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NON–EXISTENCE RESULTS AND ANALYTICAL BOUNDS OF EIGENVALUES FOR A CLASS OF FRACTIONAL EIGENVALUE PROBLEMS

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Abstract. In this paper, we study a class of linear and nonlinear fractional eigenvalue problems with fractional derivative of Caputo type. The problem is obtained by fractionalizing a term of the well-known Sturm-Liouville operator and it covers a wide class of fractional eigenvalue problems. By applying simple maximum principles, we obtain necessary conditions for the existence of eigenfunctions and analytical bounds for the eigenvalues. We also establish a uniqueness result for the nonlinear eigenvalue problem. The results in this paper are discussed in two common spaces of fractional derivatives.

1. Introduction

The Sturm-Liouville eigenvalue problems have been implemented to model many physical problems. The theory of the problems is well-developed and many results concerning the eigenvalues and eigenfunctions have been established. These results are used in large to study other types of problems. For instance, the facts that the eigenfunctions of the Sturm-Liouville eigenvalues problems form a complete set, and eigenfunctions corresponding to different eigenvalues are orthogonal, form the basis of the spectral methods that have been used to study several types of problems, analytically and numerically. In recent years, there are several analytical and numerical studies on fractional eigenvalue problems [4, 5, 9, 11, 20] in general, and on fractional Sturm-Liouville eigenvalue problems in particular. In [13, 15, 18, 21] classes of fractional Sturm-Liouville eigenvalue problems with left and right-sided fractional derivatives of Riemann-Liouville and Caputo types have been discussed. In these studies some of the well-known results of the Sturm-Liouville problems are extended to the fractional ones. These results include, orthogonality and completeness of eigenfunctions and countability of the real eigenvalues. Analogous results are obtained in [14] for a fractional eigenvalue problem of Riesz fractional derivative. However, these results are not applicable for the fractional Sturm-Liouville problems with Caputo fractional derivative. The presented fractional derivatives satisfy many elegance properties, such as the integration by parts formula, which don't hold for the Caputo fractional derivative. In [4], a class

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of fractional eigenvalue problem with fractional derivative $1<\delta<2$, of Caputo type has been studied using maximum principles and method of upper and lower solutions. Some existence, nonexistence results, as well as, analytical bounds of eigenvalues have been established. In [17], the existence of positive eigenfunctions for a coupled system of fractional Sturm-Liouville problem is established, using the Guo-Krasnoselski fixed point theorem, where the fractional derivative is of the Riemann-Liouville type. In [2, 3] spectral analysis is carried out for a class of integral operators associated with fractional order differential equations. Moreover, a connection between the eigenvalues of these operators and the zeros of the Mittag-Leffler functions is established. Several numerical techniques devoted to computing eigenvalues and eigenfunctions of certain fractional eigenvalue problems were implemented in [6, 7, 10, 12], where the fractional derivative is either of Caputo or Riemann-Liouville type. In this paper we consider the fractional eigenvalue problem

$$D_{0+}^{\delta}(p(x)y')+q(x)y=-\lambda w(x)y,\ \ 0<\delta<1,\quad 0<x<1, \tag{1.1}$$

where p,p',q and the weight function w are continuous on $(0,1)$, $p(x)>0$, $w(x)>0$ on $[0,1]$, and the fractional derivative D_{0+}^{δ} is of Caputo type. The problem in Eq. (1.1) is obtained by fractionalizing the first term of the well-known Sturm-Liouville operator, and it presents various types of fractional differential equations. For instance, if $p(x)=1$, then the problem reduces to

$$D_{0+}^{\alpha}y+q(x)y=-\lambda w(x)y,\ \ 1<\alpha<2,\quad 0<x<1.$$

This problem has been discussed in [1]. The problem has been transformed to an equivalent system of fractional equations and then existence and uniqueness results are obtained by applying method of upper and lower solutions to the resulting system.

We discuss three types of boundary conditions for Eq. (1.1):

1. Dirichlet boundary conditions

$$y(0)=y(1)=0, \tag{1.2}$$

2. Neumann boundary conditions

$$y'(0)=y'(1)=0, \tag{1.3}$$

3. Robin boundary conditions

$$y(0)-\alpha y'(0)=0,\ \ y(1)+\beta y'(1)=0,\ \ \alpha,\beta>0. \tag{1.4}$$

This article is organized as follows. In Section 2, we review basic definitions of fractional calculus and present some basic results. In Section 3, We apply simple maximum principles to establish necessary conditions for the existence of eigenfunctions and to obtain analytical bounds of eigenvalues for the linear Sturm-Liouville problems. We then apply these results to study nonlinear fractional Sturm-Liouville eigenvalue problems in Section 4. Estimates of eigenvalues, as well as, uniqueness results will be established. In Section 5, Some illustrative examples and discussions are presented. The results are discussed in two common spaces for the fractional boundary value problems of order γ, $1<\gamma<2$.

