}$$

with the right–hand side of inequality (21) *well defined, then*

$$\rho(t) \leqslant E_q(Ba^q) \left[A + \frac{Ca^q}{\Gamma(q+1)}\right] E_q(Bt^q), \quad \textit{for all } t \in [0,a]. \tag{22}$$

Proof. *Case* $B = 0$: Inequality (21) may be integrated directly to obtain (22).

Case $B > 0$: Once again, we present a new proof that is distinct from those in the preceeding results in this paper.

From (21) we have, for all $t \in [0,a]$,

$$\begin{aligned}
\rho(t) &\leqslant A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B\rho(s)\, ds + \frac{Ct^q}{\Gamma(q+1)} \\
&= A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{\rho(s)}{E_q(Bs^q)} BE_q(Bs^q)\, ds + \frac{Ct^q}{\Gamma(q+1)} \\
&\leqslant A + \left[\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)}\right] \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} BE_q(Bs^q)\, ds + \frac{Ct^q}{\Gamma(q+1)} \\
&= A + \left[\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)}\right] I^q \left[{}^C D^q \left(E_q(Bt^q)\right)\right] + \frac{Ct^q}{\Gamma(q+1)} \\
&= A + \left[\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)}\right] \left[E_q(Bt^q) - 1\right] + \frac{Ct^q}{\Gamma(q+1)}
\end{aligned}$$

where we have applied (12).

Thus, for all $t \in [0,a]$ we obtain

$$\begin{aligned}
\frac{\rho(t)}{E_q(Bt^q)} &\leqslant \frac{A}{E_q(Bt^q)} + \left[\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)}\right] \left[1 - \frac{1}{E_q(Bt^q)}\right] + \frac{Ct^q}{\Gamma(q+1)E_q(Bt^q)} \\
&\leqslant A + \left[\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)}\right] \left[1 - \frac{1}{E_q(Bt^q)}\right] + \frac{Ct^q}{\Gamma(q+1)}.
\end{aligned}$$

Taking suprema we obtain

$$\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)} \leqslant A + \left[\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)}\right] \left[1 - \frac{1}{E_q(Ba^q)}\right] + \frac{Ca^q}{\Gamma(q+1)}. \tag{23}$$

Equation (23) can be rearranged to obtain

$$\frac{1}{E_q(Ba^q)} \sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)} \leqslant A + \frac{Ca^q}{\Gamma(q+1)}$$

with a further rearrangement giving

$$\sup_{t\in[0,a]} \frac{\rho(t)}{E_q(Bt^q)} \leqslant E_q(Ba^q) \left[A + \frac{Ca^q}{\Gamma(q+1)}\right]$$

and so

$$\frac{\rho(t)}{E_q(Bt^q)} \leqslant E_q(Ba^q) \left[A + \frac{Ca^q}{\Gamma(q+1)}\right], \quad \text{for all } t \in [0,a]$$

leads to (22). $\square$

Gronwall's inequality has had an enormous positive impact in mathematics since it originally appeared in 1919. We look forward to more important generalisation, exciting new developments and critical insights that this amazing inequality will produce over the next 100 years.

REFERENCES

[1] R. P. AGARWAL AND D. O'REGAN, *An introduction to ordinary differential equations*, New York (NY): Springer, 2008.

[2] R. BELLMAN, *The stability of solutions of linear differential equations*, Duke Math. J. **10**, (1943), 643–647.

[3] R. L. BORELLI AND C. S. COLEMAN, *Differential equations. A modeling perspective*, New York (NY): John Wiley & Sons Inc, 1998.

[4] F. BRAUER AND J. A. NOHEL, *Ordinary differential equations: a first course*, Menlo Park (CA): W A Benjamin Inc, 1973.

[5] C. CORDUNEANU, *Principles of differential and integral equations*, The Bronx (NY): Chelsea Publishing Company, 1977.

