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COEFFICIENT FUNCTIONAL FOR THE KTH ROOT TRANSFORM OF ANALYTIC FUNCTION AND APPLICATIONS TO FRACTIONAL DERIVATIVES

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Abstract. In the present investigation, the authors introduce certain subclass of analytic function and obtain the sharp upper bounds for the coefficient functional $|b_{2k+1} - \nu b_{k+1}^2|$ corresponding to the kth root transformation of certain normalized analytic function defined on the unit disk Δ in the complex plane. As an application of the main results, we obtain the Fekete-Szegö inequalities for the function defined by fractional derivatives. Similar problems are investigated for the inverse function of f and for the function $\frac{z}{f(z)}$. Our results generalize and unify the work of earlier researchers in this direction.

1. Introduction and definition

Denote by $\mathscr{A}$, the class of functions of the form

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathbb{S}$ denote the family of functions $f(z) \in \mathscr{A}$ which are univalent.

For two analytic functions f and g in Δ, the function f is subordinate to g, written as $f(z) \prec g(z)$ $(z \in \Delta)$ if there exists a Schwarz function w, which (by definition) is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \Delta)$. It follows from the Schwarz lemma that $f(z) \prec g(z)$ $(z \in \Delta) \Longrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. If the function g is univalent in Δ then (see [14])

$$f(z) \prec g(z) \ (z \in \Delta) \Longleftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized univalent function $f(z)$ of the form (1) is well known for its rich history in the theory of geometric function theory. In 1933, Fekete-Szegö disproof the conjecture of Littlewood and Parley that the coefficient

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of odd univalent functions are bounded by unity (see [10]). Since then, it received a great attention among many researchers (for details, see [1, 3, 5, 6, 7, 8, 9, 12, 16]). The technique used by Ma and Minda (see [13]) for the Fekete-Szegö problem for a univalent function $f(z)$ of the form (1) for subclasses of convex and starlike functions were used by many authors to solve the same problem for other classes.

Let k be a positive integer. A domain D is said to be k-fold symmetric if a rotation of D about the origin through an angle $\frac{2\pi}{k}$ carries D to itself. A function f is said to be k-fold symmetric in Δ if $f(e^{\frac{2\pi i}{k}}z) = e^{\frac{2\pi i}{k}}f(z)$ for every $z \in \Delta$. If f is regular and k-fold symmetric in Δ, then

$$f(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \cdots. \tag{2}$$

Conversely, if f is given by (2), then f is k-fold symmetric inside the circle of convergence of the series. For an univalent function f of the form (1), the kth root transformation is defined by

$$G(z) = \left[f(z^k)\right]^{\frac{1}{k}} = z\left[\frac{f(z^k)}{z^k}\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}, \tag{3}$$

where the initial coefficients are

$$\begin{aligned} b_{k+1} &= \frac{a_2}{k}, \qquad b_{2k+1} = \frac{a_3}{k} + \frac{1-k}{2k^2}a_2^2 \\ b_{3k+1} &= \frac{a_4}{k} + \frac{1-k}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{3!k^3}a_2^3. \end{aligned}$$

Since f is univalent, so $\frac{f(z^k)}{z^k}$ is non-vanishing in Δ implies that the kth root is an analytic in Δ. The Fekete-Szegö coefficient functional of the associated function $G(z)$ is given by $|b_{2k+1} - \nu b_{k+1}^2|$. This quantity is known as Fekete-Szegö problem of the kth root transform G.

Recently, Ali et al. [2] (also see [20]) have investigated the Fekete-Szegö coefficient functional for the kth root transform of functions belonging to various subclasses of analytic functions by means of subordination. Further, Annamalai et al. [4] have obtained sharp bound of the Fekete-Szegö coefficient functional for the Janowski α-spirallike functions associated with the kth root transformation.