2. Basic definitions and preliminary results

In the following, we present the definitions and some preliminary results of the Riemann-Liouville fractional integral and the Caputo fractional derivative, see [16, 19].

DEFINITION 2.1. A real function $f(x)$, $x>0$, is said to be in the space C_{μ}, $\mu \in \mathbb{R}$ if there exists a real number $p>\mu$, such that $f(x)=x^{p}f_{1}(x)$, where $f_{1}(x)\in C[0,\infty)$, and it is said to be in the space C_{μ}^{m} if $f^{(m)}\in C_{\mu}$, $m\in\mathbb{N}$.

DEFINITION 2.2. The left Riemann-Liouville fractional integral of order $\delta>0$, of a function $f\in C_{\mu}$, $\mu\geqslant -1$, is defined by

$$I_{0+}^{\delta}f(x)=\frac{1}{\Gamma(\delta)}\int_{0}^{x}(x-s)^{\delta-1}f(s)ds, \quad x>0. \tag{2.1}$$

DEFINITION 2.3. For $\delta>0$, $m-1<\delta<m$, $m\in\mathbb{N}$, $x>0$, and $f\in C_{-1}^{m}$, the left Caputo fractional derivative is defined by

$$D_{0+}^{\delta}f(x)=\frac{1}{\Gamma(m-\delta)}\int_{0}^{x}(x-s)^{m-1-\delta}f^{(m)}(s)ds, \tag{2.2}$$

where Γ is the well-known Gamma function.

The Caputo derivative is related to the Riemann-Liouville fractional integral, I_{0+}^{δ}, of order $\delta\in\mathbb{R}^{+}$, by

$$D_{0+}^{\delta}f(x)=I_{0+}^{m-\delta}f^{(m)}(x). \tag{2.3}$$

It is known that

$$\Big((I_{0+}^{\delta}D_{0+}^{\delta})f\Big)(x)=f(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{k!}x^{k}, \tag{2.4}$$

$$\Big((D_{0+}^{\delta}I_{0+}^{\delta})f\Big)(x)=f(x). \tag{2.5}$$

The following results will be used throughout the text.

PROPOSITION 2.1. ([8]) *If* $f\in C^{1}[0,1]$, *then* $(D_{0+}^{\delta}f)(0)=0$, $0<\delta<1$.

PROPOSITION 2.2. *Let* $f\in C[0,x_{0}]$. *If* $f(x)\geqslant 0$, *and not identically zero on* $[0,x_{0}]$, *then* $(I_{0+}^{\delta}f)(x_{0})>0$, $0<\delta<1$.

Proof. Since $f(x)\geqslant 0$, is not identically equals to zero and continuous, then there exists an interval (α,β) inside $[0,x_{0}]$ where $f(x)>0$. As $r(s)=(x_{0}-s)^{\delta-1}$ is integrable, applying the mean value theorem for integrals we have

$$\begin{aligned}(I_{0+}^{\delta}f)(x_0) &= \frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}f(s)ds \geqslant \frac{1}{\Gamma(\delta)}\int_{\alpha}^{\beta}(x_0-s)^{\delta-1}f(s)ds\\ &= \frac{f(\xi)}{\Gamma(\delta)}\int_{\alpha}^{\beta}(x_0-s)^{\delta-1}ds\\ &= \frac{f(\xi)}{\Gamma(\delta+1)}((x_0-\alpha)^{\delta}-(x_0-\beta)^{\delta})>0,\ \alpha<\xi<\beta,\end{aligned}$$

which proves the result. □

PROPOSITION 2.3. *Let* $z(x)\in C^1[0,1]$ *be such that satisfy the boundary condition* $z(1)+\beta z'(1)=0$, *for some* $\beta\geqslant 0$.

1. *If* $z(0)\geqslant 0$ *and* $z(x)$ *attains its maximum* $x=1$, *then* $z(x)=0$ *on* $[0,1]$.
2. *If* $z(0)\leqslant 0$ *and* $z(x)$ *attains its minimum* $x=1$, *then* $z(x)=0$ *on* $[0,1]$.

