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ORTHOGONALITY FOR A CLASS OF GENERALISED JACOBI POLYNOMIAL $P_\nu^{\alpha,\beta}(x)$

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Abstract. This work considers g-Jacobi polynomials, a fractional generalisation of the classical Jacobi polynomials. We discuss the polynomials and compare some of their properties to the classical case. The main result of the paper is to show that one can derive an orthogonality property for a sub-class of g-Jacobi polynomials $P_\nu^{(\alpha,\beta)}$. The paper concludes with an application in modelling of ophthalmic surfaces.

1. Introduction

Orthogonal families of polynomial are important both from theoretical and practical viewpoints. They are widely used when approximating functions from mathematics and other sciences and they allow us to solve and interpret solutions of certain differential equations.

The Jacobi Polynomials $P_n^{(\alpha,\beta)}$ are a classical family of orthogonal polynomials (see [7], [23]) that has been used in many applications due to its ability to approximate general classes of function.

The Gegenbauer polynomials (which are special cases of Jacobi polynomials) are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-x^2)^{\alpha-1/2}$. They are a generalisation of the Legendre and Chebyshev polynomials. Further, the Gegenbauer polynomials $\{C_n^\lambda(t)\}$ constitute a complete orthogonal system of the real (or complex) L^2-space $L^2\left([-1,1],(1-x^2)^{\nu-\frac{1}{2}}dx\right)$ with respect to the $(1-x^2)^{\alpha-1/2}$ on the interval $[-1,1]$ (see [24]).

One can also consider the question of orthogonality over a semi-infinite interval. The authors of [6] discuss this property for the Jacobi polynomials and this was also considered by V. Romanovski [20] and (more recently) by S.D. Bajpai[4].

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In[11] the authors introduced a family of generalised Jacobi polynomials/functions with indexes $\alpha, \beta \in \mathbb{R}$ which are mutually orthogonal with respect to the corresponding Jacobi weights and which inherit selected important properties of the classical Jacobi polynomials.

In [13] the authors deal with generalised Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$, but the generalisation we consider in this paper, to the form $\{P_\nu^{\alpha,\beta}(x)\}$ with ν non integer is much more recent (see [15]). We will show that these fractional generalisations can have important properties of use in applications.

The classical Jacobi Polynomials are usually defined by a Rodrigues' type formula

$$P_n^{(\alpha,\beta)}(x) = (-2)^n (n!)^{-1} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^{n+\alpha} (1+x)^{n+\beta} \right],$$

where $\alpha > -1$, $\beta > -1$ (see [7]). In [15] the authors defined the generalised Jacobi (or g-Jacobi) functions as an extension of the Jacobi polynomials. In the previous formula they replaced the integer derivative by the Riemann-Liouville fractional of order $\nu \in \mathbb{R}$. They also derived some formal properties of this new family of polynomials.

The main theme of this paper is to find a sub-class of the g-Jacobi Polynomials, $P_\nu^{(\alpha,\beta)}$, with ν non-integer, where a orthogonality relation holds, and to show (through an application in optics) how this new class of orthogonal polynomials can have useful practical applications in function aproximation.

The paper is organised as follows. In the second section, various definitions and properties are recalled. In the third section, we prove the orthogonality property for a certain class of generalised Jacobi polynomials and some of their properties will be presented in section four. Finally, we compare the behaviour of Zernike, classical and modified Jacobi polynomials in modelling of ophthalmic surfaces.

2. Preliminaries

2.1. Special functions

Here we recall some aspects of some of the special functions that feature in this paper: the Gamma function and the Pochhammer symbol. We recall that the Gamma function $\Gamma(s)$ is defined by the integral (see [1])

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \tag{1}$$

The integral (1) defines a holomorphic function in the half-plane $Re(s) > 0$. In addition Γ satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s), \quad Re(s) > 0. \tag{2}$$

Hence, since $\Gamma(1) = 1$, we have $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

The reflection formula for the Gamma function establishes a relation between the sine and gamma functions, i.e.,

$$\Gamma(-a) = -\frac{\pi \csc(\pi a)}{\Gamma(a+1)}, \quad a \notin \mathbb{Z}.$$

The Pochhammer symbol is defined as see [1]

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)},$$

for all $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The regularised hypergeometric function ${}_2\tilde{F}_1(a,b;c;z)$ is defined by means of a hypergeometric series by the expression

$$ {}_2\tilde{F}_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{\Gamma(c+k)\,k!}\,z^k, \quad |z| \leqslant 1. \tag{3}$$

It is related to the hypergeometric function ${}_2F_1$ by the formula

$$ {}_2\tilde{F}_1(a,b;c;z) = \frac{1}{\Gamma(c)}\,{}_2F_1(a,b;c;z), \tag{4}$$

and the hypergeometric function ${}_2F_1$ has the following series representation

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!}\,z^k, \quad |z| \leqslant 1. \tag{5}$$

When $z=1$ the hypergeometric function ${}_2F_1$ assumes the particular form (see formula (7.3.5.2) in [18])

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{6}$$

2.2. Generalised Jacobi polynomials

We start by reviewing and defining the Jacobi polynomials.