Motivated by the works of Ali et al. [2], Sharma et al. [20] and Annamalai et al. [4], in this paper the authors define generalized subclass of analytic functions of complex order and investigate the Fekete-Szegö coefficient functional associated with kth root transformation of the function f and for the function defined through convolution and fractional derivative in these classes. Similar approach is used to obtain the Fekete-Szegö inequalities for the inverse function of f and for the function $\frac{z}{f(z)}$.

DEFINITION 1. Let $\phi(z)$ be a univalent analytic function with positive real part on Δ with $\phi(0) = 1$ and $\phi'(0) > 0$ where $\phi(z)$ maps Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Let b be a non-zero complex number and γ be a real number such that $0 < \gamma \leqslant 1$. A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ if

$$1+\frac{1}{b}\left[(1-\beta)\left(\frac{f(z)}{z}\right)^{\alpha}+\beta\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}-1\right] \prec [\phi(z)]^{\gamma} \quad (0\leqslant\beta\leqslant1,\ \alpha\geqslant0). \tag{4}$$

The powers are taken with their principal values.

Note that, by specializing the parameters b,α,β and γ, the class $\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ reduces to the following classes studied by various earlier researchers.

- $\mathscr{R}^{\gamma}_{b,0,1}(\phi) = S^{\gamma}_{b}(\phi)$ introduced and studied by Sharma et al. [20].
- $\mathscr{R}^{1}_{1,0,1}(\phi) = S^{*}(\phi)$ introduced and studied by Ma and Minda [13] (also see [2]).
- $\mathscr{R}^{1}_{b,0,1}(\phi) = S_{b}(\phi)$ introduced and studied by Ravichandran et al. [18].
- $\mathscr{R}^{1}_{1,\alpha,1}(\phi) = B^{\alpha}(\phi)$ introduced and studied by Ravichandran et al. [19].
- $\mathscr{R}^{1}_{b,\alpha,\beta}(\phi) = R_{b,p,\alpha,\beta}(\phi)$ (with p=1) introduced and studied by Ramachandran et al. [17].

The paper is organized in the following manner. In Section 3, sharp upper bounds for the Fekete-Szegö coefficient functional $|b_{2k+1} - \nu b^{2}_{k+1}|$ associated with kth root transform of the function f belonging to the above mentioned class is investigated. In Section 4, applications for the functions defined by fractional derivatives is obtained. Similar results have been derived for the function $\frac{z}{f(z)}$ in Section 5 and inverse of the function f in Section 6.

2. Preliminaries

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$ satisfying the condition $|w(z)| < 1$.

We need the following lemmas in order to prove our main results:

LEMMA 1. (see [3]) *If $w \in \Omega$ and*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \Delta),$$

then for any real numbers t, we have

$$|w_2 - t w_1^2| \leqslant \begin{cases} -t & t \leqslant -1 \\ 1 & -1 \leqslant t \leqslant 1 \\ t & t \geqslant 1. \end{cases}$$

For $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality *holds for* $t=-1$ *if and only if* $w(z)=z\left(\frac{\varepsilon+z}{1+\varepsilon z}\right)$ $(0\leqslant\varepsilon\leqslant 1)$ *or one of its rotations, while for* $t=1$*, equality holds if and only if* $w(z)=-z\left(\frac{\varepsilon+z}{1+\varepsilon z}\right)$ $(0\leqslant\varepsilon\leqslant 1)$ *or one of its rotations.*

LEMMA 2. (see [11]) *If* $w\in\Omega$*, then*

$$|w_2-tw_1^2|\leqslant\max\{1,|t|\},$$

for any complex number t*. The result is sharp for the function* $w(z)=z^2$ *or* $w(z)=z$*.*

3. Coefficient bounds

In this section, the bounds for the functional $|b_{2k+1}-\nu b_{k+1}^2|$ corresponding to the kth root transformation for the function f in the above class is derived.

