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BOUNDARY VALUE PROBLEMS FOR HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL MULTI–POINT BOUNDARY CONDITIONS

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Abstract. This paper is devoted to the study of the existence and uniqueness of solutions to a boundary value problem for nonlinear Hadamard fractional differential equations with nonlocal multi-point boundary conditions. The results are obtained by using a variety of fixed point theorems. The paper concludes with some illustrative examples.

1. Introduction

The theory of fractional differential equations has received much attention over the past years and has become an important field of investigation due to its extensive applications in numerous branches of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials. As a result, they are gaining much importance and attention in the literature. As recent examples, we refer the reader to [1, 3, 4, 5, 6, 7, 13, 14, 19, 20, 24] and the references contained therein.

Existence theory for real world problems that can be modeled by fractional differential equations with multi-point boundary conditions have attracted the attention of many researchers and is a rapidly growing area of investigation; for example, see, Benchohra and Hamani [5], Cui, Yu, and Mao [13], El-Sayed and Bin-Taher [14], and Houas and Dahmani [19].

The fractional derivative that Hadamard [17] introduced in 1892 is different from the often studied Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function with an arbitrary exponent. A detailed description of the Hadamard fractional derivative and integral can be found in [8, 9, 10, 24].

Examples of physical phenomena modeled by Hadamard fractional derivatives are not prevalent in the literature at this time. However, in the paper by Garra and Polito

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[15], there is a nice discussion of how they are used to solve other mathematical problems such as ones involving Laguerre operators and the Lamb-Bateman integral equation.

In this paper, we investigate the boundary value problem

$$D^r y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [1, T], \ 1 < r \leqslant 2, \tag{1}$$

$$y(1) = 0, \ D^p y(T) = \sum_{i=1}^{n} \lambda_i D^p y(\mu_i), \ 0 < p < 1, \tag{2}$$

where D^r and D^p are the Hadamard fractional derivatives of order r and p respectively, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given function, $\lambda_i \in \mathbb{R}$, $1 < \mu_i \leqslant T$, $i = 1, \ldots, n$, $n \geqslant 2$, and

$$(\log T)^{r-p-1} \neq \sum_{i=1}^{n} \lambda_i (\log \mu_i)^{r-p-1}. \tag{3}$$

We are going to prove the existence and uniqueness of solutions to (1)–(2) using Banach's fixed point theorem, and then give two additional existence results, one based on Schaefer's fixed point theorem and the other on the Leray Schauder nonlinear alternative. Examples are given in Section 4 to demonstrate the applicability of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 1 \leqslant t \leqslant T\}.$$

We begin by defining Hadamard fractional integrals and derivatives. In what follows, $[r]$ denotes the integer part of r and $\log(\cdot) = \log_e(\cdot)$.

DEFINITION 2.1. ([21]) The Hadamard fractional integral of order r for a function $h : [1, +\infty) \to \mathbb{R}$ is defined as

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds, \ r > 0,$$

where Γ is the Gamma function.

DEFINITION 2.2. ([21]) For a function h given on the interval $[1, +\infty)$, the Hadamard fractional derivative of h of order r is defined by

$$(D^r h)(t) = \frac{1}{\Gamma(n-r)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-r-1} \frac{h(s)}{s} ds, \ n-1 < r < n, \ n = [r] + 1.$$

Lemmas of the following type are rather standard in the study of fractional differential equations.

LEMMA 2.3. ([21]) *Let* $r \geqslant 0$. *Then the differential equation*

$$D^r h(t) = 0 \tag{4}$$

has as its solutions

$$h(t) = \sum_{j=1}^{n} c_j (\log t)^{r-j},$$

and moreover,

$$I^r D^r h(t) = h(t) + \sum_{j=1}^{n} c_j (\log t)^{r-j},$$

where $c_j \in \mathbb{R},\ j = 1, \ldots, n,\ n = [r] + 1$.