[6] W. R. DERRICK AND S. I. GROSSMAN, *Introduction to differential equations with boundary value problems*, St Paul (MN): West Publishing Company, 1987.

[7] K. DIETHELM AND N. J. FORD, *Multi–order fractional differential equations and their numerical solution*, Appl. Math. Comput. **154**, (2004), 621–540.

[8] R. D. DRIVER, *Introduction to ordinary differential equations*, New York (NY): Harper & Row, 1978.

[9] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher transcendental functions*, vol. III, based, in part, on notes left by Harry Bateman, New York (NY): McGraw–Hill Book Company, Inc., 1955.

[10] T. H. GRONWALL, *Note on the derivative with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. **20**, (1919), 292–296.

[11] W. KELLY, A. PETERSON, *The theory of differential equations. Classical and qualitative*, Upper Saddle River (NJ): Pearson Education Inc, 2004.

[12] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and applications of fractional differential equations*, North–Holland Mathematics Studies, **204**, Amsterdam: Elsevier Science B. V., 2006.

[13] X. LI, Z. LIU AND C. C. TISDELL, *Existence and exact controllability of fractional evolution inclusions with damping*, Math. Meth. Appl. Sci., **40**, 12 (2017) 4548–4559. `http://dx.doi.org/10.1002/mma.4325`

[14] G. MITTAG-LEFFLER, *Sur la nouvelle fonction* $E_\alpha(x)$, Comptes Rendus Acad. Sci. Paris. **137**, (1903), 554–558.

[15] G. MITTAG-LEFFLER, *Sur la représentation analytique d'une branche uniforme d'une fonction monogéne. (French) cinquiéme note*, Acta Math. **29**, (1905), 101–181.

[16] O. PLATT, *Ordinary differential equations*, San Francisco (CA): Holden-Day Inc, 1971.

[17] I. PODLUBNY, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, **198**, San Diego (CA); Academic Press, Inc.: 1999.

[18] E. O. ROXIN, *Ordinary differential equations*, Belmont (CA): Wadsworth Publishing Company Inc, 1972.

[19] W. T. REID, *Properties of solutions of an infinite system of ordinary linear differential equations of the first order with auxiliary boundary condtions*, Trans. Amer. Math. Soc. **32**, (1930), 294–318.

[20] D. A. SÁNCHEZ, *Ordinary differential equations. A brief eclectic tour*, Washington (DC): The Mathematical Association of America, 2002.

[21] C. C. TISDELL, *On the application of sequential and fixed–point methods to fractional differential equations of arbitrary order*, J. Integral Equ. Appl. **24**, (2012), 283–319.

[22] C. C. TISDELL, *Solutions to fractional differential equations that extend*, Journal of Classical Analysis, **5**, (2014), 129–136, `http://dx.doi.org/10.7153/jca-05-11`.

[23] C. C. TISDELL, *When do fractional differential equations have solutions that are bounded by the Mittag-Leffler function?*, Fractional Calculus and Applied Analysis, **18**, (2015), 642–650, `http://dx.doi.org/10.1515/fca-2015-0039`.

[24] C. C. TISDELL, *Basic existence and* a priori bound results for solutions to systems of boundary value problems for fractional differential equations, Electronic Journal of Differential Equations, vol. 2016, (2016), no. 84, pp. 1–9.

[25] C. C. TISDELL, Z. LIU AND S. MACNAMARA, *Basic existence and uniqueness results for solutions to systems of nonlinear fractional differential equations*, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, **24**, (2017), 181–193.
[26] F. VERHULST, *Nonlinear differential equations and dynamical systems*, Berlin: Springer, 1990.
[27] A. WIMAN, *Über den Fundamentalsatz in der Theorie der Funktionen* $E_a(x)$, Acta Math. **29**, (1905), 191–201.
[28] E. ZEIDLER, *Nonlinear Functional analysis and its applications I. Fixed-point theorems*, New York (NY): Springer, 1986.

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