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LYAPUNOV INEQUALITIES FOR TWO KINDS OF HIGHER–ORDER MULTI–POINT FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. We obtain Lyapunov inequality and Hartman-Wintner-type inequality for two higher-order multi-point fractional boundary value problems. The technique of order-reduction and the properties of Green's function are important to our results. As applications, we discuss the eigenvalue problem and the real zeros for Mittag-Leffler function.

1. Introduction

In 1907, Lyapunov [1] established following result.

THEOREM 1.1. *If the boundary value problem (BVP for short)*

$$\begin{cases} y''(t)+q(t)y(t)=0, & a<t<b, \\ y(a)=y(b)=0 \end{cases}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|\mathrm{d}s > \frac{4}{b-a}. \tag{1}$$

Inequality (1) is known in literature as Lyapunov inequality. Later, Wintner [2] improved (1) by replacing $|q(s)|$ to $q^+(s)=\max\{q(s),0\}$, i.e., he obtained the following inequality (known as Lyapunov-type inequality),

$$\int_a^b q^+(s)\mathrm{d}s > \frac{4}{b-a}. \tag{2}$$

In [3], Hartman and Wintner obtained a more general inequality (known as Hartman-Wintner-type inequality),

$$\int_a^b (s-a)(b-s)q^+(s)\mathrm{d}s > b-a. \tag{3}$$

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Since $(s-a)(b-s) \leqslant \frac{(b-a)^2}{4}$ for any $s \in [a,b]$, (3) implies (2).

Since the Lyapunov inequality and its generalizations have been found many applications in the study of oscillation theory, asymptotic theory, estimates for intervals of disconjugacy and bounds for eigenvalue, there have been many improvements and generalizations for integer-order (two-order as well as higher-order) BVP have appeared in literature, and here we omit these detailed conclusions but only refer the reader to a summary reference [4] given by Tiryaki in 2010, in which research results about Lyapunov-type inequalities were summarized. Some more recent results about integer-order BVP, we refer the reader to [5–7] and their references.

Recently, the study of Lyapunov inequality for fractional boundary value problem (FBVP for short) has begun in which a fractional derivative (Riemann-Liouville derivative ${}_a^R\mathscr{D}^\nu$ or Caputo derivative ${}_a^C\mathscr{D}^\nu$) is used instead of the classical ordinary derivative in differential equation. Such work was initiated by Ferreira in 2013 and he firstly obtained a Lyapunov inequality for a FBVP in which the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as following.

THEOREM 1.2. [8] *If a nontrivial continuous solution of FBVP*

$$\begin{cases} ({}_a^R\mathscr{D}^\nu y)(t) + q(t)y(t) = 0, & a < t < b, \quad 1 < \nu \leqslant 2, \\ y(a) = y(b) = 0 \end{cases}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|\mathrm{d}s > \Gamma(\nu)\left(\frac{4}{b-a}\right)^{\nu-1}.$$

Next in 2014, Ferreira obtained a Lyapunov inequality for a FBVP in which the differential equation depends on the Caputo fractional derivative, the main result is as following.

THEOREM 1.3. [9] *If a nontrivial continuous solution of FBVP*

$$\begin{cases} ({}_a^C\mathscr{D}^\nu y)(t) + q(t)y(t) = 0, & a < t < b, \quad 1 < \nu \leqslant 2, \\ y(a) = y(b) = 0 \end{cases} \tag{4}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|\mathrm{d}s > \frac{\nu^\nu \Gamma(\nu)}{[(\nu-1)(b-a)]^{\nu-1}}. \tag{5}$$

Following [8, 9], some results concerning Lyapunov-type inequalities for fractional boundary value problems appeared, we refer the reader to Jleli and Samet [10] for a Caputo fractional differential with mixed boundary condition, Jleli, Ragoub and Samet [11] for a Caputo fractional differential with Robin boundary conditions, Rong and Bai [12] for a Caputo fractional differential with Caputo fractional boundary conditions, Wang, Liang and Xia [13] fora Caputo fractional differential equation with

Sturm-liouville boundary conditions, Ferreira [14] for a sequential FBVP, Agarwal and Özbekler [15] for a FBVP with a half-linear term, Chidouh and Torresb [16] for a FBVP with a nonlinear term.

In the above cited works [8–16], we can see that the results are all about lower-order FBVP. As far as we known, few results appeared for higher-order FBVP. Then in 2015, O'Regan and Samet gave the following result for a higher-order two-point FBVP with Riemann-Liouville derivative.

THEOREM 1.4. [17] *If a nontrivial continuous solution of FBVP*

$$\begin{cases} ({}_a^R\mathscr{D}^{\nu}y)(t)+q(t)y(t)=0, \quad a<t<b, \quad 3<\nu\leqslant 4, \\ y(a)=y'(a)=y''(a)=y''(b)=0 \end{cases} \tag{6}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|\mathrm{d}s > \frac{\Gamma(\nu)(\nu-2)^{\nu-2}}{2(\nu-3)^{\nu-3}(b-a)^{\nu-1}}.$$

The most difficulty in [17] is to prove the maximum value of the Green's function corresponding to (6).