LEMMA 2.4. ([21]) *If* $r > 0,\ p > 0$, *and* $0 < a < \infty$, *then*

$$\left(I^p \left(\log \frac{t}{a}\right)^{r-1}\right)(x) = \frac{\Gamma(r)}{\Gamma(r+p)} \left(\log \frac{x}{a}\right)^{r+p-1}$$

and

$$\left(D^p \left(\log \frac{t}{a}\right)^{r-1}\right)(x) = \frac{\Gamma(r)}{\Gamma(r-p)} \left(\log \frac{x}{a}\right)^{r-p-1}.$$

The following result will be used in Section 3.3.

THEOREM 2.5. ([16]) (Nonlinear alternative for single valued maps) *Let* E *be a Banach space,* C *a closed, convex subset of* E, *and* U *be an open subset of* C *with* $0 \in U$. *Suppose that* $N : U \to C$ *is a continuous, compact (that is,* $N(U)$ *is a relatively compact subset of* C*) map. Then:*

(i) Either N *has a fixed point in* U, *or*

(ii) There exists $x \in \partial U$ *(the boundary of* U *in* C*) and* $\lambda \in (0,1)$ *with* $x = \lambda N(x)$.

3. Main results

We start by defining what is meant by a solution of the problem (1)–(2).

DEFINITION 3.1. A function $y \in C^1([1,T],\mathbb{R})$ is said to be a solution of (1)–(2) if y satisfies the equation $D^r y(t) = f(t, y(t))$ on J and the boundary conditions (2).

For the existence of solutions to the problem (1)–(2), we need the following auxiliary lemma.

LEMMA 3.2. *Let $h : [1,+\infty) \to \mathbb{R}$ be a continuous function. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) = {} & \frac{(\log t)^{r-1}}{\Gamma(r)[(\log T)^{r-p-1} - \sum_{i=1}^{n} \lambda_i (\log \mu_i)^{r-p-1}]} \\ & \times \Big(\sum_{i=1}^{n} \lambda_i \int_1^{\mu_i} \Big(\log \frac{\mu_i}{s}\Big)^{r-p-1} h(s) \frac{ds}{s} - \int_1^T \Big(\log \frac{T}{s}\Big)^{r-p-1} h(s) \frac{ds}{s} \Big) \\ & + \frac{1}{\Gamma(r)} \int_1^t \Big(\log \frac{t}{s}\Big)^{r-1} h(s) \frac{ds}{s} \end{aligned} \tag{5}$$

if and only if y is a solution of the fractional BVP

$$D^r y(t) = h(t),\ 1 < r \leqslant 2, \tag{6}$$

$$y(1) = 0, \quad D^p y(T) = \sum_{i=1}^{n} \lambda_i D^p y(\mu_i),\ 0 < p \leqslant 1. \tag{7}$$

Proof. Assume y satisfies (6); then Lemma 2.3 implies that

$$y(t) = I^r h(t) + c_1 (\log t)^{r-1} + c_2 (\log t)^{r-2}. \tag{8}$$

The first condition in (7) implies that

$$c_2 = 0.$$

To find c_1, differentiate (8) and apply Lemma 2.4 to obtain

$$D^p y(t) = D^p I^r h(t) + c_1 D^p (\log t)^{r-1} = I^{r-p} h(t) + c_1 \frac{\Gamma(r)}{\Gamma(r-p)} (\log t)^{r-p-1}. \tag{9}$$

Setting $t = T$ gives

$$D^p y(T) = I^{r-p} h(T) + c_1 \frac{\Gamma(r)}{\Gamma(r-p)} (\log T)^{r-p-1}. \tag{10}$$

Evaluating (9) at μ_i, multiplying by λ_i, and summing, we have

$$\sum_{i=1}^{n} \lambda_i D^p y(\mu_i) = \sum_{i=1}^{n} \lambda_i I^{r-p} h(\mu_i) + c_1 \frac{\Gamma(r)}{\Gamma(r-p)} \sum_{i=1}^{n} \lambda_i (\log \mu_i)^{r-p-1}. \tag{11}$$