Proof.

1. Assume by contradiction $z(x)$ is not identically zero on $[0,1]$. Since $z(x)$ attains its maximum at $x=1$, then simple maximum principle implies $z'(1^-)\geqslant 0$. Since $z(0)\geqslant 0$, we have $z(1)>0$, and thus

$$z(1)+\beta z'(1)\geqslant z(1)>0,$$

and a contradiction is reached.

2. Assume by contradiction $z(x)$ is not identically zero on $[0,1]$. Since $z(x)$ attains its minimum at $x=1$, then simple maximum principle implies $z'(1^-)\leqslant 0$. Since $z(0)\leqslant 0$, we have $z(1)<0$, and thus

$$z(1)+\beta z'(1)\leqslant z(1)<0,$$

and a contradiction is reached. □

3. The linear fractional eigenvalue problem

We start with the linear fractional eigenvalue problem (1.1–1.4) and obtain analytical results in the spaces $C^2_{-1}[0,1]$ and $C^2[0,1]$. Applying the Riemann-Liuoville fractional integral operator I^{δ}_{0+} to Eq. (1.1) yields

$$\begin{aligned}p(x)y' &= p(0)y'(0)-I_{0+}^{\delta}\Big((q(x)+\lambda w(x))y(x)\Big)\\ &= p(0)y'(0)-\frac{1}{\Gamma(\delta)}\int_0^x(x-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds. \qquad (3.1)\end{aligned}$$

3.1. Analytical results on the space $C^2_{-1}[0,1]$

LEMMA 3.1. *Consider the fractional eigenvalue problem (1.1) subject to the Dirichlet boundary conditions (1.2). If*

$$q(x)+\lambda w(x)<0, \quad x\in[0,1]$$

then the problem has no eigenfunctions in the space $C^2_{-1}[0,1]$.

Proof. Assume that the problem possesses a nonzero eigenfunction $y(x)$ on $[0,1]$, we shall reach a contradiction. Since $y\in C^2_{-1}$ then y' is continuous on $[0,1]$ and $y'(0)$ is well defined. We consider three cases for $y'(0)$, we have

1. If $y'(0)<0$. Let $x_0\in(0,1)$ be the first point such that $y'(x_0)=0$. Because $y(x)$ is not identically zero on $[0,1]$ and $y(0)=y(1)=0$, then $y(x)$ attains its maximum or minimum on $(0,1)$, and thus the existence of $x_0\in(0,1)$ is guaranteed. Since $y'(0)<0$ it holds that
$$y(x)\leqslant 0, \text{ on } [0,x_0].$$
Substituting in Eq. (3.1) yields
$$\begin{aligned}0=p(x_0)y'(x_0) &= p(0)y'(0)-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\\ &< -\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\leqslant 0,\end{aligned}$$
and a contradiction is reached.

2. If $y'(0)>0$. Let $x_0\in(0,1)$ be the first point such that $y'(x_0)=0$. Since $y'(0)>0$, it holds that
$$y(x)\geqslant 0, \text{ on } [0,x_0].$$
Substituting in Eq. (3.1) yields
$$\begin{aligned}0=p(x_0)y'(x_0) &= p(0)y'(0)-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\\ &> -\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\geqslant 0,\end{aligned}$$
and a contradiction is reached.

3. If $y'(0)=0$, then Eq. (3.1) reduces to
$$0=p(x_0)y'(x_0)=-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds. \tag{3.2}$$
Let $x_0\in(0,1)$ be the first point such that $y'(x_0)=0$, and $y(x_0)\neq 0$. Then $y(x)$ is not identically zero and of one sign on $[0,x_0]$. If
$$y(x)\geqslant 0, \text{ on } [0,x_0],$$

then by Proposition 2.2 we have

$$-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds>0,$$

which contradicts Eq. (3.2). Analogously, if

$$y(x)\leqslant 0, \text{ on } [0,x_0],$$

then by Proposition 2.2 we have

$$-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds<0,$$

which contradicts Eq. (3.2). □

LEMMA 3.2. *Consider the fractional eigenvalue problem (1.1) subject to the Robin boundary conditions (1.4). If $q(x)+\lambda w(x)<0$ on $[0,1]$, then the problem has no eigenfunctions in the space $C^2_{-1}[0,1]$.*