Recently, Arifi et al. [18] studied a Lyapunov inequality for the following higher-order two-point fractional p-Laplacian equation:

$$\begin{cases} ({}_a^R\mathscr{D}^{\gamma}\Phi_p({}_a^R\mathscr{D}^{\alpha})y)(t)+q(t)\Phi_p(y)(t)=0, \quad a<t<b, \quad 2<\alpha\leqslant 3,\ 1<\gamma\leqslant 2, \\ y(a)=y'(a)=y'(b)=0, ({}_a^R\mathscr{D}^{\alpha}y)(a)=({}_a^R\mathscr{D}^{\alpha}y)(b)=0. \end{cases} \tag{7}$$

Using the boundary value conditions, (7) is easily transformed to a lower-order problem in [18].

In a recent paper [19], Cabrera, Sadarangani and Samet gave the following result for a higher-order multi-point FBVP with Riemann-Liouville derivative.

THEOREM 1.5. [19] *If there exists nontrivial solution for*

$$\begin{cases} ({}_a^R\mathscr{D}^{\nu}y)(t)+q(t)y(t)=0, \quad a<t<b, \quad 2<\nu\leqslant 3, \\ y(a)=y'(a)=0, y'(b)=\beta y(\xi), \end{cases}$$

where q is a real and continuous function, then

$$\int_a^b |q(s)|\mathrm{d}s > \frac{\Gamma(\nu)(\nu-1)^{\nu-1}}{(\nu-2)^{\nu-2}(b-a)^{\nu-1}\left(1+\frac{\beta(b-a)^{\nu-1}}{(\nu-1)(b-a)^{\nu-2}-\beta(\xi-a)^{\nu-1}}\right)}.$$

As far as we known, [19] is the first paper to get Lyapunov inequality for a higher-order multi-point FBVP.

For other recent research results about Lyapunov inequality for differential equation, see [20] and [21].

Motivated by [8–19], we will study Lyapunov inequality and Hartman-Wintner-type inequality for

$$\begin{cases} ({}_a^C\mathscr{D}^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b, \quad 2 < \beta \leqslant 3, \\ y'(a) = y'(b) = y(c) = 0, \;\; a < b, \; c \in [a,b] \end{cases} \tag{8}$$

and

$$\begin{cases} ({}_a^C\mathscr{D}^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b_2, \quad 2 < \beta \leqslant 3, \\ y'(a) = y(b_1) = y(b_2) = 0, \; a \leqslant b_1 < b_2, \end{cases} \tag{9}$$

where q is a real and continuous function.

Just as mentioned in [9], the classical strategy known so far in literature to get the Lyapunov inequality for FBVP, which was also used in [8–16] for lower-order FBVP as well as in [17–19] for higher-order FBVP, is to find the maximum value of a Green's function corresponding to the original FBVP. But the strategy is not necessarily valid for our problem since the Green's function corresponding to (8) or (9) is too complicated owning to the higher-order and multi-point. In this paper, we will reform the classical strategy and avoid getting a complicated Green's function, but focus on a technique by which the higher-order multi-point FBVP problem reduce to a low-order two-point one. Accurate properties for the Green's function corresponding to a low-order FBVP are the basis of our paper. By the way, we give a more succinct proof of the result in [9] as an extra result.

For convenience, we recall some basic concepts on fractional calculus and the definitions can be found in literature such as [22] and [23].

DEFINITION 1.1. Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α for f is defined by

$$({}_a\mathscr{I}^\alpha f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \; t \in [a,b], \; \alpha > 0, \\ f(t), \; t \in [a,b], \; \alpha = 0, \end{cases}$$

where $\Gamma(\alpha)$ is the Gamma Function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \;\; \alpha > 0$.

DEFINITION 1.2. Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Caputo fractional derivative of order α for f is defined by

$$({}_a^C\mathscr{D}^\alpha f)(t) = \begin{cases} ({}_a\mathscr{I}^{m-\alpha}(f^{(m)}))(t), \; t \in [a,b], \; \alpha > 0, \\ f(t), \; t \in [a,b], \; \alpha = 0, \end{cases}$$

where m is the smallest integer greater or equal than α.

REMARK 1.1. For $\alpha \in \mathbb{N}$ with $\alpha \geqslant 0$, we have ${}_a^C\mathscr{D}^\alpha f = f^{(\alpha)}$ and ${}_a^C\mathscr{D}^{\alpha+\beta} f = {}_a^C\mathscr{D}^\beta f^{(\alpha)}$ for any $\beta \geqslant 0$.

The paper is organized as following. In Section 2, we give some key preliminary conclusions. Main results are given in Section 3 and we give some applications of our results in Section 4.