THEOREM 1. *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$ *with* $B_1>0$, $B_2\geqslant 0$ *and* B_n*'s real. If* $f\in\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ *and* G *is the* k*th root transformation of* f *given by (3), then for any complex number* ν*, we have*

$$|b_{2k+1}-\nu b_{k+1}^2|\leqslant\frac{|b|\gamma B_1}{(\alpha+2\beta)k}\max\left\{1,\left|\frac{b\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}[2\nu-(1-\alpha k)]-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right|\right\}. \tag{5}$$

The estimate is sharp.

Proof. Let $f\in\mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$. Then by Definition 1, there exists a Schwarz's function $w(z)\in\Omega$ with $w(0)=0$ and $|w(z)|<1$ such that

$$1+\frac{1}{b}\left[(1-\beta)\left(\frac{f(z)}{z}\right)^{\alpha}+\beta\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}-1\right]=[\phi(w(z))]^{\gamma}. \tag{6}$$

Since

$$\left(\frac{f(z)}{z}\right)^{\alpha}=1+\alpha a_2z+\left(\alpha a_3+\frac{\alpha(\alpha-1)}{2}a_2^2\right)z^2+\cdots$$

and

$$\frac{zf'(z)}{f(z)}=1+a_2z+(2a_3-a_2^2)z^2+\cdots,$$

it follows from above that

$$\begin{aligned}&1+\frac{1}{b}\left[(1-\beta)\left(\frac{f(z)}{z}\right)^{\alpha}+\beta\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}-1\right]\\&=1+\frac{\alpha+\beta}{b}a_2z+\left(\frac{\alpha+2\beta}{b}a_3+\frac{(\alpha-1)(\alpha+2\beta)}{2b}a_2^2\right)z^2+\cdots.\end{aligned} \tag{7}$$

Also,

$$
\begin{aligned}
[\phi(w(z))]^{\gamma} &= [1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\cdots]^{\gamma} \\
&= 1+\gamma B_1w_1z+\{\gamma B_1w_2+\gamma B_2w_1^2+\frac{\gamma(\gamma-1)}{2}B_1^2w_1^2\}z^2+\cdots. \qquad (8)
\end{aligned}
$$

Making use of (7) and (8) in (6), we obtain

$$
a_2=\frac{b\gamma B_1w_1}{\alpha+\beta} \qquad (9)
$$

and

$$
a_3=\frac{b\gamma}{\alpha+2\beta}\left[B_1w_2+B_2w_1^2+\frac{\gamma-1}{2}B_1^2w_1^2-\frac{(\alpha-1)(\alpha+2\beta)}{2(\alpha+\beta)^2}b\gamma B_1^2w_1^2\right]. \qquad (10)
$$

If $G(z)$ is the kth root transformation of $f(z)$, then

$$
\begin{aligned}
G(z)=[f(z^k)]^{\frac{1}{k}} &= z+\frac{a_2}{k}z^{k+1}+\left[\frac{a_3}{k}-\frac{k-1}{2k^2}a_2^2\right]z^{2k+1}+\cdots \\
&= z+\sum_{n=1}^{\infty}b_{nk+1}z^{nk+1}.
\end{aligned}
$$

Equating the coefficients of z^{k+1} and z^{2k+1}, we get

$$
b_{k+1}=\frac{a_2}{k}=\frac{b\gamma B_1w_1}{k(\alpha+\beta)}, \qquad (11)
$$

and

$$
\begin{aligned}
b_{2k+1}=\frac{a_3}{k}-\frac{k-1}{2k^2}a_2^2 &= \frac{b\gamma B_1}{(\alpha+2\beta)k}\left[w_2+\frac{B_2}{B_1}w_1^2+\frac{\gamma-1}{2}B_1w_1^2\right. \\
&\left.+\frac{(1-\alpha)(\alpha+2\beta)}{2(\alpha+\beta)^2}b\gamma B_1w_1^2-\left(1-\frac{1}{k}\right)\frac{\alpha+2\beta}{2(\alpha+\beta)^2}b\gamma B_1w_1^2\right]. \qquad (12)
\end{aligned}
$$

