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DETERMINATION OF A DISTRIBUTION IN A SOURCE TERM OF A TIME FRACTIONAL DIFFUSION–WAVE EQUATION

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Abstract. We study the inverse Cauchy problem to a time fractional diffusion-wave equation with distributions in right-hand sides. This problem is to find a generalized solution of direct problem and an unknown time-dependent part of a source from the space of distributions. The unique solvability of the problem is established.

1. Introduction

The conditions of classical solvability of the Cauchy and boundary value problems to equations with the regularized time fractional derivative were obtained in [3, 4, 10, 12, 13, 14, 19, 20] and other works. In such of these works the method of the Green function was used.

The inverse boundary value problems to a time fractional diffusion equations with different unknown functions or parameters were investigated, for example, in [1, 2, 5, 7, 8, 9, 11, 16, 17, 21]. Most papers were devoted to the inverse source problems (see, for example, [1, 5, 8, 16, 21]). Mainly such problems were studied at regular data.

In this paper we study the inverse Cauchy problem

$$u_t^{(\beta)} - \Delta u = g(t)F_0(x), \quad (x,t) \in \mathbb{R}^n \times (0,T] := Q, \tag{1}$$

$$\frac{\partial^{j-1}}{\partial t^{j-1}} u(x,0) = F_j(x), \ \ x \in \mathbb{R}^n, \quad j = \overline{1,m}, \tag{2}$$

$$(u(\cdot,t), \varphi_0(\cdot)) = F(t), \quad t \in (0,T] \tag{3}$$

with the Riemann-Liouville fractional derivative of order $\beta \in (m-1,m)$, $m,n \in \mathbb{N}$, given distributions F, F_j, $j=\overline{0,m}$, the unknown distribution $g(t)$. Here $(u(\cdot,t),\varphi_0(\cdot))$ stands for the value of an unknown distribution u on given test function φ_0 for every $t \in [0,T]$ and defines the distribution

$$\big((u(x,t),\varphi_0(x)),\eta(t)\big) = \big(u(x,t),\varphi_0(x)\eta(t)\big)$$

for every test function η. We prove the existence and uniqueness of a solution (u,g) of the problem in the cases $m = 1,2$.

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Note that elliptic and parabolic initial and boundary value problems to differential and pseudo-differential equations having distributions in right-hand sides are sufficiently investigated (see, for example, [6, 15] and references therein), the inverse Cauchy problem on determination a pair (u,g) under given distributions in right-hand sides of the direct problem, similar over-determination condition and the unknown continuous $g(t)$, $t\in[0,T]$ was studied in [11].

2. Definitions and auxiliary results

We use the following: $\mathscr{D}(\mathbb{R}^n)$ is the space of indefinitely differentiable functions with compact supports in $\mathbb{R}^n$, $\mathscr{D}[0,T]=\{v\in C^\infty[0,T]: v^{(s)}(T)=0,\ s\in\mathbb{Z}_+\}$, $C^{\infty,(0)}(\bar{Q})=\{v\in C^\infty(\overline{Q}):(\frac{\partial}{\partial t})^k v|_{t=T}=0,\ k\in\mathbb{Z}_+\}$, $\mathscr{D}(\overline{Q})$ is the space of functions from $C^{\infty,(0)}(\bar{Q})$ having compact supports, $\mathscr{D}'[0,T]$, $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{D}'(\overline{Q})$ are the spaces of linear continuous functionals (distributions [18, p. 13-15]) on $\mathscr{D}[0,T]$, $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{D}(\overline{Q})$, respectively, $\mathscr{E}'(\mathbb{R}^n)=[C^\infty(\mathbb{R}^n)]'$ is the space of distributions with compact supports, the symbol (f,φ) stands for the value of the distribution f on the test function φ.