2. Preliminaries

LEMMA 2.1. [9] *$y \in C[a,b]$ is a solution of FBVP (4) if and only if y satisfies the integral equation*

$$y(t) = \int_a^b G(t,s)q(s)y(s)\mathrm{d}s,$$

where $G(t,s)$ which is named as the Green's function of FBVP (4) is continuous on $[a,b]\times[a,b]$ and defined as

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t-a}{b-a}(b-s)^{\nu-1} - (t-s)^{\nu-1}, & a \leqslant s < t \leqslant b, \\ \frac{t-a}{b-a}(b-s)^{\nu-1}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{10}$$

LEMMA 2.2. *Let*

$$H(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t-a}{b-a} - \left(\frac{t-s}{b-s}\right)^{\nu-1}, & a \leqslant s < t \leqslant b, \\ \frac{t-a}{b-a}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{11}$$

i.e., $G(t,s) = (b-s)^{\nu-1}H(t,s)$. Then $H(t,s)$ is continuous on $[a,b)\times[a,b]$; Furthermore, we have

i) $\max\limits_{(t,s)\in[a,b]\times[a,b]} |H(t,s)| = \frac{1}{\Gamma(\nu)}$;

ii) $\int_a^b \max\limits_{s\in[a,b]} |H(t,s)|dt = \frac{(b-a)}{\Gamma(\nu)}\left\{\frac{2-\nu}{\nu}2^{\frac{2}{\nu-2}} + \frac{1}{2}\right\}$ *(denote* $2^{\frac{2}{\nu-2}}|_{\nu=2} = \lim\limits_{\nu\to 2^-} 2^{\frac{2}{\nu-2}} = 0$*);*

iii) $\max\limits_{(t,s)\in[a,b]\times[a,b]} |G(t,s)| = \frac{(\nu-1)^{\nu-1}(b-a)^{\nu-1}}{\nu^\nu \Gamma(\nu)}$;

iv) $\int_a^b \max\limits_{s\in[a,b]} |G(t,s)|dt = \frac{(b-a)^\nu}{\Gamma(\nu)}\left\{\frac{s_0^\nu}{\nu} - \frac{s_0^2}{2} + \frac{(1-s_0)^\nu}{\nu} - \frac{(1-s_0)^{\nu+1}}{\nu+1}\right\}$*, where s_0 is the unique solution of $1 - s^{2-\nu} - s^{2-\nu}(1-s)^{\nu-1} = 0$ on $(0,1)$ when $\nu \neq 2$ and $s_0 = 0$ when $\nu = 2$.*

Proof. It is easy to see $H(t,s)$ is continuous on $[a,b)\times[a,b]$.

We prove i). Clearly,

$$\max_{s\in[a,b]} |H(t,s)| = \frac{1}{\Gamma(\nu)} \max\left\{\left(\frac{t-a}{b-a}\right)^{\nu-1} - \frac{t-a}{b-a},\ \frac{t-a}{b-a}\right\},\ \forall t \in [a,b]. \tag{12}$$

By (12), we know

$$\begin{aligned} \max_{(t,s)\in[a,b]\times[a,b]} |H(t,s)| &= \max_{t\in[a,b]} \max_{s\in[a,b]} |H(t,s)| \\ &= \max_{t\in[a,b]} \frac{1}{\Gamma(\nu)} \max\left\{\left(\frac{t-a}{b-a}\right)^{\nu-1} - \frac{t-a}{b-a},\ \frac{t-a}{b-a}\right\} \\ &= \frac{1}{\Gamma(\nu)} \max\left\{\max_{t\in[a,b]}\left\{\left(\frac{t-a}{b-a}\right)^{\nu-1} - \frac{t-a}{b-a}\right\},\ \max_{t\in[a,b]} \frac{t-a}{b-a}\right\} = \frac{1}{\Gamma(\nu)}. \end{aligned}$$

For ii), by (12), we have

$$\begin{aligned}\int_a^b \max_{s\in[a,b]} |H(t,s)|dt &= \frac{1}{\Gamma(v)}\int_a^b \max\left\{\left(\frac{t-a}{b-a}\right)^{v-1}-\frac{t-a}{b-a},\ \ \frac{t-a}{b-a}\right\}dt\\ &= \frac{b-a}{\Gamma(v)}\int_0^1 \max\left\{s^{v-1}-s,\ \ s\right\}\mathrm{d}s\\ &= \frac{(b-a)}{\Gamma(v)}\left(\int_0^{2^{\frac{1}{v-2}}}(s^{v-1}-s)\mathrm{d}s+\int_{2^{\frac{1}{v-2}}}^1 s\mathrm{d}s\right)\\ &= \frac{(b-a)}{\Gamma(v)}\left(\frac{2-v}{v}2^{\frac{2}{v-2}}+\frac{1}{2}\right).\end{aligned}$$

Next, we prove iii). Firstly, we prove

$$\max_{s\in[a,b]}|G(t,s)| = \frac{1}{\Gamma(v)}\max\left\{(t-a)^{v-1}-\frac{(t-a)}{(b-a)^{2-v}},\ \ \frac{(t-a)(b-t)^{v-1}}{b-a}\right\},\ \forall t\in[a,b]. \tag{13}$$