Proof. Assume that the problem possesses a nonzero eigenfunction $y(x)$ on $[0,1]$, we shall reach a contradiction. We consider three cases for $y'(0)$, we have

1. If $y'(0)<0$, then the boundary conditions yield $y(0)<0$. Let $x_0\in(0,1]$ be the first point such that $y'(x_0)=0$. By Proposition 2.3 we have $x_0\in(0,1)$. Since $y'(0)<0$ it holds that
$$y(x)\leqslant 0, \text{ on } [0,x_0].$$
Substituting in Eq. (3.1) yields
$$\begin{aligned}0=p(x_0)y'(x_0)&=p(0)y'(0)-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\\&<-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\leqslant 0,\end{aligned}$$
and a contradiction is reached.

2. If $y'(0)>0$, then the boundary conditions yield $y(0)>0$. Let $x_0\in(0,1]$ be the first point such that $y'(x_0)=0$. By Proposition 2.3 we have $x_0\in(0,1)$. Since $y'(0)>0$, it holds that
$$y(x)\geqslant 0, \text{ on } [0,x_0].$$
Substituting in Eq. (3.1) yields
$$\begin{aligned}0=p(x_0)y'(x_0)&=p(0)y'(0)-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\\&>-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds\geqslant 0,\end{aligned}$$
and a contradiction is reached.

3. If $y'(0) = 0$, then the boundary conditions yield $y(0) = 0$. Let $x_0 \in (0,1]$ be the first point such that $y'(x_0) = 0$, and $y(x_0) \neq 0$. By Proposition 2.3 we have $x_0 \in (0,1)$. Thus $y(x)$ is not identically zero and of one sign on $[0,x_0]$, and Eq. (3.1) reduces to

$$0 = p(x_0)y'(x_0) = -\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds. \tag{3.3}$$

If

$$y(x) \geqslant 0, \text{ on } [0,x_0],$$

then by Proposition 2.2 we have

$$-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds > 0,$$

which contradicts Eq. (3.3). Analogously, if

$$y(x) \leqslant 0, \text{ on } [0,x_0],$$

then by Proposition 2.2 we have

$$-\frac{1}{\Gamma(\delta)}\int_0^{x_0}(x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds < 0,$$

which contradicts Eq. (3.3). □

COROLLARY 3.1. *Consider the fractional eigenvalue problem (1.1) subject to the Dirichlet (1.2) or Robin (1.4) boundary conditions. If the problem possesses an eigenfunction* $y \in C^2_{-1}[0,1]$, *then the following holds true for the corresponding eigenvalue* λ,

$$\lambda \geqslant \inf\Big\{-\frac{q(x)}{w(x)}\Big\}, \quad x \in [0,1].$$

Proof. Assume by contradiction that $\lambda < \inf\{-\frac{q(x)}{w(x)}\}$, $x \in [0,1]$, then it holds that

$$\lambda < -\frac{q(x)}{w(x)}, \quad \text{for all } x \in [0,1],$$

or

$$\lambda w(x) + q(x) < 0, \text{ for all } \quad x \in [0,1],$$

which contradicts the result in Lemma 3.1 for the Dirichlet boundary conditions and the result in Lemma 3.2 for the Robin boundary conditions. □

REMARK 3.1. For the case of the Neumann boundary conditions, no analytical results for eigenvalues and eigenfunctions are achieved in the space $C^2_{-1}[0,1]$. However, in the next section, some results will be established in the space $C^2[0,1]$.

3.2. Analytical results on the space $C^2[0,1]$

We first mention that, the results obtained in Section 3.1 are valid for the space $C^2[0,1]$, since $C^2[0,1] \subseteq C^2_{-1}[0,1]$, while the converse is not true. In the space $C^2[0,1]$, we establish new results for the case of Neumann boundary conditions, and sharper bounds for the case of Dirichlet and Robin conditions.

LEMMA 3.3. *Consider the fractional eigenvalue problem (1.1) subject to the Robin boundary conditions (1.4). If* $q(x)+\lambda w(x)$, *is* >0 *or* <0 *on* $[0,1]$, *then the problem has no eigenfunctions in the space* $C^2[0,1]$.