We denote $(g\hat{*}\varphi)(x)=\big(g(\xi),\varphi(x+\xi)\big)$, by $f*g$ the convolution of the distributions f and g: $(f*g,\varphi)=(f,g\hat{*}\varphi)$ for any test function φ, by $f\times g=f\cdot g=fg$ the direct product of the distributions f and g: $(fg,\varphi)=(f(x),(g(t),\varphi(x,t))$ for any test function $\varphi(x,t)$, use the function

$$f_\lambda(t)=\begin{cases}\frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)}, & \lambda>0\\ f'_{1+\lambda}(t), & \lambda\leqslant 0\end{cases},$$

where $\Gamma(\lambda)$ is the Gamma-function, $\theta(t)$ is the Heaviside function, and the relations

$$f_\lambda * f_\mu = f_{\lambda+\mu},\qquad f_\lambda\hat{*}f_\mu=f_{\lambda+\mu}.$$

Note that the Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta>0$ is defined by the formula

$$v^{(\beta)}(t)=f_{-\beta}(t)*v(t),$$

the Djrbashian-Caputo fractional derivative (regularized fractional derivative)

$$D^\beta v(t)=\frac{1}{\Gamma(m-\beta)}\int_0^t(t-\tau)^{m-\beta-1}\frac{d^m}{d\tau^m}v(\tau)d\tau \quad\text{for}\quad m-1<\beta<m,\ m\in\mathbb{N},$$

and therefore,

$$D^\beta v(t)=v^{(\beta)}(t)-\sum_{j=0}^{m-1}f_{j+1-\beta}(t)v^{(j)}(0)\quad\text{for}\quad \beta\in(m-1,m).$$

Let $C_{2,\beta}(Q) = \{v \in C(Q) \mid \Delta v, D_t^{\beta} v \in C(Q)\}$, and for $\beta \in (m-1, m)$

$$
\begin{aligned}
C_{2,\beta}(\bar{Q}) &= \left\{ v \in C_{2,\beta}(Q) \mid \frac{\partial^j}{\partial t^j} v \in C(\bar{Q}),\ j = \overline{0, m-1} \right\}, \\
(Lv)(x,t) &\equiv v_t^{(\beta)}(x,t) - \Delta v(x,t), \\
(L^{reg} v)(x,t) &\equiv D_t^{\beta} v(x,t) - \Delta v(x,t), \\
(\widehat{L}v)(x,t) &\equiv f_{-\beta}(t) \hat{*} v(x,t) - \Delta v(x,t), \quad (x,t) \in Q, \\
\mathscr{X}(\overline{Q}) &= \{ v \in C^{\infty,(0)}(\bar{Q}) : \widehat{L}v \in \mathscr{D}(\bar{Q}) \}.
\end{aligned}
$$

$D^{\bar{\alpha}} v(x,t) = \frac{\partial^{\alpha_0 + |\alpha|} v(x,t)}{\partial t^{\alpha_0} \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ for $\bar{\alpha} = (\alpha_0, \alpha)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathrm{Z}_+$, $j \in \{0, 1, \ldots, n\}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

REMARK 1. It follows from [11, Lemma 2.5] that the space $\mathscr{X}(\overline{Q})$ does not empty. For any $\varphi \in \mathscr{D}(\bar{Q})$ the function

$$
\psi(y,\tau) = \int_{\tau}^{T} dt \int_{\mathbb{R}^n} G_0(x-y, t-\tau) \varphi(x,t) dx, \ (y,\tau) \in \bar{Q}
$$

belongs to $\mathscr{X}(\bar{Q})$ and satisfies the equation

$$
(\widehat{L}\psi)(x,t) = \varphi(x,t), \ (x,t) \in \bar{Q}.
$$

ASSUMPTIONS.

(A1) $F_j \in \mathscr{E}'(\mathbb{R}^n)$, $j = \overline{0,m}$, $g \in \mathscr{D}'[0,T]$;
(A2) $F_j \in \mathscr{E}'(\mathbb{R}^n)$, $j = \overline{0,m}$, $F \in \mathscr{D}'[0,T]$, $\varphi_0 \in \mathscr{D}(\mathbb{R}^n)$, $(F_0, \varphi_0) \neq 0$.

DEFINITION 1. Under assumption (A1) the function $u \in \mathscr{D}'(\bar{Q})$ is called a solution of the Cauchy problem (1), (2) if the identity

$$
\big(u, (\hat{L}\psi)\big) = \big(g(t) F_0(y), \psi(y,t)\big) + \sum_{j=1}^{m} \big(F_j(y) \times f_{j-\beta}(t), \psi(y,t)\big) \quad \forall \psi \in X(\bar{Q}) \tag{4}
$$

holds.