We recall that the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(n \geqslant 0)$ are defined by

$$P_n^{(\alpha,\beta)}(x) = (-2)^n (n!)^{-1} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{7}$$

Let $\omega_{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}$ be the Jacobi weight function. For $\alpha,\beta > -1$ the Jacobi polynomials are mutually orthogonal in $L^2_{\omega_{\alpha,\beta}}([-1,1])$, i.e.,

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\omega_{\alpha,\beta}(x)dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{\mu\nu}, \tag{8}$$

where $\delta_{\mu\nu}$ denotes the Kronecker delta $m,n \in \mathbb{N}_0$. This class of Jacobi weight functions leads to Jacobi polynomials with many attractive properties that are not shared by general orthogonal polynomials.

The generalised Jacobi polynomials may be considered as fractional index generalisations of the classical Jacobi polynomials. These polynomials appear as a special case of fractional Gauss functions, defined as solutions of the fractional generalisation of the Gauss hypergeometric equation (see [10]).

The representation of the generalised Jacobi polynomial in terms of the hypergeometric function ${}_2F_1$ is given by

THEOREM 1. (c.f. [15]) *The g-Jacobi Polynomial can be represented by*

$$P_\nu^{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\nu+1))\Gamma(1+\alpha)}\,{}_2F_1\left(-\nu,\alpha+\beta+\nu+1;\alpha+1;\frac{(1-x)}{2}\right). \tag{9}$$

Moreover, (9) satisfies the following differential equation

$$(1-x^2)Y''(x)+(\beta-\alpha-(\alpha+\beta+2)x)Y'(x)+\nu(\nu+2\lambda)Y(x)=0. \tag{10}$$

The g-Jacobi polynomials were introduced in [15]. There, the authors replaced the integer derivatives in (7) by the Riemann-Liouville fractional derivative D^ν, obtaining

$$P_\nu^{\alpha,\beta}(x) = \frac{(-1)^\nu}{\Gamma(n+1)2^\nu(1-x)^\alpha(1+x)^\beta}D^\nu\left[(1-x)^{\nu+\alpha}(1+x)^{\nu+\beta}\right], \tag{11}$$

where the Riemann-Liouville fractional derivative of order α is defined as

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, & n-1<\alpha<n\in\mathbb{N} \\ \frac{d^n}{dt^n}, & \alpha=n\in\mathbb{N} \end{cases} \tag{12}$$

(see [21]).

In [15] the authors presented the following explicit formula for the g-Jacobi polynomials

THEOREM 2. (cf. [15]) *For the generalised Jacobi functions the following representation holds*

$$P_\nu^{(\alpha,\beta)}(x) = 2^{-\nu}\sum_{k=0}^{\infty}\binom{\nu+\beta}{k}\binom{\nu+\alpha}{\nu-k}(x+1)^{\nu-k}(x-1)^k \tag{13}$$

with $\nu>0$, $\alpha>-1$, $\beta>-1$ *and where*

$$\binom{\alpha}{\beta} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)\Gamma(1+\alpha-\beta)}$$

is the binomial coefficient with real arguments.

We know that the classical Jacobi polynomials satisfy an orthogonality property. In the following example we can see that the orthogonality property (in general) for the g-Jacobi polynomials is not applicable.

EXAMPLE 1. For $\alpha=1$, $\beta=\frac{1}{2}, \nu=\frac{1}{4}$ and $\mu=\frac{1}{2}$

$$\int_{-1}^{1}P_\nu^{(\alpha,\beta)}(x)P_\mu^{(\alpha,\beta)}(x)\omega_{\alpha,\beta}(x)dx = \frac{72\sqrt{2\pi}}{13\Gamma\left(\frac{1}{4}\right)^2},$$

i.e., we cannot say that the g-Jacobi polynomials are mutually orthogonal in $L^2_{\omega_{\alpha,\beta}}([-1,1])$.

It is therefore interesting to explore the existence (or otherwise) of an orthogonal subclass of the g-Jacobi polynomials.

3. Orthogonality for a certain class of the g-Jacobi polynomials

Our aim in this section is to present the orthogonality property for a certain class of the g-Jacobi polynomials $P_\nu^{(\alpha,\beta)}$ with ν non-integer.

Let $w \in C([-1,1])$ with $w(x) > 0$ on $[-1,1]$ and $\int_{-1}^{1} w(x)dx < \infty$. The function w is called the weight function.

We can define

$$L^2_{\omega_{\alpha,\beta}}([-1,1]) = \{f : f \text{ is measurable on } [-1,1] \text{ and } \int_{-1}^{1} |f(x)|^2 w(x)dx < \infty\}$$

and we define $\langle g, f\rangle_w = \int_{-1}^{1} g(x)f(x)w(x)dx$, for $f, g \in L^2_{\omega_{\alpha,\beta}}([-1,1])$. Then (after identifying f and g when $f = g$ a.e.), $L^2_{\omega_{\alpha,\beta}}([-1,1])$ is a Hilbert space.

In the following result we prove that $P_\nu^{(\alpha,\beta)}(x) \in L^2_{\omega_{\alpha,\beta}}([-1,1])$. We will also obtain an explicit expression for $\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_\nu^{(\alpha,\beta)}(x)P_\nu^{(\alpha,\beta)}(x)dx$.