Combining (10), (11), and the second condition in (7), we find that

$$c_1 = \frac{\Gamma(r-p)\big(\sum_{i=1}^{n} \lambda_i I^{r-p} h(\mu_i) - I^{r-p} h(T)\big)}{\Gamma(r)\big((\log T)^{r-p-1} - \sum_{i=1}^{n} \lambda_i (\log \mu_i)^{r-p-1}\big)}.$$

Substituting the values of c_1 and c_2 into (8), we obtain the solution (5). A direct substitution shows that $y(t)$ in (5) satisfies (6)–(7). $\square$

In the following, for the sake of convenience, we set

$$\Omega = (\log T)^{r-p-1} - \sum_{i=1}^{n} \lambda_i (\log \mu_i)^{r-p-1}.$$

Note that $\Omega \neq 0$ in view of (3). It is important to point out that, in view of (5), the problem (1)–(2) has a solution if and only if the operator

$$\begin{aligned}(Ny)(t) = {} & \frac{(\log t)^{r-1}}{\Gamma(r)\Omega}\Big(\sum_{i=1}^{n} \lambda_i \int_1^{\mu_i} \left(\log \frac{\mu_i}{s}\right)^{r-p-1} f(s,y(s))\frac{ds}{s} \\ & - \int_1^{T} \left(\log \frac{T}{s}\right)^{r-p-1} f(s,y(s))\frac{ds}{s}\Big) \\ & + \frac{1}{\Gamma(r)} \int_1^{t} \left(\log \frac{t}{s}\right)^{r-1} f(s,y(s))\frac{ds}{s} \end{aligned} \tag{12}$$

has a fixed point.

In the following subsections we prove existence, as well as existence and uniqueness, results for the boundary value problem (1)–(2) by using a variety of fixed point theorems.

3.1. Existence and uniqueness result via Banach's fixed point theorem

THEOREM 3.3. *Assume that:*

(H1) There exists a constant $k > 0$ such that

$$|f(t,x) - f(t,y)| \leqslant k|x-y| \ \textit{for a.e. } t \in J \textit{ and } x,y \in \mathbb{R}.$$

If

$$\frac{(\log T)^{r-1}k}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^{n} |\lambda_i| \frac{(\log \mu_i)^{r-p}}{r-p} + \frac{(\log T)^{r-p}}{r-p}\Big) + \frac{k}{\Gamma(r+1)}(\log T)^r < 1,$$

then the BVP (1)–(2) has a unique solution on $[1,T]$.

Proof. We transform the problem (1)–(2) into a fixed point problem using the operator N defined in (12). In order to apply Banach's contraction mapping principle, we need to show that N is a contraction.

Let $x,\ y \in C^1([1,T],\mathbb{R})$; then for each $t \in J$, we have

$$\begin{aligned}|(Nx)(t) - (Ny)(t)| \leqslant {} & \frac{(\log T)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^{n} |\lambda_i| \int_1^{\mu_i} \left(\log \frac{\mu_i}{s}\right)^{r-p-1} |f(s,x(s)) - f(s,y(s))|\frac{ds}{s} \\ & + \int_1^{T} \left(\log \frac{T}{s}\right)^{r-p-1} |f(s,x(s)) - f(s,y(s))|\frac{ds}{s}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r)}\int_1^t\Big(\log\frac{t}{s}\Big)^{r-1}|f(s,x(s))-f(s,y(s))|\frac{ds}{s}\\
&\leqslant \frac{(\log T)^{r-1}k\|x-y\|_\infty}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^n|\lambda_i|\int_1^{\mu_i}\Big(\log\frac{\mu_i}{s}\Big)^{r-p-1}\frac{ds}{s}\\
&\quad+\int_1^T\Big(\log\frac{T}{s}\Big)^{r-p-1}\frac{ds}{s}\Big)+\frac{k\|x-y\|_\infty}{\Gamma(r)}\int_1^t\Big(\log\frac{t}{s}\Big)^{r-1}\frac{ds}{s}\\
&\leqslant \frac{(\log T)^{r-1}k\|x-y\|_\infty}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^n|\lambda_i|\frac{(\log\mu_i)^{r-p}}{r-p}+\frac{(\log T)^{r-p}}{r-p}\Big)\\
&\quad+\frac{k\|x-y\|_\infty}{\Gamma(r+1)}(\log T)^r\\
&\leqslant \Big[\frac{(\log T)^{r-1}k}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^n|\lambda_i|\frac{(\log\mu_i)^{r-p}}{r-p}+\frac{(\log T)^{r-p}}{r-p}\Big)\\
&\quad+\frac{k}{\Gamma(r+1)}(\log T)^r\Big]\|x-y\|_\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|N(y_n)-N(y)\|_\infty \leqslant \Big[&\frac{(\log T)^{r-1}k}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^n|\lambda_i|\frac{(\log\mu_i)^{r-p}}{r-p}+\frac{(\log T)^{r-p}}{r-p}\Big)\\
&+\frac{k}{\Gamma(r+1)}(\log T)^r\Big]\|x-y\|_\infty.
\end{aligned}
$$