Proof. As $C^2[0,1] \subseteq C^2_{-1}[0,1]$, by Lemma 3.2 the result is true for the case $q(x)+\lambda w(x)<0,\ x\in[0,1]$. We consider the case $q(x)+\lambda w(x)>0,\ x\in[0,1]$. By Proposition 2.1 we have $\left(D^{\delta}_{0+}(p(x)y')\right)(0)=0$, which together with $q(0)+\lambda w(0)>0$ implies that $y(0)=0$, by the continuity of the solution y in Eq. (1.1). Since $\alpha>0$ the boundary conditions yield $y'(0)=0$. Thus Eq. (3.1) yields

$$p(x)y'(x) = -\frac{1}{\Gamma(\delta)}\int_0^x (x-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds. \tag{3.4}$$

Assume that the problem possesses a nonzero eigenfunction $y(x)$ on $[0,1]$, we shall reach a contradiction. Let $x_0\in[0,1]$ be the first point such that $y'(x_0)=0$, and $y(x_0)\neq 0$. As $y(0)=0$ and using the result in Proposition 2.3 we have $x_0\in(0,1)$. If $y(x_0)>0$ then it holds that $y(x)$ is not identically zero on $[0,x_0]$ and $y(x)\geqslant 0$, on $[0,x_0]$. Substituting in Eq. (3.4) and using Proposition 2.2, we have

$$0=p(x_0)y'(x_0) = -\frac{1}{\Gamma(\delta)}\int_0^{x_0} (x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds<0, \tag{3.5}$$

and a contradiction is reached. Analogously, if $y(x_0)<0$, then it holds that $y(x)$ is not identically zero on $[0,x_0]$ and $y(x)\leqslant 0$, on $[0,x_0]$. Substituting in Eq. (3.4) and using Proposition 2.2, we have

$$0=p(x_0)y'(x_0) = -\frac{1}{\Gamma(\delta)}\int_0^{x_0} (x_0-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds>0, \tag{3.6}$$

and a contradiction is reached. □

LEMMA 3.4. *Consider the fractional eigenvalue problem (1.1) subject to the Neumann boundary conditions (1.3). If* $q(x)+\lambda w(x)$, *is* >0 *or* <0 *on* $[0,1]$, *then the problem has no eigenfunctions in the space* $C^2[0,1]$.

Proof. We first assume that $q(x)+\lambda w(x)>0,\ x\in[0,1]$. By Proposition 2.1 we have $\left(D^{\delta}_{0+}(p(x)y')\right)(0)=0$, which together with $q(0)+\lambda w(0)>0$ implies that $y(0)=0$, by the continuity of the solution y in Eq. (1.1). Thus Eq. (3.1) yields

$$0=p(1)y'(1) = -\frac{1}{\Gamma(\delta)}\int_0^{1} (1-s)^{\delta-1}(q(s)+\lambda w(s))y(s)\,ds. \tag{3.7}$$

Assume that the problem possesses a nonzero eigenfunction $y(x)$ on $[0,1]$, we shall reach a contradiction. We have two cases;

1. There exits $x_0 \in (0,1)$ such that $y'(x_0) = 0$, and $y(x_0) \neq 0$. Then a contradiction is reached by applying analogues steps in the proof of Lemma 3.3.
2. Otherwise $y(x)$ is of one sign on $[0,1]$. Then the right hand side of Eq. (3.7) is either negative or positive and a contradiction is reached.

By applying analogous steps one can prove the result for $q(x) + \lambda w(x) < 0$, $x \in [0,1]$. □

COROLLARY 3.2. *Consider the fractional eigenvalue problem (1.1) subject to the Neumann (1.3) or Robin (1.4) boundary conditions. If the problem possesses an eigenfunction* $y \in C^2[0,1]$, *then the following holds true for the corresponding eigenvalue* λ,

$$\inf\Big\{-\frac{q(x)}{w(x)}\Big\} \leqslant \lambda \leqslant \sup\Big\{-\frac{q(x)}{w(x)}\Big\}, \ \ x \in [0,1].$$

Proof. The proof is analogous to the proof of Corollary 3.1. □

4. The nonlinear fractional eigenvalue problem

We consider the following class of nonlinear fractional eigenvalue problem

$$D_{0^+}^{\delta}(p(x)y') + q(x)y = -\lambda k(x,y), \ \ 0 < \delta < 1, \quad 0 < x < 1, \tag{4.1}$$

where $k(x,y)$ is continuous and smooth with respect to the variable y. We establish uniqueness results and analytical bounds of eigenvalues in the spaces $C^2_{-1}[0,1]$ and $C^2[0,1]$. We have

LEMMA 4.1. *Consider the fractional eigenvalue problem (4.1) subject to the Dirichlet (1.2) or Robin (1.4) boundary conditions. If* $q(x) + \lambda \frac{\partial k(x,y)}{\partial y} < 0$, *for all* $y \in C^2_{-1}[0,1]$ *and* $x \in [0,1]$, *then the problem has at most one eigenfunction* $y \in C^2_{-1}[0,1]$.