THEOREM 3. *The following relation holds*

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_{n+\theta}^{(\alpha,\beta)}(x)P_{n+\theta}^{(\alpha,\beta)}(x)dx = \psi(n,\theta,\alpha,\beta), \tag{14}$$

where

$$\psi(n,\theta,\alpha,\beta) = \frac{(-1)^\theta 2^{\alpha+\beta+1}\Gamma(\alpha+n+\theta+1)\Gamma(\beta+n+\theta+1)}{\Gamma(1-\theta)\Gamma(1+\theta)(2n+\alpha+\beta+\theta+1)\Gamma(n+\alpha+\beta+\theta+1)\Gamma(n+1+\theta)}$$

for $0 \leqslant \theta < 1,\ \alpha > -1$ *and* $\beta > -1$.

Proof. Taking into account (11), we have

$$\begin{aligned}
&\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\, P_{n+\theta}^{(\alpha,\beta)}(x)\, P_{n+\theta}^{(\alpha,\beta)}(x)\, dx \\
&\qquad = \int_{-1}^{1} P_{n+\theta}^{(\alpha,\beta)}(x)\, \frac{(-1)^{n+\theta}}{2^{n+\theta}\Gamma(n+\theta+1)}\, D^{n+\theta}\left[(1-x)^{n+\theta+\alpha}\, (1+x)^{n+\theta+\beta}\right] dx.
\end{aligned}$$

Applying the definition of the Riemann-Liouville fractional derivatives (12) and taking into account formula (2.2.6.1) in [19] the right hand side of the last expression becomes equal to

$$\begin{aligned}
&\int_{-1}^{1} P_{n+\theta}^{(\alpha,\beta)}(x)\, \frac{(-1)^{n+\theta}}{2^{n+\theta}\,\Gamma(n+\theta+1)} \left[\frac{d^{n+1}}{dx^{n+1}} \int_{-1}^{x} \frac{(1-t)^{n+\theta+\alpha}\,(1+t)^{n+\theta+\beta}}{(x-t)^\theta}\, dt\right] dx \\
&\quad = \int_{-1}^{1} P_{n+\theta}^{(\alpha,\beta)}(x)\, \frac{(-1)^{n+\theta}}{2^{n+\theta}\,\Gamma(n+\theta+1)} \left[\frac{d^{n+1}}{dx^{n+1}} \left(2^{n+\theta+\alpha}\,(1+x)^{1+\beta+n}\,\Gamma(n+\theta+\beta+1)\right.\right. \\
&\qquad \left.\left.\times\, {}_2\tilde{F}_1\left(-\alpha-\theta-n, 1+\beta+\theta+n; 2+\beta+n; \frac{1+x}{2}\right)\right)\right] dx.
\end{aligned}$$

Using the series representation (3) the previous expression becomes equal to

$$\int_{-1}^{1} P_{n+\theta}^{(\alpha,\beta)}(x)\, \frac{(-1)^{n+\theta}}{2^{n+\theta}\,\Gamma(n+\theta+1)}\, 2^{n+\theta+\alpha}\,\Gamma(n+\theta+\beta+1) \times \sum_{k=0}^{\infty} \frac{(-n-\theta-\alpha)_k\,(n+\theta+\beta+1)_k}{\Gamma(2+\beta+n+k)\,k!\,2^k}\, \frac{d^{n+1}}{dx^{n+1}}\left[(1+x)^{k+\beta+n+1}\right] dx. \tag{15}$$

Making the calculation of the integer derivative we arrive at

$$\int_{-1}^{1} P_{n+\theta}^{(\alpha,\beta)}(x)\, \frac{(-1)^{n+\theta}}{2^{n+\theta}\,\Gamma(n+\theta+1)}\, 2^{n+\theta+\alpha}\,\Gamma(n+\theta+\beta+1) \times \sum_{k=0}^{\infty} \frac{(-n-\theta-\alpha)_k\,(n+\theta+\beta+1)_k}{\Gamma(2+\beta+n+k)\,k!\,2^k} \prod_{i=0}^{n}(k+\beta+n+1-i)(1+x)^{\beta+k}\,dx. \tag{16}$$

Taking into account that the g-Jacobi polynomials satisfy (10), we can consider equation (9), together with the series representation (5) of ${}_2F_1$, and Theorem 4.2.2 in [23]. We consider the term of order n of $P_{n+\theta}^{(\alpha,\beta)}(x)$ in (16). It follows, after straightforward calculations, that the last expression becomes equal to

$$\begin{aligned} &\frac{(-1)^{n+\theta}\,2^{\alpha-n}\,\Gamma(n+\theta+\alpha+1)\,\Gamma(n+\theta+\beta+1)\,(-n-\theta)_n\,(n+\theta+\alpha+\beta+1)_n}{(\Gamma(n+\theta+1))^2\,\Gamma(n+\alpha+1)n!} \\ &\qquad \times \sum_{k=0}^{\infty} \frac{(-n-\theta-\alpha)_k\,(n+\theta+\beta+1)_k}{\Gamma(\beta+k+1)\,k!\,2^k} \int_{-1}^{1}(1+x)^{k+\beta}\,(1-x)^n\,dx \\ &= \frac{(-1)^{n+\theta}\,2^{\alpha}\,\Gamma(n+\theta+\alpha+1)\,\Gamma(n+\theta+\beta+1)\,(-n-\theta)_n\,(n+\theta+\alpha+\beta+1)_n}{(\Gamma(n+\theta+1))^2\,\Gamma(n+\alpha+1)} \\ &\qquad \times \sum_{k=0}^{\infty} \frac{(-n-\theta-\alpha)_k\,(n+\theta+\beta+1)_k}{\Gamma(n+\beta+k+2)\,k!}. \end{aligned}$$