This shows that N is a contraction. Therefore, by the Banach contraction mapping principle, N has a fixed point that is the unique solution of our boundary value problem. □

3.2. Existence result via Schaefer's fixed point theorem

Our main result in this subsection is the following.

THEOREM 3.4. *Assume that:*

(H2) The function $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ *is continuous.*

(H3) There exists a constant $M>0$ *such that*

$$|f(t,u))|\leqslant M \text{ for a.e. } t\in J \text{ and each } u\in\mathbb{R}.$$

Then the BVP (1)–(2) has at least one solution on $[1,T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that N defined by (6) has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C^1(J,\mathbb{R})$. Then, for each $t \in J$,

$$\begin{aligned}
|(Ny_n)(t)-(Ny)(t)| \leqslant{}& \frac{(\log t)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^{n}|\lambda_i|\int_1^{\mu_i}\Big(\log\frac{\mu_i}{s}\Big)^{r-p-1}|f(s,y_n(s))-f(s,y(s))|\frac{ds}{s}\\
&+\int_1^{T}\Big(\log\frac{T}{s}\Big)^{r-p-1}|f(s,y_n(s))-f(s,y(s))|\frac{ds}{s}\Big)\\
&+\frac{1}{\Gamma(r)}\int_1^{t}\Big(\log\frac{t}{s}\Big)^{r-1}|f(s,y_n(s))-f(s,y(s))|\frac{ds}{s}\\
\leqslant{}&\Big[\frac{(\log T)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^{n}|\lambda_i|\frac{(\log\mu_i)^{r-p}}{r-p}+\frac{(\log T)^{r-p}}{r-p}\Big)\\
&+\frac{1}{\Gamma(r+1)}(\log T)^{r}\Big]\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty.
\end{aligned}$$

Since f is continuous, we have $\|N(y_n)-N(y)\|_\infty \to 0$ as $n\to\infty$, so N is continuous.

Step 2: *N maps bounded sets into bonded sets in* $C([1,T],\mathbb{R})$.

It suffices to show that for any $\mu^*>0$, there exists a positive constant l such that for each $y\in B_{\mu^*}=\{y\in C([1,T],R):\|y\|_\infty\leqslant\mu^*\}$, we have $\|N(y)\|_\infty<l$. Now

$$\begin{aligned}
|N(y)(t)| \leqslant{}& \frac{(\log t)^{r-1}}{\Gamma(r)|\Omega|}\left(\sum_{i=1}^{n}|\lambda_i|\int_1^{\mu_i}\Big(\log\frac{\mu_i}{s}\Big)^{r-p-1}|f(s,y(s))|\frac{ds}{s}\right.\\
&\left.+\int_1^{T}\Big(\log\frac{T}{s}\Big)^{r-p-1}|f(s,y(s))|\frac{ds}{s}\right)+\frac{1}{\Gamma(r)}\int_1^{t}\Big(\log\frac{t}{s}\Big)^{r-1}|f(s,y(s))|\frac{ds}{s},
\end{aligned}$$

so

$$\|N(y)(t)\|_\infty\leqslant\left[\frac{(\log T)^{r-1}M}{\Gamma(r)|\Omega|}\left(\sum_{i=1}^{n}|\lambda_i|\frac{(\log\mu_i)^{r-p}}{r-p}+\frac{(\log T)^{r-p}}{r-p}\right)+\frac{M}{\Gamma(r+1)}(\log T)^{r}\right]=l.$$

Step 3: *N maps bounded sets into equicontinuous sets of* $C([1,T],\mathbb{R})$.