Proof. Assume that the problem possesses two solutions y_1 and y_2, and let $\mu = y_1 - y_2$. We have

$$D_{0^+}^{\delta}(p(x)\mu') + q(x)\mu = -\lambda(k(x,y_1) - k(x,y_2)), \ \ 0 < \delta < 1, \quad 0 < x < 1. \tag{4.2}$$

Applying the mean value theorem we have

$$k(x,y_1) - k(x,y_2) = \frac{\partial k}{\partial y}(\psi)(y_1 - y_2) = \frac{\partial k}{\partial y}(\psi)\mu,$$

where $\psi = \nu y_1 + (1-\nu)y_2$ for some $0 < \nu < 1$. Thus Eq. (4.2) reduces to

$$D_{0^+}^{\delta}(p(x)\mu') + \Big(q(x) + \lambda\frac{\partial k}{\partial y}(\psi)\Big)\mu = 0.$$

If y_1 and y_2 are of Dirichlet type then $\mu = y_1 - y_2$ satisfies the same homogenous Dirichlet conditions (1.2). Since $q(x) + \lambda \frac{\partial k}{\partial y}(\psi) < 0$, then by Lemma 3.1 we have $\mu = 0$ and thus $y_1 = y_2$. If y_1 and y_2 are of Robin type then so is $\mu = y_1 - y_2$, which together with $q(x) + \lambda \frac{\partial k}{\partial y}(\psi) < 0$, proves that $\mu = 0$ by virtue of Lemma 3.2. □

COROLLARY 4.1. *Consider the fractional eigenvalue problem (4.1) with* $k(x,0) = 0$, *and subject to the Dirichlet (1.2) or Robin boundary conditions (1.4). Assuming that the problem possesses an eigenfunction then for the corresponding eigenvalue* λ, *we have*

1. *If there exists a positive constant* ρ *such that* $\frac{\partial k(x,y)}{\partial y} \geqslant \rho$, *for all* $y \in C^2_{-1}[0,1]$ *and* $x \in [0,1]$, *then it holds that*

$$\lambda \geqslant \inf\left\{\frac{-q(x)}{\partial k/\partial y}\right\}.$$

2. *If there exists a negative constant* κ *such that* $\frac{\partial k(x,y)}{\partial y} \leqslant \kappa$, *for all* $y \in C^2_{-1}[0,1]$ *and* $x \in [0,1]$, *then it holds that*

$$\lambda \leqslant \sup\left\{\frac{-q(x)}{\partial k/\partial y}\right\}.$$

Proof.

1. Assume by contradiction that there exists and eigenvalue λ with $\lambda < \inf\{\frac{-q(x)}{\partial k/\partial y}\}$. Then $\lambda < \frac{-q(x)}{\partial k/\partial y}$, for all $x \in [0,1]$. Since $\frac{\partial k(x,y)}{\partial y} \geqslant \rho > 0$, we have $q(x) + \lambda \partial k/\partial y < 0$, and thus the problem possesses at most one solution by Lemma 4.1. Since $k(x,0) = 0$, then $y = 0$ is the unique solution and thus the problem has no eigenfunction and a contradictions is reached.

2. Assume by contradiction that there exists and eigenvalue λ with $\lambda > \sup\{\frac{-q(x)}{\partial k/\partial y}\}$. Then $\lambda > \frac{-q(x)}{\partial k/\partial y}$, for all $x \in [0,1]$. Since $\frac{\partial k(x,y)}{\partial y} \leqslant \rho < 0$, we have $q(x) + \lambda \partial k/\partial y < 0$, and thus the problem possesses at most one solution by Lemma 4.1. Since $k(x,0) = 0$, then $y = 0$ is the unique solution and thus the problem has no eigenfunction and a contradictions is reached. □

LEMMA 4.2. *Consider the fractional eigenvalue problem (4.1) subject to the Neumann (1.3) or Robin (1.4) boundary conditions. If* $q(x) + \lambda \frac{\partial k(x,y)}{\partial y}$, *is* > 0 *or* < 0, *for all* $y \in C^2[0,1]$ *and* $x \in [0,1]$, *then the problem has at most one eigenfunction* $y \in C^2[0,1]$.