For given $t\in[a,b]$,

$$G_s'(t,s) = \frac{1}{\Gamma(v)}\begin{cases}(v-1)\left[\frac{1}{(t-s)^{2-v}}-\frac{t-a}{(b-a)(b-s)^{2-v}}\right]>0, & a<s<t,\\ -\frac{(v-1)(t-a)}{(b-a)(b-s)^{2-v}}<0, & t<s<b,\end{cases}$$

which means

$$\begin{aligned}\max_{s\in[a,b]}|G(t,s)| &= \frac{1}{\Gamma(v)}\max\{|G(t,a)|,\ \ |G(t,t)|,\ \ |G(t,b)|\}\\ &= \frac{1}{\Gamma(v)}\max\{|G(t,a)|,\ \ |G(t,t)|\}\\ &= \frac{1}{\Gamma(v)}\max\left\{(t-a)^{v-1}-\frac{(t-a)}{(b-a)^{2-v}},\ \ \frac{(t-a)(b-t)^{v-1}}{b-a}\right\}\end{aligned}$$

For convenience, we define two nonnegative and continuous functions g and f as following,

$$g(t) = \frac{(t-a)(b-t)^{v-1}}{b-a},\ \ t\in[a,b];$$

$$h(t) = (t-a)^{v-1}-\frac{(t-a)}{(b-a)^{2-v}},\ \ t\in[a,b].$$

Then,

$$g'(t) = \frac{v(b-t)^{v-2}\left[(a+\frac{b-a}{v})-t\right]}{b-a} = \begin{cases}\leqslant 0, & t\geqslant a+\frac{b-a}{v},\\ \geqslant 0, & t\leqslant a+\frac{b-a}{v};\end{cases}$$

$$h'(t) = \frac{v-1}{(t-a)^{2-v}}-\frac{1}{(b-a)^{2-v}} = \begin{cases}\leqslant 0, & t\geqslant a+(b-a)(v-1)^{\frac{1}{2-v}},\\ \leqslant 0, & t\leqslant a+(b-a)(v-1)^{\frac{1}{2-v}}.\end{cases}$$

Since $a+\frac{b-a}{v}\in[\frac{a+b}{2},\ b)$, we get

$$\max_{t\in[a,b]} g(t) = g\left(a+\frac{b-a}{v}\right) = \frac{(v-1)^{v-1}}{v^v}(b-a)^{v-1}; \tag{14}$$

Since $a+(b-a)(v-1)^{\frac{1}{2-v}} \in (a,\, b]$, we get

$$\max_{t\in[a,b]} h(t) = h\left(a+(b-a)(v-1)^{\frac{1}{2-v}}\right) = (v-1)^{\frac{v-1}{2-v}}(2-v)(b-a)^{v-1}. \tag{15}$$

From (14) and (15), we get

$$\begin{aligned}
\max_{(t,s)\in[a,b]\times[a,b]} |G(t,s)| &= \max_{t\in[a,b]} \max_{s\in[a,b]} |G(t,s)| \\
&= \max_{t\in[a,b]} \tfrac{1}{\Gamma(v)} \max\left\{(t-a)^{v-1} - \tfrac{(t-a)}{(b-a)^{2-v}},\ \ \tfrac{(t-a)(b-t)^{v-1}}{b-a}\right\} \\
&= \tfrac{1}{\Gamma(v)} \max\left\{\max_{t\in[a,b]} h(t),\ \ \max_{t\in[a,b]} g(t)\right\} \\
&= \tfrac{1}{\Gamma(v)} \max\left\{(v-1)^{\frac{v-1}{2-v}}(2-v)(b-a)^{v-1},\ \ \tfrac{(v-1)^{v-1}}{v^v}(b-a)^{v-1}\right\} \\
&= \tfrac{(v-1)^{v-1}(b-a)^{v-1}}{v^v\Gamma(v)} \max\left\{(v-1)^{\frac{(v-1)^2}{2-v}} v^v(2-v),\ 1\right\}.
\end{aligned}$$

We next show $\max\left\{(v-1)^{\frac{(v-1)^2}{2-v}} v^v(2-v),\, 1\right\} = 1$. In fact, let $k(t) = t^t(2-t),\ \ t \in [1,2]$, it is easy to prove $k'(t) \leqslant 0$ which means that $k(t) \leqslant k(1) = 1$ for any $t \in [1,2]$, and then we have $v^v(2-v) < 1$ since $v \in (1,2]$. Combining $(v-1)^{\frac{(v-1)^2}{2-v}} < 1$, we get $\max\left\{(v-1)^{\frac{(v-1)^2}{2-v}} v^v(2-v),\ \ 1\right\} = 1$. Therefore

$$\max_{(t,s)\in[a,b]\times[a,b]} |G(t,s)| = \frac{(v-1)^{v-1}(b-a)^{v-1}}{v^v\Gamma(v)}.$$

In the end, we prove iv). When $v \neq 2$, by (13), we know

$$\begin{aligned}
\int_a^b \max_{s\in[a,b]} |G(t,s)|dt &= \tfrac{1}{\Gamma(v)} \int_a^b \max\left\{(t-a)^{v-1} - \tfrac{(t-a)}{(b-a)^{2-v}},\ \ \tfrac{(t-a)(b-t)^{v-1}}{b-a}\right\} dt \\
&= \tfrac{(b-a)^v}{\Gamma(v)} \int_0^1 \max\{s^{v-1} - s,\ s(1-s)^{v-1}\}\mathrm{d}s \\
&= \tfrac{(b-a)^v}{\Gamma(v)} \int_0^1 s^{v-1} \max\{1-s^{2-v},\ s^{2-v}(1-s)^{v-1}\}\mathrm{d}s.
\end{aligned}$$