The last equality results from the change of variable $1-t=2w$ and formula (2.2.4.8) in [19]. In fact

$$\int_{-1}^{1}(1+x)^{k+\beta}\,(1-x)^n\,dx = 2\int_{0}^{1}(2-2w)^{k+\beta}\,(2w)^n\,dw = 2^{n+\beta+k+1}\,\frac{\Gamma(n+1)\,\Gamma(\beta+k+1)}{\Gamma(n+\beta+k+2)}.$$

Furthermore, from (4) and (6) we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-n-\theta-\alpha)_k(n+\theta+\beta+1)_k}{\Gamma(n+\beta+k+2)k!} &= {}_2\tilde{F}_1\,(-\alpha-\theta-n, 1+\beta+\theta+n; 2+\beta+n; 1) \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(2n+\alpha+\theta+\beta+2)\Gamma(1-\theta)}, \end{aligned}$$

and therefore we obtain our result. □

Under the conditions of the Theorem 3, we seek a class of g-Jacobi polynomials that satisfies

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_{\nu}^{(\alpha,\beta)}(x)P_{\mu}^{(\alpha,\beta)}(x)dx=\begin{cases}\psi(\nu,\alpha,\beta),\ \text{if}\ \nu=\mu\\ 0,\ \text{if}\ \nu\neq\mu\end{cases},$$

i.e., a class of g-Jacobi polynomials where the orthogonality property holds.

REMARK 1. Note that, for $\beta=\frac{1}{2}$ the g-Jacobi polynomials can be rewritten in terms of Gegenbauer polynomials, i.e.

$$P_{\nu}^{(\alpha,\frac{1}{2})}(x)=\sqrt{\frac{2}{\pi}}\frac{\Gamma(\alpha+1)\Gamma(\nu+\frac{3}{2})}{\sqrt{x+1}\,\Gamma\left(\alpha+\nu+\frac{3}{2}\right)}C_{2\nu+1}^{\alpha+\frac{1}{2}}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right),$$

(see e.g. [23]). Taking into account the following:

- that the g-Jacobi polynomials can be rewritten in terms of Gegenbauer polynomials
- that the Gegenbauer polynomials are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-x^2)^{\alpha-1/2}$
- that the Gegenbauer polynomials of even degree are even and Gegenbauer polynomials of odd degree are odd

$$\begin{aligned}
&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\frac{1}{2}}P_{\nu}^{(\alpha,\frac{1}{2})}(x)P_{\mu}^{(\alpha,\frac{1}{2})}(x)dx\\
&=c_{\mu,\alpha}c_{\nu,\alpha}\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}C_{2\nu+1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right)C_{2\mu+1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right)\\
&=c_{\mu,\alpha}c_{\nu,\alpha}\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{-\frac{1}{2}}C_{2\nu+1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right)C_{2\mu+1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right)\\
&=c_{\mu,\alpha}c_{\nu,\alpha}\,4\sqrt{2}\int_{0}^{1}(1-t^2)^{\alpha}C_{2\nu+1}^{\alpha+\frac{1}{2}}(t)C_{2\mu+1}^{\alpha+\frac{1}{2}}(t)dt\\
&=c_{\mu,\alpha}c_{\nu,\alpha}\,2\sqrt{2}\int_{-1}^{1}(1-t^2)^{\alpha}C_{2\nu+1}^{\alpha+\frac{1}{2}}(t)C_{2\mu+1}^{\alpha+\frac{1}{2}}(t)dt=0,
\end{aligned}\tag{17}$$

when 2ν and 2μ are odd, and where

$$c_{\mu,\alpha}=\sqrt{\frac{2}{\pi}}\frac{\Gamma(\alpha+1)\Gamma(\mu+\frac{3}{2})}{\Gamma\left(\alpha+\mu+\frac{3}{2}\right)}\quad\text{and}\quad c_{\nu,\alpha}=\sqrt{\frac{2}{\pi}}\frac{\Gamma(\alpha+1)\Gamma(\nu+\frac{3}{2})}{\sqrt{x+1}\Gamma\left(\alpha+\nu+\frac{3}{2}\right)}.$$

Combining Theorem 3 and (17), we can establish an orthogonality class for the g-Jacobi polynomials.

COROLLARY 1. *For $\beta=\frac{1}{2}$, $\alpha>-1$ and $\nu=n+\frac{1}{2}$, $\mu=m+\frac{1}{2}$ we have*

$$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}\,P_{n+\frac{1}{2}}^{(\alpha,\beta)}(x)\,P_{m+\frac{1}{2}}^{(\alpha,\beta)}(x)\,dx=\psi(n,\alpha,\beta)\,\delta_{n,m},$$

where $\delta_{n,m}$ denotes the Kronecker number.