Proof. The proof is analogous to the proof of Lemma 4.1 and using the results of Lemmas 3.3 and 3.4. □

COROLLARY 4.2. *Consider the fractional eigenvalue problem (4.1) with* $k(x,0) = 0$*, and subject to the Neumann (1.3) or Robin (1.4) boundary conditions. Assuming that the problem possesses an eigenfunction then for the corresponding eigenvalue* λ*, we have*

1. *If there exists a positive constant* ρ *such that* $\frac{\partial k(x,y)}{\partial y} \geqslant \rho$*, for all* $y \in C^2[0,1]$ *and* $x \in [0,1]$*, then it holds that*

$$\inf\left\{\frac{-q(x)}{\partial k/\partial y}\right\} \leqslant \lambda \leqslant \sup\left\{\frac{-q(x)}{\partial k/\partial y}\right\}.$$

2. *If there exists a negative constant* κ *such that* $\frac{\partial k(x,y)}{\partial y} \leqslant \kappa$*, for all* $y \in C^2[0,1]$ *and* $x \in [0,1]$*, then it holds that*

$$\inf\left\{\frac{-q(x)}{\partial k/\partial y}\right\} \leqslant \lambda \leqslant \sup\left\{\frac{-q(x)}{\partial k/\partial y}\right\}.$$

Proof. The proof is analogous to the proof of Corollary 4.1 and using the results of Lemma 4.2. □

5. Illustrative examples and discussions

We first consider Eq. (1.1) with $p(x) = w(x) = 1$ and $q(x) = 0$, subject to the Dirichlet boundary conditions. We have

$$D_{0^+}^{\delta+1} y = -\lambda y, \ \ 0 < \delta < 1. \tag{5.1}$$

The results of Corollary 3.1 implies the eigenvalue(s) $\lambda \geqslant 0$. It is not difficult to see that

$$\phi(x) = xE_\gamma(-\lambda x^\gamma) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(n\gamma+1)} x^{n\gamma+1}$$

is a solution to the above fractional eigenvalue problem that satisfies $\phi(0) = 0$, where $\gamma = \delta + 1$, and E_γ is the well-known Mittag-Leffler function. To find the eigenvalues we impose the condition $\phi(1) = 0$, that yields

$$E_\gamma(-\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(n\gamma+1)} = 0. \tag{5.2}$$

The above equation possesses solution for certain values of λ, which are the eigenvalues of the problem.

REMARK 5.1. The eigenvalue problem in Eq. (5.1) subject to the Neumann (1.3) or Robin (1.4) boundary conditions has no solution $y \in C^2[0,1]$ by Corollary 3.2. However, the eigenvalue problem with the Neumann boundary conditions possesses the solution

$$\phi(x) = E_\gamma(-\lambda x^\gamma) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(n\gamma+1)} x^{n\gamma},$$

where λ_i are the solutions of Eg. (5.2). This solution ϕ doesn't belong to the space $C^2[0,1]$. That is the space $C^2[0,1]$ is too restrictive, and a wider space, such as $C^2_{-1}[0,1]$, should be considered for the case of Neumann and Robin boundary conditions.

As a second example we consider the nonlinear fractional eigenvalue problem

$$D^{\delta}_{0+}(p(x)y') + q(x)y = -\lambda \sinh(y), \ \ 0 < \delta < 1, \quad 0 < x < 1, \tag{5.3}$$

$y \in C^2_{-1}[0,1]$, subject to the boundary conditions of Dirichlet or Robin type. Since $k(x,y) = \sinh(y)$ satisfies $k(x,0) = 0$, and $\frac{\partial k}{\partial y} = \cosh(y) > 1$, then Corollary 4.1 implies the following. If $q(x) \geqslant 0$, we have

$$-q(x) \leqslant \frac{-q(x)}{\cosh(y)} < 0,$$

and thus

$$\lambda \geqslant \inf_{[0,1]}\{-q(x)\}.$$

If $q(x) \leqslant 0$, we have

$$0 < \frac{-q(x)}{\cosh(y)} < -q(x),$$

and thus

$$\lambda \geqslant 0.$$

If $q(x) = 0$, then $\lambda = 0$ and the only solution of Eq. (5.3) is, $y(x) = 0$, and thus the problem has no eigenvalues.

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