Let t_1, $t_2\in J$ with $t_1<t_2$, and let B_{μ^*} be a bounded set in $C([1,T],\mathbb{R})$ as in Step 2, and let $y\in B_{\mu^*}$. Then,

$$\begin{aligned}
&|(Ny)(t_2)-(Ny)(t_1)|\\
\leqslant{}&\frac{1}{\Gamma(r)}\int_1^{t_1}\left[\Big(\log\frac{t_2}{s}\Big)^{r-1}-\Big(\log\frac{t_1}{s}\Big)^{r-1}\right]|f(s,y(s))|\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{r-1}|f(s,y(s))|\frac{ds}{s}+\frac{(\log t_2)^{r-1}-(\log t_1)^{r-1}}{\Gamma(r)|\Omega|}\\
&\times\Big(\sum_{i=1}^{n}|\lambda_i|\int_1^{\mu_i}\Big(\log\frac{\mu_i}{s}\Big)^{r-p-1}f(s,y(s))\frac{ds}{s}+\int_1^{T}\Big(\log\frac{T}{s}\Big)^{r-p-1}f(s,y(s))\frac{ds}{s}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{M}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1}-\left(\log\frac{t_1}{s}\right)^{r-1}\right]\frac{ds}{s}+\frac{M}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}\frac{ds}{s}\\
&\quad+\frac{M(\log t_2)^{r-1}-(\log t_1)^{r-1}}{\Gamma(r)|\Omega|}\\
&\quad\times\Big(\sum_{i=1}^{n}|\lambda_i|\int_1^{\mu_i}\left(\log\frac{\mu_i}{s}\right)^{r-p-1}\frac{ds}{s}+\int_1^{T}\left(\log\frac{T}{s}\right)^{r-p-1}\frac{ds}{s}\Big)\\
&\leqslant \frac{M}{\Gamma(r+1)}[(\log t_2)^{r}-(\log t_1)^{r}]+\frac{M\Big((\log t_2)^{r-1}-(\log t_1)^{r-1}\Big)}{\Gamma(r)|\Omega|}\\
&\quad\times\Big(\sum_{i=1}^{n}|\lambda_i|\int_1^{\mu_i}\left(\log\frac{\mu_i}{s}\right)^{r-p-1}\frac{ds}{s}+\int_1^{T}\left(\log\frac{T}{s}\right)^{r-p-1}\frac{ds}{s}\Big)
\end{aligned}
$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzelà-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Step 4: *A priori bounds*

Now it remains to show that the set

$$
\varepsilon = \{y \in C(J,\mathbb{R}) : y = \mu N(y) \text{ for some } 0 < \mu < 1\}
$$

is bounded. We have

$$
\begin{aligned}
y(t) = \mu\Big[&\frac{(\log t)^{r-1}}{\Gamma(r)\Omega}\Big(\sum_{i=1}^{n}\lambda_i\int_1^{\mu_i}\left(\log\frac{\mu_i}{s}\right)^{r-p-1}f(s,y(s))\frac{ds}{s}\\
&-\int_1^{T}\left(\log\frac{T}{s}\right)^{r-p-1}f(s,y(s))\frac{ds}{s}\Big)+\frac{1}{\Gamma(r)}\int_1^{t}\left(\log\frac{t}{s}\right)^{r-1}f(s,y(s))\frac{ds}{s}\Big].
\end{aligned}
$$