REMARK 2. Since the Gegenbauer polynomials $\{C_n^{\lambda}(x)\}$ constitute a complete orthogonal system of the real (or complex) L^2- space $L^2\left([-1,1],(1-x^2)^{\nu-\frac{1}{2}}dx\right)$ with respect to the $(1-x^2)^{\alpha-1/2}$ on the interval $[-1,1]$ (see [24]), and taking into account the Theorem 3, Remark 1 and Corollary 1, we can say that the $\{P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}(x)\}$ constitute a complete orthogonal system of the real L^2- space $L^2\left([-1,1],(1-x)^{\alpha}(1+x)^{\frac{1}{2}}dx\right)$ with respect to the $(1-x)^{\alpha}(1+x)^{\frac{1}{2}}$ on the interval $[-1,1]$.

Now, we are ready to present a class of g-Jacobi polynomials which are orthogonal functions in $[-1,1]$ with respect to the weight function $w_{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$ which is defined by

$$\mathscr{C}=\left\{\left\{P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}(x)\right\},\ \ \alpha>-1,\ n\in\mathbb{N}\right\}.$$

We present some graphs of the g-Jacobi polynomials for varying n,α,β. We can observe from the graphs that for $\alpha=\beta$, as expected, the functions $P_n^{(\alpha,\beta)}(x)$, $x\in[-1,1]$ are odd for n odd and even for n even. Additionally, it is evident that in these examples the maximum absolute values of the functions are attained at the endpoints of the interval.

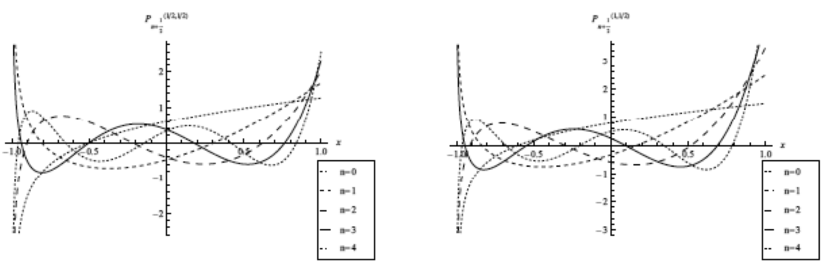


Figure 1: $P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}(x)$ *for* $\alpha=\frac{1}{2}$ *and* $\alpha=1$

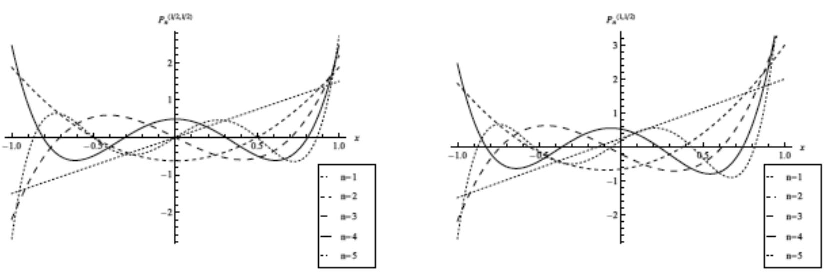


Figure 2: $P_n^{(\alpha,\frac{1}{2})}(x)$ *for* $\alpha=\frac{1}{2}$ *and* $\alpha=1$

4. Some properties of the g-Jacobi polynomials

The aim of this section is to present some properties for the g-Jacobi polynomials making use of the orthogonality property. In [8] the authors presented some properties of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. In what follows, We will extend some of these properties presented in [8] for the g-Jacobi polynomials, $P_{\nu}^{(\alpha,\beta)}(x)$, with ν non integer.

We start by recalling the following result:

LEMMA 1. (cf. [8]) *If $u(x)$ is real and continuously differentiable on $[-1,1]$ with $u(1)=0$ and $\omega_{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$, then*

$$\int_{-1}^{1}\frac{du^2(x)}{dx}\,\omega_{\alpha,\beta}(x)\leqslant 0,\quad -2<\alpha\leqslant 0,\beta>0. \tag{18}$$

Equality holds only if $u(x)=0$ for all $x\in[-1,1]$.

We start by recalling a definition of the fractional Jacobi polynomial of order $n+\theta$, i.e., the series representation of the fractional Jacobi polynomial(13) can be rewritten and given by

$$P_{n+\theta}^{(\alpha,\beta)}(x)=2^{-(n+\theta)}\sum_{k=0}^{\infty}\binom{\theta+\alpha}{k}\binom{\theta+\beta}{\theta-k}(x-1)^{\theta+k}(x+1)^{k} \tag{19}$$

with $n\in\mathbb{N}$, $\theta>0$, and $\alpha,\beta>-1$.

We have the following relation

LEMMA 2. *Let $f_{n+\theta}(x)$ be a polynomial of order $n+\theta$ with $f_{n+\theta}(1)=0$ and $P_{n+\theta}^{\alpha,\beta}(x)$ be the g-Jacobi polynomial of order $n+\theta$ with weight function $\omega_{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$, $\alpha,\beta>-1$. Then*

$$\int_{-1}^{1}f_{n+\frac{1}{2}}(x)\,P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}(x)\,\omega_{\alpha-1,\frac{1}{2}+1}(x)\,dx=-\int_{-1}^{1}f_{n+\frac{1}{2}}(x)\,P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}(x)\,\omega_{\alpha,\frac{1}{2}}(x)\,dx. \tag{20}$$