Thus, for any complex number ν, we have

$$
b_{2k+1}-\nu b_{k+1}^2=\frac{b\gamma B_1}{(\alpha+2\beta)k}[w_2-tw_1^2], \qquad (13)
$$

where

$$
t=\frac{b\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}[2\nu-(1-\alpha k)]-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1.
$$

Therefore,

$$
|b_{2k+1}-\nu b_{k+1}^2|=\frac{|b|\gamma B_1}{(\alpha+2\beta)k}|w_2-tw_1^2|. \qquad (14)
$$

An application of Lemma 2 to the right hand side of (14) gives the desire result as stated in Theorem 1. The estimation is sharp and followed by

$$|b_{2k+1}-\nu b_{k+1}^2| = \begin{cases} \frac{|b|\gamma B_1}{(\alpha+2\beta)k} & (w(z)=z^2) \\ \frac{|b|\gamma B_1}{(\alpha+2\beta)k}\left\{\left|\frac{b\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right|\right\} & (w(z)=z). \end{cases}$$

This complete the proof of Theorem 1. □

REMARK 1. (i) Putting $\alpha=0$ and $\beta=1$ in Theorem 1 gives the result obtained by Sharma et al. [20].

(ii) Letting $\alpha=0$, $\beta=b=\gamma=1$ we get the result due to Ma and Minda [13] (also, see [2]).

Now, we determine the bounds for the functional $|b_{2k+1}-\nu b_{k+1}^2|$ for real ν for the class $\mathscr{R}_{1,\alpha,\beta}^{\gamma}(\phi)$. We denote such class by $\mathscr{R}_{\alpha,\beta}^{\gamma}(\phi)$.

THEOREM 2. *If $f\in\mathscr{R}_{\alpha,\beta}^{\gamma}(\phi)$ and G is the kth root transformation of the function f defined by (3), then for any real number ν and for*

$$d_1 = \frac{2k(\alpha+\beta)^2[B_2-B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

$$d_2 = \frac{2k(\alpha+\beta)^2[B_2+B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

we have

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \begin{cases} \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))\right] & \nu\leqslant d_1, \\ \frac{\gamma B_1}{(\alpha+2\beta)k} & d_1\leqslant\nu\leqslant d_2 \\ \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right] & \nu\geqslant d_2. \end{cases} \tag{15}$$

Each of the estimates in (15) is sharp.

Proof. Let $f\in\mathscr{R}_{\alpha,\beta}^{\gamma}(\phi)$. From (13) we have

$$b_{2k+1}-\nu b_{k+1}^2 = \frac{\gamma B_1}{(\alpha+2\beta)k}[w_2-tw_1^2], \tag{16}$$

where

$$t=\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}[2\nu-(1-\alpha k)]-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1.$$

Taking modulus on both sides of (16), we get

$$|b_{2k+1} - \nu b_{k+1}^2| = \frac{\gamma B_1}{(\alpha+2\beta)k}|w_2 - t w_1^2|. \tag{17}$$

An application of Lemma 1 on right hand side of (17) gives the following cases.

Case 1: If $\nu \leqslant d_1$, then

$$\nu \leqslant \frac{2k(\alpha+\beta)^2[(B_2-B_1)+\frac{\gamma-1}{2}B_1^2]+(1-\alpha k)(\alpha+2\beta)\gamma B_1^2}{2(\alpha+2\beta)\gamma B_1^2}.$$

$$\Longrightarrow t \leqslant -1 \Longrightarrow |w_2 - t w_1^2| \leqslant -t \Longrightarrow$$

$$|b_{2k+1} - \nu b_{k+1}^2| \leqslant \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1 - \frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k))\right]. \tag{18}$$