For $\mu \in [0,1]$, let y be such that for each $t \in [1,T]$,

$$
\begin{aligned}
|(Ny)(t)| &\leqslant \frac{M(\log t)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^{n}|\lambda_i|\int_1^{\mu_i}(\log\frac{\mu_i}{s})^{r-p-1}\frac{ds}{s}+\int_1^{T}(\log\frac{T}{s})^{r-p-1}\frac{ds}{s}\Big)\\
&\quad+\frac{M}{\Gamma(r)}\int_1^{T}\left(\log\frac{T}{s}\right)^{r-1}\frac{ds}{s}\\
&\leqslant \frac{M(\log T)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^{n}|\lambda_i|\frac{(\log\mu_i)^{r-p}}{r-p}+\frac{(\log T)^{r-p}}{r-p}\Big)+\frac{M}{\Gamma(r+1)}(\log T)^{r}\\
&= R.
\end{aligned}
$$

Thus,

$$
\|N(y)\|_\infty \leqslant R.
$$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem, N has a fixed point that is a solution of the problem (1)–(2). $\square$

3.3. Existence result via the Leray-Schauder nonlinear alternative

Our final existence result in this paper is the following.

THEOREM 3.5. *Assume that the following conditions hold:*

(H4) There exists $\phi_f \in L^1(J,\mathbb{R}_+)$ *and a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *such that*

$$|f(t,u)| \leqslant \phi_f(t)\psi(|u|) \text{ for a.e. } t \in J \text{ and } u \in \mathbb{R}. \tag{13}$$

(H5) There exists $M^* > 0$ *such that*

$$\frac{\Gamma(r)|\Omega|M^*}{\Gamma(r)|\Omega|\psi(M^*)I^r\phi_f(T)+\Gamma(r-p)\psi(M^*)(\log T)^{r-1}\Big(\sum_{i=1}^n|\lambda_i|I^{r-p}\phi_f(\mu_i)+I^{r-p}\phi_f(T)\Big)} > 1.$$

Then the BVP (1)–(2) has at least one solution on $[1,T]$.

Proof. We shall use the Leary-Schauder theorem to prove that N defined by (12) has a fixed point. As shown in the proof of Theorem 3.4, we see that the operator N is continuous, uniformly bounded, and equicontinous. So by the Arzelà-Ascoli theorem, N is completely continuous.

Now in view of (13), for each $t \in [1,T]$,

$$\begin{aligned}
|y(t)| \leqslant{}& \frac{(\log T)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^n|\lambda_i|\int_1^{\mu_i}\left(\log\frac{\mu_i}{s}\right)^{r-p-1}\phi_f(s)\psi(|y|)\frac{ds}{s} \\
&+\int_1^T\left(\log\frac{T}{s}\right)^{r-p-1}\phi_f(s)\psi(|y|)\frac{ds}{s}\Big) \\
&+\frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}\phi_f(s)\psi(|y|)\frac{ds}{s} \\
\leqslant{}& \frac{\Gamma(r-p)\psi(\|y\|_\infty)(\log T)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^n|\lambda_i|I^{r-p}\phi_f(\mu_i)+I^{r-p}\phi_f(T)\Big)+\psi(\|y\|_\infty)I^r\phi_f(T).
\end{aligned}$$

Thus,

$$\frac{\Gamma(r)|\Omega|\|y\|_\infty}{\Gamma(r)|\Omega|\psi(\|y\|_\infty)I^r\phi_f(T)+\Gamma(r-p)\psi(\|y\|_\infty)(\log T)^{r-1}\Big(\sum_{i=1}^n|\lambda_i|I^{r-p}\phi_f(\mu_i)+I^{r-p}\phi_f(T)\Big)} \leqslant 1.$$

By condition (H5), there exists M^* such that $\|y\|_\infty \neq M^*$.

Let $B_{M^*} = \{y \in C([1,T],\mathbb{R}) : \|y\|_\infty < M^*\}$. The operator N is continuous and completely continuous. By the choice of B_{M^*} there is no $y \in \partial B_{M^*}$ such that $y = \mu N(y)$ for some $\mu \in (0,1)$.

As a consequence of the nonlinear alternative of Leary-Schauder type (Theorem 2.5 above), N has a fixed point $y \in B_{M^*}$ that is a solution of the problem, and this completes the proof of the theorem. □

4. Examples

In this section, we present some examples to illustrate our results in the previous section.