Proof. Let us consider a polynomial of order $n+\theta$ denoted by $f_{n+\theta}(x)$ such that $f_{n+\theta}(1)=0$. Then, there exist n coefficients $c_k\in\mathbb{C}$, $k=0,\dots,n-1$, such that

$$f_{n+\theta}(x)=(1-x)\sum_{k=0}^{n-1}c_kP_{k+\theta}^{(\alpha,\beta)}(x). \tag{21}$$

Taking into account (21) and the relation $\omega_{\alpha-1,\beta+1}(x)=\frac{1+x}{1-x}\,\omega_{\alpha,\beta}(x)$, we obtain

$$\begin{aligned}
&\int_{-1}^{1}\frac{1+x}{1-x}\,f_{n+\theta}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx\\
&\quad=\int_{-1}^{1}(1+x)\sum_{k=0}^{n-1}c_k\,P_{k+\theta}^{(\alpha,\beta)}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx\\
&\quad=\int_{-1}^{1}(x+1)\sum_{k=0}^{n-1}c_k\,P_{k+\theta}^{(\alpha,\beta)}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx\\
&\quad=\sum_{k=0}^{n-1}c_k\int_{-1}^{1}(x-1)\,P_{k+\theta}^{(\alpha,\beta)}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx
\end{aligned}\tag{22}$$

For $\beta=\frac{1}{2}$ and $\theta=\frac{1}{2}$ the previous expression becomes equal to

$$c_{n-1}\int_{-1}^{1}x\,P_{n-1+\theta}^{(\alpha,\beta)}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx,\tag{23}$$

because, in that case, $P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}(x)$ is orthogonal, with respect to the weight function $\omega_{\alpha,\frac{1}{2}}(x)$, to all polynomials of order less than $n+\frac{1}{2}$ that belong to $\mathscr{C}$ (see Theorem 3 and Corollary 1).

On the other hand we have

$$\begin{aligned}
&\int_{-1}^{1}f_{n+\theta}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx\\
&\quad=\int_{-1}^{1}(1-x)\sum_{k=0}^{n-1}c_k\,P_{k+\theta}^{(\alpha,\beta)}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx\\
&\quad=-\int_{-1}^{1}(x-1)\sum_{k=0}^{n-1}c_k\,P_{k+\theta}^{(\alpha,\beta)}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx\\
&\quad=-c_{n-1}\int_{-1}^{1}x\,P_{n-1+\theta}^{(\alpha,\beta)}(x)\,P_{n+\theta}^{(\alpha,\beta)}(x)\,\omega_{\alpha,\beta}(x)\,dx. \quad\square
\end{aligned}$$

Now, we will present a result about the stability of certain polynomials.

LEMMA 3. *Let $P_{n+\theta}^{\alpha,\beta}(x)$ the Jacobi polynomial of order $n+\theta$, where $n\geqslant 2$. For $\alpha\in]-1,1]$ and $\beta=\frac{1}{2}$ then the zeros of the polynomial*

$$\varphi_{n+\theta}(u):=\sum_{k=0}^{n}\left(\frac{d^k}{dx^k}P_{n+\theta}^{(\alpha,\beta)}(x)\right)_{x=1}u^k$$

lie in the left half plane, i.e., $\varphi_{n+\theta}(u)$ is a stable polynomial.

Proof. Now, let us define

$$\varphi_{n+\theta}(u) := \sum_{k=0}^{n} \left(\frac{d^k}{dx^k} P_{n+\theta}^{(\alpha,\beta)}(x) \right)_{x=1} u^k \tag{24}$$

and

$$f_{n+\theta}(x) := \sum_{k=0}^{n} u^k \frac{d^k}{dx^k} P_{n+\theta}^{(\alpha,\beta)}(x), \tag{25}$$

where u is such that $\varphi_{n+\theta}(u) = 0$ and so that $f_{n+\theta}(1) = 0$. Remark that

$$\varphi_{n+\theta}(0) = P_{n+\theta}^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+\theta+n+1)}{\Gamma(\theta+n+1)\Gamma(\alpha-1)} \neq 0,$$

hence $u \neq 0$. Which can be verified by

$$\begin{aligned} \left(\sum_{k=0}^{n} u^k \frac{d^k}{dx^k} P_{n+\theta}^{(\alpha,\beta)}(x) - P_{n+\theta}^{(\alpha,\beta)}(x) \right) &= \sum_{k=1}^{n} u^k \frac{d^k}{dx^k} P_{n+\theta}^{(\alpha,\beta)}(x) \\ &= u \frac{d}{dx} f_{n+\theta}(x), \end{aligned}$$

i.e.,

$$f_{n+\theta}(x) - P_{n+\theta}^{(\alpha,\beta)}(x) = u \frac{d}{dx} f_{n+\theta}(x). \tag{26}$$

For $\beta = \frac{1}{2}$ and $\theta = \frac{1}{2}$ and using (25) and (26) we get

$$\begin{aligned} &\int_{-1}^{1} f_{n+\theta}(x) P_{n+\theta}^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) dx \\ &= \int_{-1}^{1} \left(u \frac{d}{dx} f_{n+\theta}(x) + P_{n+\theta}^{(\alpha,\beta)}(x) \right) P_{n+\theta}^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) dx \\ &= \int_{-1}^{1} u \frac{d}{dx} f_{n+\theta}(x) P_{n+\theta}^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) dx + \int_{-1}^{1} \left(P_{n+\theta}^{(\alpha,\beta)}(x) \right)^2 \omega_{\alpha,\beta}(x) dx \\ &= \int_{-1}^{1} \left(P_{n+\theta}^{(\alpha,\beta)}(x) \right)^2 \omega_{\alpha,\beta}(x) dx, \end{aligned} \tag{27}$$

because, in that case, $P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}(x)$ is orthogonal, with respect to the weight function $\omega_{\alpha,\frac{1}{2}}(x)$, to all polynomials of order less than $n+\frac{1}{2}$ that belong to $\mathscr{C}$ (see Theorem 3 and Corollary 1).