Case 2: If $d_1 \leqslant \nu \leqslant d_2$, then

$$\frac{2k(\alpha+\beta)^2[B_2-B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2} \leqslant \nu$$

$$\leqslant \frac{2k(\alpha+\beta)^2[B_2+B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

$$\Longrightarrow -1 \leqslant t \leqslant 1 \Longrightarrow$$

$$|b_{2k+1} - \nu b_{k+1}^2| \leqslant \frac{\gamma B_1}{(\alpha+2\beta)k}. \tag{19}$$

Case-III: If $\nu \geqslant d_2$, then

$$\nu \geqslant \frac{2k(\alpha+\beta)^2[B_2+B_1+\frac{\gamma-1}{2}B_1^2]+\gamma B_1^2(\alpha+2\beta)(1-\alpha k)}{2(\alpha+2\beta)\gamma B_1^2}$$

$$\Longrightarrow t \geqslant 1 \Longrightarrow |w_2 - t w_1^2| \leqslant t \Longrightarrow$$

$$|b_{2k+1} - \nu b_{k+1}^2| \leqslant \frac{\gamma B_1}{(\alpha+2\beta)k}\left[\frac{\gamma B_1(\alpha+2\beta)}{2k(\alpha+\beta)^2}(2\nu-(1-\alpha k)) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right]. \tag{20}$$

The results in (15) follows from (18), (19) and (20). We also note the following:

(i) When $\nu \leqslant d_1$, the equality holds if and only if $w(z) = z$ or one of its rotation.

(ii) When $d_1 \leqslant \nu \leqslant d_2$, then the equality holds if and only if $w(z) = z^2$ or one of its rotation.

(iii) When $\nu \geqslant d_2$, then the equality holds when $w(z) = \frac{z(\varepsilon+z)}{1+\varepsilon z}$ $(0 \leqslant \varepsilon \leqslant 1)$ or one of its rotation.

This complete the proof of Theorem 2. □

REMARK 2. (i) Putting $\alpha = 0$ and $\beta = 1$ in Theorem 2 gives the result due to Sharma et al. [20].

(ii) Taking $\alpha = 0$ and $\beta = b = \gamma = 1$ in Theorem 2 we obtain the first part result of Ali et al. ([2], Theorem 2.1, p. 122).

4. Applications to functions defined by fractional derivatives

For a fixed $g \in \mathscr{A}$, we define $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$ to be the class of functions $f \in \mathscr{A}$ for which $(f*g) \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$. In order to introduce the class $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$, we need the following:

DEFINITION 2. (see [15]) Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order δ is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi \quad (0 \leqslant \delta < 1),$$

where the multiplicity of $(z-\xi)^{\delta}$ is removed by requiring that $\log(z-\xi)$ is real for $(z-\xi) > 0$.

Using Definition 2, Owa and Srivastava [15] introduced a fractional derivative operator $\Omega^{\delta} : \mathscr{A} \longrightarrow \mathscr{A}$ defined by

$$\Omega^{\delta} f(z) = \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z) \quad (\delta \neq 2,3,4,\cdots).$$

The class $\mathscr{R}^{\gamma,\delta}_{b,\alpha,\beta}(\phi)$ consists of function $f \in \mathscr{A}$ for which $\Omega^{\delta} f \in \mathscr{T}^{\gamma}_{b,\alpha,\beta}(\phi)$. The class $\mathscr{R}^{\gamma,\delta}_{b,\alpha,\beta}(\phi)$ is the special case of the class $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n.$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0).$$

Since $f \in \mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi) \Longrightarrow (f*g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$. Applying Theorem 1 and Theorem 2 for the function

$$(f*g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$$

we get Theorem 3 and Theorem 4 (mentioned below) respectively.