Let

$$\varphi(s) = 1 - s^{2-v} - s^{2-v}(1-s)^{v-1},\ \ s \in [0,1].$$

We conclude that $\varphi(s)$ has only one zero point s_0 on $(0,1)$. In fact, from $\varphi(0) = 1$ and $\varphi(\frac{1}{2}) = \frac{1}{2} - \left(\frac{1}{2}\right)^{2-v} < 0$, we know the existence of the zero point. The uniqueness is obvious from $\varphi(0) = 1, \varphi(1) = 0$ and

$$\varphi''(s) = (2-v)(v-1)s^{-v}\left[1+(1-s)^{v-1} + 2s(1-s)^{v-2} + s^2(1-s)^{v-3}\right] > 0,\ \ s \in (0,1).$$

Considering $\varphi(0) = 1$, we have

$$\begin{aligned}
\int_a^b \max_{s\in[a,b]} |G(t,s)|dt &= \tfrac{(b-a)^v}{\Gamma(v)} \left\{\int_0^{s_0} s^{v-1}(1-s^{2-v})\mathrm{d}s + \int_{s_0}^1 s^{v-1}s^{2-v}(1-s)^{v-1}\}\mathrm{d}s\right\} \\
&= \tfrac{(b-a)^v}{\Gamma(v)} \left\{\tfrac{s_0^v}{v} - \tfrac{s_0^2}{2} + \tfrac{(1-s_0)^v}{v} - \tfrac{(1-s_0)^{v+1}}{v+1}\right\},
\end{aligned} \tag{16}$$

where s_0 is the unique solution of $1-s^{2-\nu}-s^{2-\nu}(1-s)^{\nu-1}=0$ on $(0,1)$.

When $\nu=2$, $G(t,s)$ reduces to

$$G(t,s)=\begin{cases}\frac{(s-a)(b-t)}{b-a}, & a\leqslant s<t\leqslant b,\\ \frac{(t-a)(b-s)}{b-a}, & a\leqslant t\leqslant s\leqslant b,\end{cases}$$

and thus $\int_a^b \max\limits_{s\in[a,b]}|G(t,s)|dt=\frac{(b-a)^2}{6}$, which can be unified in (16) if we choose $s_0=0$. □

REMARK 2.1. iii) was already proved in [9]. But our proof is more brief and clear than that in [9].

3. Main results

THEOREM 3.1. *If FBVP (4) has a nontrivial solution, then Lyapunov inequality*

$$\int_a^b |q(s)|\mathrm{d}s\geqslant\frac{\nu^\nu\Gamma(\nu)}{(\nu-1)^{\nu-1}(b-a)^{\nu-1}} \tag{17}$$

and Hartman-Wintner-type inequality

$$\int_a^b (b-s)^{\nu-1}|q(s)|\mathrm{d}s\geqslant\Gamma(\nu) \tag{18}$$

hold.

Proof. Let $E=C[a,b]$ be the Banach space endowed with norm $\|y\|=\max\limits_{t\in[a,b]}|y(t)|$. It follows from Lemma 2.1 that a nontrivial solution y to FBVP (4) satisfies the integral equation

$$y(t)=\int_a^b G(t,s)q(s)y(s)\mathrm{d}s,\ \ t\in[a,b],$$

where $G(t,s)$, and the following $H(t,s)$ are defined as in (10) and (11). Then,

$$|y(t)|\leqslant\int_a^b |G(t,s)||q(s)||y(s)|\mathrm{d}s,\ \ t\in[a,b]. \tag{19}$$

On one hand, an application of iii) in Lemma 2.2 for (19) yields

$$\|y\|\leqslant\int_a^b \max_{(t,s)\in[a,b]\times[a,b]}|G(t,s)||q(s)|\mathrm{d}s\|y\|=\frac{(\nu-1)^{\nu-1}(b-a)^{\nu-1}}{\nu^\nu\Gamma(\nu)}\|y\|\int_a^b|q(s)|\mathrm{d}s,$$

which means

$$\int_a^b |q(s)|\mathrm{d}s\geqslant\frac{\nu^\nu\Gamma(\nu)}{(\nu-1)^{\nu-1}(b-a)^{\nu-1}}.$$

On the other hand, an application of i) in Lemma 2.2 for (19) yields

$$\begin{aligned}\|y\| &\leqslant \int_a^b |H(t,s)|(b-s)^{\nu-1}|q(s)||y(s)|\mathrm{d}s \\ &\leqslant \int_a^b \max_{(t,s)\in[a,b]\times[a,b]} |H(t,s)|(b-s)^{\nu-1}|q(s)|\mathrm{d}s\|y\| \\ &= \frac{1}{\Gamma(\nu)}\int_a^b (b-s)^{\nu-1}|q(s)|\mathrm{d}s\|y\|,\end{aligned}$$

which means

$$\int_a^b (b-s)^{\nu-1}|q(s)|\mathrm{d}s \geqslant \Gamma(\nu). \quad \square$$

REMARK 3.1. Lyapunov inequality (17) for FBVP (4) was the main result in [9], we still state it here only to show that our proof is so brief comparing with the complicate proof in [9]. Most of all, we get a Hartman-Wintner-type inequality (18) for FBVP (4) which is not in [9].