Multiplying (26) by the complex conjugated of $f_{n+\theta}(x)$, and adding to this its complex conjugate, we have

$$\left(\frac{1}{u} + \frac{1}{\overline{u}} \right) |f_{n+\theta}(x)|^2 - \left(\frac{\overline{f_{n+\theta}(x)}}{u} + \frac{f_{n+\theta}(x)}{\overline{u}} \right) P_{n+\theta}^{(\alpha,\beta)}(x) = \frac{d}{dx} |f_{n+\theta}(x)|^2. \tag{28}$$

Now, we multiply both members of the previous identity by $\omega_{\alpha-1,\beta+1}(x)$, integrate over $[-1,1]$, and using (20) and (27), we get

$$\left(\frac{1}{u}+\frac{1}{\overline{u}}\right)\left(\int_{-1}^{1}|f_{n+\theta}(x)|^{2}\,\omega_{\alpha-1,\beta+1}(x)dx+\int_{-1}^{1}\left(P_{n+\theta}^{(\alpha,\beta)}(x)\right)^{2}\omega_{\alpha,\beta}(x)dx\right)$$
$$=\int_{-1}^{1}\frac{d}{dx}|f_{n+\theta}(x)|^{2}\,\omega_{\alpha-1,\beta+1}(x)dx. \tag{29}$$

both integrals on the left hand side are positive, and the integral on the right hand side is negative if $-1<\alpha\leqslant 1$, $\beta>-1$, and therefore

$$\frac{1}{u}+\frac{1}{\overline{u}}=\frac{2Re(u)}{|u|^{2}}<0. \quad \square \tag{30}$$

5. Application in modelling of ophthalmic surfaces

A significant application of classes of orthogonal polynomials is in practical approximation problems. Here we will consider the problem of ocular aberrations that are commonly represented in terms of a series of Zernike polynomials that offer distinct advances to the application because of their normalisation on a circular pupil. However, they exhibit slow convergence in some cases and so they may not be the most appropriate choice, particularly when applied to complex ophthalmic surfaces with high spatial frequency content [12].

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}$ are a family of orthogonal polynomials ([7, 23]) that have been used in many applications due to its ability for approximate general classes of functions. They provide complete orthogonal sets [23, 16] on a circular pupil and so they are a reasonable candidate for this problem.

We have established the orthogonality and completeness of a particular case of g-Jacobi polynomials $P_{n+\frac{1}{2}}^{(\alpha,\frac{1}{2})}$ (modified Jacobi polynomials) for the particular case where $\beta=\frac{1}{2}$ and $\alpha>-1$.

In this section, we aim to compare the behaviour of these sets of orthogonal functions (Zernike, classical and modified Jacobi polynomials) when used to model ophthalmic surfaces. Evaluation of the RMS error in fitting these polynomials with the specific surfaces is chosen as an effective indication for the comparison of those polynomials, using (in each case) 15 polynomials from the orthogonal set.

The surface of interest can be modelled by

$$C(r,\phi)=\sum_{p=1}^{P}a_{p}\psi_{p}(r,\phi)+\varepsilon_{p}(r,\phi), \tag{31}$$

where the index p is a polynomial ordering number, $\psi_p(r,\phi)$, $p=1,\dots,P$ is the pth polynomial, a_p, $p=1,\dots,P$ is the coefficient associated with $\psi_p(r,\phi)$, P is the order, r is the normalised distance from the origin, ϕ is the angle and $\varepsilon_p(r,\phi)$ represents the modeling error. Throughout this analysis, we choose the polar coordinate system for convenience.

Using a set of such orthogonal discrete polynomials, we can form a linear model

$$C = \psi a + \varepsilon, \tag{32}$$

where C is a $D-$element column vector of surface evaluated at discrete points (r_d, ϕ_d), $d = 1,\ldots,D$, ψ is a $(D \times P)$ matrix of discrete, orthogonal polynomials $\psi_p(r_d, \phi_d)$, a is a $P-$element column vector of coefficients, and ε represents a D-element column vector of the measurement and modeling error. For such a model, the coefficient vector a can be estimated using the method of least-squares, i.e.,

$$\widehat{a} = (\psi^T \psi)^{-1} \psi^T C, \tag{33}$$

where T denotes the transposition, providing that the inverse exists. The RMS error can be found by

$$RMS_{error} = \frac{\sqrt{\sum_{p=1}^{P} (\varepsilon_p(r,\phi))^2}}{P}. \tag{34}$$