THEOREM 3. *Let* $g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \quad (B_1 > 0)$. *If* $f(z) \in \mathscr{A}$ *given by* (1) *belong to the class* $\mathscr{R}^{\gamma,g}_{b,\alpha,\beta}(\phi)$ *and* F *is the* k*th root transformation of* f *given by* (3)*. Then for any complex number* ν*, we have*

$$|b_{2k+1} - \nu b_{k+1}^2| \leqslant \frac{|b|\gamma B_1}{k(\alpha+2\beta)g_3} \max\left\{1, \left|\frac{b\gamma B_1(\alpha+2\beta)g_3}{2k(\alpha+\beta)^2 g_2^2}\left[(2\nu-1)+k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right)\right] - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right|\right\}.$$

The estimate is sharp.

THEOREM 4. *If $f \in \mathscr{R}^{\gamma,g}_{\alpha,\beta}(\phi)$ and F is the kth root transform of the function f given by (3), then for any real number ν and for*

$$e_1 = \frac{1}{2}\left[1+\frac{2k(\alpha+\beta)^2 g_2^2}{\gamma B_1(\alpha+2\beta)g_3}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-1\right)-k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right)\right],$$

and

$$e_2 = \frac{1}{2}\left[1+\frac{2k(\alpha+\beta)^2 g_2^2}{\gamma B_1(\alpha+2\beta)g_3}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right)\right],$$

we have

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \begin{cases} \frac{\gamma B_1}{k(\alpha+2\beta)g_3}\left[\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{\gamma B_1(\alpha+2\beta)g_3}{2k(\alpha+\beta)^2 g_2^2}\left\{(2\nu-1)+k(1-(1-\alpha)\frac{g_2^2}{g_3})\right\}\right] & (\nu \leqslant e_1) \\ \frac{\gamma B_1}{k(\alpha+2\beta)g_3} & (e_1 \leqslant \nu \leqslant e_2) \\ \frac{\gamma B_1}{k(\alpha+2\beta)g_3}\left[\frac{\gamma B_1(\alpha+2\beta)g_3}{2k(\alpha+\beta)^2 g_2^2}\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{g_2^2}{g_3}\right)-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right\}\right] & (\nu \geqslant e_2). \end{cases}$$

The result is sharp.

As

$$\Omega^{\delta} f(z) = z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_n z^n \tag{21}$$

we have

$$g_2 = \frac{2}{2-\delta} \tag{22}$$

$$g_3 = \frac{6}{(2-\delta)(3-\delta)}. \tag{23}$$

For g_2 and g_3 given by (22) and (23) respectively, Theorem 3 and Theorem 4 reduces to the following results.

THEOREM 5. *If $f \in \mathscr{R}^{\gamma,\delta}_{b,\alpha,\beta}(\phi)$ and F is the kth root transform of f given by (3), then for any complex number ν, we have*

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \frac{|b|\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)}\max\left\{1,\left|\frac{3b\gamma B_1(\alpha+2\beta)(2-\delta)}{4k(3-\delta)(\alpha+\beta)^2}\right.\right.$$
$$\left.\left.\times\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right\}-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right|\right\}.$$

THEOREM 6. *If $f \in \mathscr{R}^{\gamma,\delta}_{\alpha,\beta}(\phi)$ and $g_n > 0$ and F is the kth root transform of f given by (3), then for any real number ν and for*

$$e_1 = \frac{1}{2}\left[1+\frac{4k(\alpha+\beta)^2(3-\delta)}{3(2-\delta)\gamma B_1(\alpha+2\beta)}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}-1\right)-k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right],$$

and

$$e_2 = \frac{1}{2}\left[1+\frac{4k(\alpha+\beta)^2(3-\delta)}{3(2-\delta)\gamma B_1(\alpha+2\beta)}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right]$$

we have

$$|b_{2k+1}-\nu b_{k+1}^2| \leqslant \begin{cases} \frac{\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)}\left[\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{3\gamma B_1(\alpha+2\beta)(2-\delta)}{4k(\alpha+\beta)^2(3-\delta)}\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)\right\}\right] & (\nu \leqslant e_1) \\ \frac{\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)} & (e_1 \leqslant \nu \leqslant e_2) \\ \frac{\gamma B_1(2-\delta)(3-\delta)}{6k(\alpha+2\beta)}\left[\frac{3\gamma B_1(\alpha+2\beta)(2-\delta)}{4k(\alpha+\beta)^2(3-\delta)}\left\{(2\nu-1)+k\left(1-(1-\alpha)\frac{2(3-\delta)}{3(2-\delta)}\right)-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right\}\right] & (\nu \geqslant e_2). \end{cases}$$

The result is sharp.