Following, we denote

$$\begin{aligned}\triangle &= \left\{\frac{s_0^\nu}{\nu} - \frac{s_0^2}{2} + \frac{(1-s_0)^\nu}{\nu} - \frac{(1-s_0)^{\nu+1}}{\nu+1}\right\}\Big|_{\nu=\beta-1} \\ &= \frac{s_0^{\beta-1}}{\beta-1} - \frac{s_0^2}{2} + \frac{(1-s_0)^{\beta-1}}{\beta-1} - \frac{(1-s_0)^\beta}{\beta},\end{aligned} \tag{20}$$

where s_0 is the unique solution of $1-s^{3-\beta}-s^{3-\beta}(1-s)^{\beta-2}=0$ on $(0,1)$ when $\beta\neq 3$ and $s_0=0$ when $\beta=3$.

THEOREM 3.2. *If FBVP (8) has a nontrivial solution, then Lyapunov inequality*

$$\int_a^b |q(s)|\mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b-a)^{\beta-1}\triangle} \tag{21}$$

and Hartman-Wintner-type inequality

$$\int_a^b (b-s)^{\beta-2}|q(s)|\mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b-a)\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}. \tag{22}$$

hold.

Proof. Let $E=C[a,b]$ be the Banach space endowed with norm $\|y\|=\max_{t\in[a,b]}|y(t)|$. Assume that $y(t)$ is a nontrivial solution to FBVP (8), then we have

$$\begin{cases} ({}_a^C D^\beta y)(t)+q(t)y(t)=0, \quad a<t<b, \quad 2<\beta\leqslant 3, \\ y'(a)=y'(b)=y(c)=0, \ \ a<b, \ c\in[a,b]. \end{cases} \tag{23}$$

By the definition of Caputo fractional derivative, we know $({}_a^C D^\beta y)(t)=({}_a^C D^{\beta-1}(y'))(t)$, thus we rewrite (23) as

$$\begin{cases} ({}_a^C D^{\beta-1}(y'))(t)+q(t)y(t)=0, \quad a<t<b \quad 1<\beta-1\leqslant 2, \\ (y')(a)=(y')(b)=0. \end{cases} \tag{24}$$

Then from Lemma 2.1 we know

$$y'(t) = \int_a^b G(t,s)q(s)y(s)\mathrm{d}s, \ \ t \in [a,b], \tag{25}$$

where $G(t,s)$, and the following $H(t,s)$ are defined as in (10) and (11) with parameter v changing to $(\beta-1)$.

For any $t \in [a,b]$, since $c \in [a,b]$ and $y(c)=0$, so integrating (25) from c to t, we get

$$y(t) = \int_c^t \left[\int_a^b G(u,s)q(s)y(s)\mathrm{d}s\right]\mathrm{d}u.$$

Taking the absolute value of above equality, we get

$$\begin{aligned} |y(t)| &= |\textstyle\int_c^t \left[\int_a^b G(u,s)q(s)y(s)\mathrm{d}s\right]\mathrm{d}u| \\ &\leqslant \textstyle\int_a^b \left[\int_a^b |G(u,s)||q(s)|\mathrm{d}s\right]\mathrm{d}u\|y\|. \end{aligned} \tag{26}$$

Now, exchanging integral order for (26), and then an application of iv) in Lemma 2.2, we have

$$\begin{aligned} |y(t)| &\leqslant \textstyle\int_a^b \left[\int_a^b |G(u,s)|\mathrm{d}u\right]|q(s)|\mathrm{d}s\|y\| \\ &\leqslant \textstyle\int_a^b \left[\int_a^b \max\limits_{s\in[a,b]} |G(u,s)|\mathrm{d}u\right]|q(s)|\mathrm{d}s\|y\| \\ &= \tfrac{(b-a)^{\beta-1}\triangle}{\Gamma(\beta-1)} \textstyle\int_a^b |q(s)|\mathrm{d}s\|y\|, \end{aligned}$$

and thus,

$$\int_a^b |q(s)|\mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b-a)^{\beta-1}\triangle}. \tag{27}$$

Exchanging integral order for (26), and then an application of ii) in Lemma 2.2 , we have

$$\begin{aligned} |y(t)| &\leqslant \textstyle\int_a^b \left[\int_a^b |H(u,s)|(b-s)^{\beta-2}|q(s)|\mathrm{d}s\right]\mathrm{d}u\|y\| \\ &= \textstyle\int_a^b \left[\int_a^b |H(u,s)|\mathrm{d}u\right](b-s)^{\beta-2}|q(s)|\mathrm{d}s\|y\| \\ &\leqslant \textstyle\int_a^b \left[\int_a^b \max\limits_{s\in[a,b]} |H(u,s)|\mathrm{d}u\right](b-s)^{\beta-2}|q(s)|\mathrm{d}s\|y\| \\ &= \tfrac{(b-a)}{\Gamma(\beta-1)}\left\{\tfrac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\tfrac{1}{2}\right\}\textstyle\int_a^b (b-s)^{\beta-2}|q(s)|\mathrm{d}s\|y\|, \end{aligned}$$

and thus,

$$\int_a^b (b-s)^{\beta-2}|q(s)|\mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b-a)\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}. \quad \square \tag{28}$$