For the Zernike polynomials, the most accepted representation of their radial function is [25]

$$R_n^{|m|}(r) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s!(\frac{n-|m|}{2} - s)!(\frac{n+|m|}{2} - s)!} r^{n-2s}, \tag{35}$$

where $n \geqslant m$ so the parity of a polynomial is the same as the corresponding n. Putting together the two variable modes in this case yields the Zernike polynomials which are given in normalised form by:

$$Z_n^m(r,\phi) = \sqrt{\frac{2(n+1)}{1+\delta_{m,0}}} R_n^m(r) \begin{cases} \sin(m\phi), \text{ for odd m} \\ \cos(m\phi), \text{ for even m} \end{cases}. \tag{36}$$

As before, the Jacobi polynomials are defined by the Rodrigues' type formula

$$J_n^{(\alpha,\beta)}(x) = (-2)^n (n!)^{-1} (1-x)^{\alpha} (1-x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1-x)^{n+\beta}] \tag{37}$$

where (for integrability purposes) $\alpha > -1$, $-\beta > -1$ [7]. The consideration of Jacobi polynomials on a circular disk is given by [22]

$$F_n^{(\alpha,\beta)}(r,\phi) = r^{|m|} J_n^{(\alpha,\beta)}(2r^2 - 1) \exp(mi\phi), \tag{38}$$

where $F_n^{(\alpha,\beta)}(r,\phi)$ is one-sided Jacobi polynomials and $J_n^{(\alpha,\beta)}(2r^2-1)$ is an even polynomial of degree $2n$.

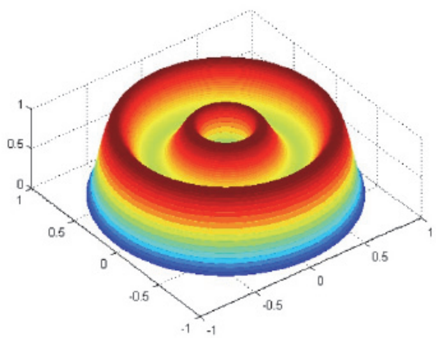
In order to evaluate the RMS error in fitting these polynomials with the specific surfaces, we selected 15 modes from each orthogonal set. In case of Zernike polynomials with respect to Zernike standard pyramid the first 15 modes were under analysis

($m = -4, -3, \ldots, 3, 4$ and $n = 0, 1, 2, 3, 4$). For the classical Jacobi polynomials $\alpha = 1$, $\beta = 1$ and $n = 1, \ldots, 15$ and in case of modified Jacobi $\beta = 1/2$ and $n = 1, \ldots, 15$. The number of modes (15) was chosen arbitrarily with similar results for other choices.

In our analysis, we have considered surfaces with rings and total anterior eye in detection of the RMS error. A ring structure can be obtained by a radial Gaussian function centred at a radius r_0 and a constant angular factor,

$$C(r, \phi) = exp\left\{\frac{-(r - r_0)^2 \times (r - r_1)^2}{W^2}\right\}. \tag{39}$$

This type of surface could represent the effects of wearing a certain type of contact lens. A multiple ringed surface, for instance a sum of two or more similar functions, could also represent a wavefront generated by a multifocal lens. We performed the comparison of Zernike, classical and modified Jacobi polynomials where the same number of modes fitted the surface with rings. Figure 3 shows the surface of Gaussian rings and evaluation of the RMS error for each polynomial type when fitting this surface. As the results show, modified and classical Jacobi polynomials have lower fitting error in comparison with Zernike polynomials.

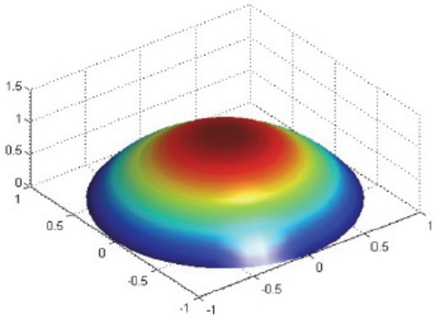


Polynomial	Total RMS error
Zernike	57 μm
Classical Jacobi	40 μm
Modified Jacobi	38 μm

Figure 3: *The model of Gaussian rings ($r_0 = 0.8$ - $r_1 = 0.25$ - $W = 0.1$) and the RMS error for different functions of Zernike, classical and modified Jacobi.*

A second interesting case is the model of the total anterior eye surface (figure 4), which includes the anterior surface of the cornea, limbus and sclera. This was done by etching together two spherical surfaces of different radii. To produce this model we employed typical parameters for anterior corneal radius, visible iris and diameter of the eye. As shown in figure 4, fitting errors for modified and classical Jacobi polynomials are similar to some extent, although the modified Jacobi shows the smaller error. On the other hand, by a large margin, the Zernike polynomials show the greatest fitting error.

Based on these examples, it is clear that both the classical and modified Jacobi polynomials are competitive for the modelling of ophthalmic surfaces with some limited evidence that the modified Jacobi polynomial outperform the classical version. Further work needs to be undertaken to establish whether the modified Jacobi polynomials continue to outperform the classical Jacobi polynomials for a range of other problems.



Polynomial	Total RMS error
Zernike	9.5 μm
Classical Jacobi	3.2 μm
Modified Jacobi	2.9 μm

Figure 4: *Total eye model with typical parameters and the RMS error for different functions of Zernike, classical and modified Jacobi.*

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