5. Coefficient functional associated with $\frac{z}{f(z)}$

In this section, bounds for Fekete-Szegö coefficient functional associated with the function H defined by

$$H(z) = \frac{z}{f(z)} = 1+\sum_{n=1}^{\infty} s_n z^n \tag{24}$$

where f belongs to the class $\mathscr{R}_{b,\alpha,\beta}^{\gamma}(\phi)$ are obtained. The proof of the results obtained in this section are similar to those given in Section 3 and hence we chose to omit the details.

THEOREM 7. *Let $f \in \mathscr{R}_{b,\alpha,\beta}^{\gamma}(\phi)$ and $H(z) = \frac{z}{f(z)} = 1+\sum_{n=1}^{\infty} s_n z^n$. Then for any complex number ν, we have*

$$|s_2-\nu s_1^2| \leqslant \frac{|b|\gamma B_1}{\alpha+2\beta}\max\left\{1,\left|(2\nu-1-\alpha)\frac{(\alpha+2\beta)b\gamma B_1}{2(\alpha+\beta)^2}+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right|\right\}. \tag{25}$$

The result is sharp.

Proof. A simple computation gives

$$H(z) = \frac{z}{f(z)} = 1-a_2 z+(a_2^2-a_3)z^2+\cdots \tag{26}$$

From (24) and (26) we have

$$s_1 = -a_2, \tag{27}$$

and

$$s_2 = a_2^2-a_3. \tag{28}$$

Using (9) and (10) in (27) and (28), we obtain

$$s_1 = -\frac{b\gamma B_1 w_1}{\alpha+\beta},$$
$$s_2 = \frac{b^2\gamma^2 B_1^2 w_1^2}{(\alpha+\beta)^2} - \frac{b\gamma}{\alpha+2\beta}\left[B_1 w_2 + \left\{B_2 + \frac{\gamma-1}{2}B_1^2 + \frac{(1-\alpha)(\alpha+2\beta)}{2(\alpha+\beta)^2} b\gamma B_1^2\right\} w_1^2\right].$$

For any complex number ν,

$$|s_2 - \nu s_1^2| = \frac{|b|\gamma B_1}{\alpha+2\beta}\left| w_2 - \left\{(1+\alpha-2\nu)\frac{(\alpha+2\beta)b\gamma B_1}{2(\alpha+\beta)^2} - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right\} w_1^2\right|.$$

The desire result can be obtained by application of Lemma 2. This complete the proof of Theorem 7. □

Restricting ν to be real and taking $b=1$, we now obtain the coefficient inequality for the function f in the class $\mathscr{R}^{\gamma}_{\alpha,\beta}(\phi)$. Proceeding similar manner as in Theorem 2 for the function $\frac{z}{f(z)}$ we can obtain the following result.

THEOREM 8. *IF $f \in \mathscr{R}^{\gamma}_{\alpha,\beta}(\phi)$ and $H(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} s_n z^n$, then for any real number ν, and for*

$$p_1 = \frac{(1+\alpha)(\alpha+2\beta)\gamma B_1^2 - 2(\alpha+\beta)^2(B_1 + B_2 + \frac{\gamma-1}{2}B_1^2)}{2(\alpha+2\beta)\gamma B_1^2}$$

and

$$p_2 = \frac{(1+\alpha)(\alpha+2\beta)\gamma B_1^2 - 2(\alpha+\beta)^2(B_2 - B_1 + \frac{\gamma-1}{2}B_1^2)}{2(\alpha+2\beta)\gamma B_1^2}$$