THEOREM 3.3. *If FBVP (9) has a nontrivial solution, then Lyapunov inequality*

$$\int_a^{b_2} |q(s)|\mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b_2-a)^{\beta-1}\triangle} \tag{29}$$

and Hartman-Wintner-type inequality

$$\int_a^{b_2} (b_2 - s)^{\beta-2} |q(s)| \mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b_2-a)\left(\frac{3-\beta}{\beta-1} 2^{\frac{2}{\beta-3}} + \frac{1}{2}\right)}. \tag{30}$$

hold.

Proof. Let $E = C[a, b_2]$ be the Banach space endowed with norm $\|y\| = \max_{t \in [a,b_2]} |y(t)|$. Assume that $y(t)$ is a nontrivial solution to FBVP (9), then we have

$$\begin{cases} ({}_a^C D^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b_2, \quad 2 < \beta \leqslant 3, \\ y'(a) = y(b_1) = y(b_2) = 0, \ a \leqslant b_1 < b_2. \end{cases} \tag{31}$$

From Rolle's Theorem, there exists $b \in (b_1, b_2)$ such that $y'(b) = 0$. By the definition of Caputo fractional derivative, we know $({}_a^C D^\beta y)(t) = ({}_a^C D^{\beta-1}(y'))(t)$, thus we rewrite (31) as

$$\begin{cases} ({}_a^C D^{\beta-1}(y'))(t) + q(t)y(t) = 0, \quad a < t < b \quad 1 < \beta - 1 \leqslant 2, \\ (y')(a) = (y')(b) = 0. \end{cases} \tag{32}$$

Then from Lemma 2.1 we know

$$y'(t) = \int_a^b G(t,s) q(s) y(s) \mathrm{d}s, \ \ t \in [a,b]. \tag{33}$$

where $G(t,s)$, and the following $H(t,s)$ are defined as in (10) and (11) with parameters v changing to $(\beta - 1)$.

For any $t \in [a,b]$, since $b_1 \in [a,b]$ and $y(b_1) = 0$, so integrating (33) from b_1 to t, we get

$$y(t) = \int_{b_1}^t \left[\int_a^b G(u,s) q(s) y(s) \mathrm{d}s \right] \mathrm{d}u.$$

Taking the absolute value of above equality, we get

$$\begin{aligned} |y(t)| &= | \textstyle\int_{b_1}^t \left[\int_a^b G(u,s) q(s) y(s) \mathrm{d}s \right] \mathrm{d}u | \\ &\leqslant \textstyle\int_a^b \left[\int_a^b |G(u,s)| |q(s)| \mathrm{d}s \right] \mathrm{d}u \|y\|. \end{aligned} \tag{34}$$

Next, just similar to the proof in Theorem 3.2, we exchange integral order for (34). Then we firstly apply iv) of Lemma 2.2 to get

$$\int_a^b |q(s)| \mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b-a)^{\beta-1} \triangle}; \tag{35}$$

we secondly apply ii) of Lemma 2.2 to get

$$\int_a^b (b-s)^{\beta-2} |q(s)| \mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b-a)\left(\frac{3-\beta}{\beta-1} 2^{\frac{2}{\beta-3}} + \frac{1}{2}\right)}. \tag{36}$$

Considering $a \leqslant b_1 < b < b_2$, inequality (35) and (36) still hold if we substitute b with b_2 which means we have

$$\int_a^{b_2} |q(s)|\mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b_2-a)^{\beta-1}\triangle};$$

$$\int_a^{b_2} (b_2-s)^{\beta-2}|q(s)|\mathrm{d}s \geqslant \frac{\Gamma(\beta-1)}{(b_2-a)\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}. \quad \square$$

REMARK 3.2. If we take $c = a$ in (8), by Theorem 3.2, we can get Lyapunov inequality and Hartman-Wintner-type inequality for the following higher-order two-point FBVP,

$$\begin{cases} ({}_a^C\mathscr{D}^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b, \quad 2 < \beta \leqslant 3, \\ y(a) = y'(a) = y'(b) = 0. \end{cases} \tag{37}$$

REMARK 3.3. If we take $b_1 = a$ in (9), by Theorem 3.3, we can get Lyapunov inequality and Hartman-Wintner-type inequality for the following higher-order two-point FBVP,

$$\begin{cases} ({}_a^C\mathscr{D}^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b, \quad 2 < \beta \leqslant 3, \\ y(a) = y'(a) = y(b) = 0. \end{cases} \tag{38}$$

4. Applications

In this section, we will discuss the eigenvalue problem and the real zeros for Mittag-Leffler function as applications of our main results.