EXAMPLE 4.1. Consider the fractional boundary value problem

$$D^{\frac{3}{2}}y(t) = \frac{1}{t^2+8}\sin y, \text{ for a.e. } (t,y) \in ([1,e],\mathbb{R}_+), \tag{14}$$

$$y(1) = 0,\ D^{\frac{1}{2}}y(e) = D^{\frac{1}{2}}y(e) + 2D^{\frac{1}{2}}y(e), \tag{15}$$

where $r = \frac{3}{2}$, $p = \frac{1}{2}$, $T = e$, $n = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, and $\mu_1 = \mu_2 = e$. We have

$$|f(t,y)| = \left|\frac{1}{t^2+8}\sin y\right| \leqslant \frac{1}{9}.$$

Choosing $M = \frac{1}{9}$, we see that

$$\begin{aligned}\|N(y)\|_\infty &\leqslant \frac{M}{\Gamma(r+1)}(\log T)^r + \frac{M(\log T)^{r-1}}{\Gamma(r)|\Omega|}\Big(\sum_{i=1}^{n}|\lambda_i|\frac{(\log\mu_i)^{r-p}}{r-p} + \frac{(\log T)^{r-p}}{r-p}\Big)\\ &= \frac{M}{\Gamma(r+1)} + \frac{2M}{\Gamma(r)} = \frac{4M}{\Gamma(r+1)} = \frac{16}{27\sqrt{\pi}} < 1\end{aligned}$$

is bounded. Then by Theorem 3.4, problem (14)–(15) has a solution on $[1,e]$.

EXAMPLE 4.2. Consider the fractional boundary value problem

$${}^{H}D^{\frac{3}{2}}y(t) = (\log t)^2\frac{y^2}{8|y|+1}, \text{ for a.e. } (t,y) \in ([1,e],\mathbb{R}_+), \tag{16}$$

$$y(1) = 0,\ D^{\frac{1}{2}}y(e) = D^{\frac{1}{2}}y(e) + 2D^{\frac{1}{2}}y(e), \tag{17}$$

where $r = \frac{3}{2}$, $p = \frac{1}{2}$, $T = e$, $n = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = \mu_2 = e$, and

$$f(t,y) = (\log t)^2\frac{y^2}{8|y|+1}.$$

Now,

$$|f(t,y)| = \left|(\log t)^2\frac{y^2}{8|y|+1}\right| \leqslant (\log t)^2\frac{1}{8}|y|,$$

so choosing $\psi(|y|) = \dfrac{1}{8}|y|$ and $\phi_f(t) = (\log t)^2$, we have $I^1\phi_f(t) = \frac{(\log t)^3}{3}$, and $I^{\frac{3}{2}}\phi_f(e)$

$= \dfrac{32}{35\sqrt{\pi}}$. We see that

$$\frac{\Gamma(r)|\Omega|M^*}{\Gamma(r)|\Omega|\psi(M^*)I^r\phi_f(T)+\Gamma(r-p)\psi(M^*)(\log T)^{r-1}\Big(\sum_{i=1}^n|\lambda_i|I^{r-p}\phi_f(\mu_i)+I^{r-p}\phi_f(T)\Big)}$$
$$=\frac{\sqrt{\pi}M^*}{\sqrt{\pi}\psi(M^*)I^{\frac{3}{2}}\phi_f(e)+\psi(M^*)\Big(\sum_{i=1}^2|\lambda_i|I^1\phi_f(\mu_i)+I^1\phi_f(e)\Big)}$$
$$=\frac{8\sqrt{\pi}}{\sqrt{\pi}I^{\frac{3}{2}}\phi_f(e)+\Big(\sum_{i=1}^2|\lambda_i|I^1\phi_f(\mu_i)+I^1\phi_f(e)\Big)}=\frac{840\sqrt{\pi}}{209}>1.$$

Hence, all the conditions of Theorem 3.5 are satisfied, so there exists at least one solution of the problem (16)–(17) on $[1,e]$.

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