we have

$$|s_2 - \nu s_1^2| \leqslant \begin{cases} \frac{\gamma B_1}{\alpha+2\beta}\left[\frac{(\alpha+2\beta)\gamma B_1}{2(\alpha+\beta)^2}(1+\alpha-2\nu) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right] & \nu \leqslant p_1 \\ \frac{\gamma B_1}{\alpha+2\beta} & p_1 \leqslant \nu \leqslant p_2 \\ \frac{\gamma B_1}{\alpha+2\beta}\left[\frac{(\alpha+2\beta)\gamma B_1}{2(\alpha+\beta)^2}(2\nu-\alpha-1) + \frac{B_2}{B_1} + \frac{\gamma-1}{2}B_1\right] & \nu \geqslant p_2. \end{cases}$$

6. Coefficient inequality for the inverse of the function $f(z)$

THEOREM 9. *If $f \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n$ is the inverse function of f with $|w| < r_0$, where r_0 is the greater than the radius of the Koebe domain of the class $f \in \mathscr{R}^{\gamma}_{b,\alpha,\beta}(\phi)$, then for any complex number ν, we have*

$$|l_3 - \nu l_2^2| \leqslant \frac{|b|\gamma B_1}{\alpha+2\beta}\max\left\{1, \left|(\alpha+3-2\nu)\frac{b\gamma B_1(\alpha+2\beta)}{2(\alpha+\beta)^2} - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right|\right\}. \quad (29)$$

The result is sharp.

Proof. Since

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n \tag{30}$$

is the inverse function of f, we have

$$f^{-1}(f(z)) = f(f^{-1}(z)) = z \tag{31}$$

From (30) and (31) we have

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z \tag{32}$$

Equating the coefficients of z and z^2 from equations (30) and (32) we obtain

$$l_2 = -a_2 = -\frac{b\gamma B_1 w_1}{\alpha+\beta},$$

$$l_3 = 2a_2^2 - a_3 = 2\frac{b^2\gamma^2 B_1^2 w_1^2}{(\alpha+\beta)^2} - \frac{b\gamma}{\alpha+2\beta}\left[B_1 w_2 + \left\{B_2 + \frac{\gamma-1}{2}B_1^2 + \frac{(1-\alpha)(\alpha+2\beta)}{2(\alpha+\beta)^2} b\gamma B_1^2\right\} w_1^2\right].$$

For any complex number ν, we have

$$|l_3 - \nu l_2^2| = \frac{|b|\gamma B_1}{(\alpha+2\beta)}\left|w_2 - \left\{\frac{b\gamma B_1(\alpha+2\beta)}{2(\alpha+\beta)^2}(\alpha+3-2\nu) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right\} w_1^2\right|. \tag{33}$$

Applying Lemma 2 on right hand side of (33) we obtain the require result as stated in (29). The result is sharp and followed by

$$|l_3 - \nu l_2^2| = \begin{cases} \frac{|b|\gamma B_1}{\alpha+2\beta} & w(z) = z^2 \\ \frac{|b|\gamma B_1}{\alpha+2\beta}\left|\frac{b\gamma B_1(\alpha+2\beta)}{2(\alpha+\beta)^2}(\alpha+3-2\nu) - \frac{B_2}{B_1} - \frac{\gamma-1}{2}B_1\right| & w(z) = z. \end{cases} \qquad \square$$

CONCLUDING REMARK. For $k = 1$, the kth root transformation of f reduces to the given function f itself. Therefore, the estimate given in equations (5) and (15) (with $b = \gamma = 1$) is an extension of the corresponding results for the Fekete-Szegö functional with $p = 1$ studied by Ramachandran et al. [17]. For the class defined in (4), an attempt has made for finding second Hankel determinant for the kth root transform of f by making use of Chebyshev polynomial.

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