4.1. Eigenvalue problem

COROLLARY 4.1. *If λ is an eigenvalue of*

$$\begin{cases} ({}_a^C\mathscr{D}^\beta y)(t) + \lambda y(t) = 0, \quad a < t < b, \quad 2 < \beta \leqslant 3, \\ y'(a) = y'(b) = y(c) = 0, \ a < b, \ c \in [a,b], \end{cases} \tag{39}$$

then

$$|\lambda| \geqslant \frac{\Gamma(\beta-1)}{(b-a)^\beta} \times \max\left\{\frac{1}{\triangle}, \frac{\beta-1}{\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}\right\}; \tag{40}$$

If λ is an eigenvalue of

$$\begin{cases} ({}_a^C\mathscr{D}^\beta y)(t) + \lambda y(t) = 0, \quad a < t < b_2, \quad 2 < \beta \leqslant 3, \\ y'(a) = y(b_1) = y(b_2) = 0, \ a \leqslant b_1 < b_2, \end{cases} \tag{41}$$

then

$$|\lambda| \geqslant \frac{\Gamma(\beta-1)}{(b_2-a)^\beta} \times \max\left\{\frac{1}{\triangle}, \frac{\beta-1}{\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}\right\}. \tag{42}$$

Proof. Let λ be an eigenvalue of FBVP (39) and y_λ be the eigenfunction corresponding to λ. By (21) and (22) in Theoren 3.2, we get

$$|\lambda| \geqslant \frac{\Gamma(\beta-1)}{(b-a)^\beta \triangle} \quad \text{and} \quad |\lambda| \geqslant \frac{(\beta-1)\Gamma(\beta-1)}{(b-a)^\beta\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)},$$

which means that (40) holds. Similarly, (42) can be obtained from (29) and (30) in Theoren 3.3. $\square$

4.2. Real zeros for Mittag-Leffler function

In order to discuss Mittag-Leffler function problem, we first let $a = b_1 = 0$ and $b_2 = 1$ in (41), and then we get the following corollary by Corollary 4.1.

COROLLARY 4.2. *If λ is an eigenvalue of*

$$\begin{cases} ({}_0^C\mathscr{D}^\beta y)(t) + \lambda y(t) = 0, \quad 0 < t < 1, \quad 2 < \beta \leqslant 3, \\ y(0) = y'(0) = y(1) = 0, \end{cases} \tag{43}$$

then

$$|\lambda| \geqslant \max\left\{\frac{\Gamma(\beta-1)}{\triangle}, \frac{\Gamma(\beta)}{\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}\right\}. \tag{44}$$

Consider now the two parameter Mittag-Leffler function

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+\gamma)}, \; \beta > 0, \; \gamma > 0, \; z \in R. \tag{45}$$

Obviously, $E_{\beta,\gamma}(z) > 0$ for all $z \geqslant 0$. Hence, the real zeros of $E_{\beta,\gamma}(z)$, if they exist, must be negative real numbers. In the following, we will use Corollary 4.2 to obtain an interval in which the Mittag-Leffler function (45) with $2 < \beta \leqslant 3$, $\gamma = 3$ has no real zeros.

COROLLARY 4.3. *Let $2 < \beta \leqslant 3$. Then, the Mittag-Leffler function $E_{\beta,3}(z)$ has no real zeros for* $z \in \left(-\max\left\{\frac{\Gamma(\beta-1)}{\triangle}, \frac{\Gamma(\beta)}{\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}\right\}, \; +\infty\right)$.

Proof. We firstly recall some elementary knowledges for eigenvalue problem (43). By Theorem 1 in [24], the general solution $y(t)$ satisfying the fractional differential equation in (43) is

$$y(t) = AE_{\beta,1}(-\lambda t^{\nu}) + BtE_{\beta,2}(-\lambda t^{\nu}) + Ct^2E_{\beta,3}(-\lambda t^{\nu}),$$

combining the boundary conditions in (43), we get

$$A = 0, \ \ B = 0, \ \ CE_{\beta,3}(-\lambda) = 0.$$

Thus, if there exists a real number λ satisfying $E_{\beta,3}(-\lambda) = 0$, then λ must be an eigenvalue of (43) and the corresponding eigenfunction is given by

$$y(t) = t^2E_{\beta,3}(-\lambda t^{\nu}).$$

Now, we suppose $\lambda_0 \in \left(-\max\left\{\frac{\Gamma(\beta-1)}{\triangle}, \frac{\Gamma(\beta)}{\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}\right\}, +\infty\right)$ satisfying $E_{\beta,3}(\lambda_0) = 0$. Then λ_0 must be negative. From $E_{\beta,3}(-(-\lambda_0)) = 0$, we know $-\lambda_0 > 0$ must be an eigenvalue of (43), thus from Corollary 4.2, we know

$$-\lambda_0 \geqslant \max\left\{\frac{\Gamma(\beta-1)}{\triangle}, \frac{\Gamma(\beta)}{\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}\right\},$$

hence

$$\lambda_0 \leqslant -\max\left\{\frac{\Gamma(\beta-1)}{\triangle}, \frac{\Gamma(\beta)}{\left(\frac{3-\beta}{\beta-1}2^{\frac{2}{\beta-3}}+\frac{1}{2}\right)}\right\},$$

which is a contradiction, and we end the proof. □

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