Note that the identity (4) is the generalization of the Green formula [11]

$$
\begin{aligned}
\int_Q v(x,\tau)(\widehat{L}\psi)(x,\tau) dx d\tau = &\int_Q (L^{reg} v)(x,\tau) \psi(x,\tau) dx d\tau \\
&+ \sum_{j=1}^{m} \int_{\mathbb{R}^n} \frac{\partial^{j-1}}{\partial \tau^{j-1}} v(x,0) \big(f_{j-\beta}(\tau), \psi(x,\tau)\big) dx,
\end{aligned}
$$

$$
\beta \in (m-1, m), \ m \in \mathbb{N}, \quad v \in C_{2,\beta}(\bar{Q}) \cap \mathscr{D}(\mathrm{R}^n), \ \psi \in \mathscr{X}(\bar{Q}).
$$

DEFINITION 2. Under assumption (A2) the pair $(u,g)\in \mathscr{D}'(\bar{Q})\times\mathscr{D}'[0,T]$ is called a solution of the inverse Cauchy problem (1)–(3) if the identity (4) and the condition (3) hold.

DEFINITION 3. The vector-function $(G_0(x,t),G_1(x,t),\dots,G_m(x,t))$ is called a Green vector-function of the Cauchy problem (2) to the equation

$$(Lu)(x,t)=F(x,t),\quad (x,t)\in Q,$$

and also of such problem to the equation

$$(L^{reg}u)(x,t)=F(x,t),\quad (x,t)\in Q, \tag{5}$$

if under rather regular F, F_j, $j=\overline{1,m}$ the function

$$u(x,t)=\int_0^t d\tau\int_{\mathbb{R}^n} G_0(x-y,t-\tau)F(y,\tau)dy+\sum_{j=1}^m\int_{\mathbb{R}^n} G_j(x-y,t)F_j(y)dy,\quad (x,t)\in\bar{Q} \tag{6}$$

is a classical (from $C_{2,\beta}(\bar{Q})$) solution of the Cauchy problem (5), (2).

The Green function $G_0(x,t)$ of the Cauchy problem (1), (2) with $m=1,2$ exists [19, 11] and

$$G_j(x,t)=f_{j-\beta}(t)*G_0(x,t),\quad (x,t)\in Q,\quad j=\overline{1,m},\quad m=1,2. \tag{7}$$

Let

$$(\widehat{G_j}\varphi)(y,t)=\int_{\mathrm{R}^n} G_j(x-y,t)\varphi(x,t)\,dx,\ \ (y,t)\in\bar{Q},\ \ j=\overline{0,m}, \tag{8}$$

$$(\widehat{\mathscr{G}_0}\varphi)(y,\tau)=\int_\tau^T dt\int_{\mathbb{R}^n}\varphi(x,t)G_0(x-y,t-\tau)dx=\int_\tau^T(\widehat{G_0}\varphi)(y,t-\tau)dt,\ \ (y,\tau)\in\bar{Q},$$

$$(\widehat{\mathscr{G}_j}\varphi)(y)=\int_0^T dt\int_{\mathbb{R}^n}\varphi(x,t)G_j(x-y,t)dx=\int_0^T(\widehat{G_j}\varphi)(y,t)dt,\ \ y\in\mathbb{R}^n,\ \ j=\overline{1,m}.$$

3. Existence and uniqueness theorems

THEOREM 1. *Assume that (A1) with $m=1,2$ holds. Then there exists the unique solution $u\in\mathscr{D}(\bar{Q})$ of the Cauchy problem (1), (2) (with $m=1,2$). It is defined by*

$$(u,\varphi)=\Big(g(t)F_0(y),(\widehat{\mathscr{G}_0}\varphi)(y,t)\Big)+\sum_{j=1}^m\Big(F_j(y),(\widehat{\mathscr{G}_j}\varphi)(y,t)\Big),\ \forall\varphi\in\mathscr{D}(\bar{Q}). \tag{9}$$

Proof. By [11, Lemma 2.9], the functions $D_y^{\alpha}(\widehat{G_j}\varphi)(y,t)$, $j=\overline{0,m}$ belong to $C(\mathbb{R}^n)$ for every $t\in(0,T]$, multi-index α, $\varphi\in\mathscr{D}(\bar{Q})$ and the following bounds hold:

$$\begin{aligned}&\left|D_y^{\alpha}(\widehat{G_0}\varphi)(y,t)\right|\leqslant c_\alpha t^{\beta-1}(1+|\ln t|),\\&\left|D_y^{\alpha}(\widehat{G_j}\varphi)(y,t)\right|\leqslant c_\alpha t^{j-1},\ \ (y,t)\in\bar{Q},\ \ j=\overline{1,m},\ \ m=1,2,\end{aligned}$$

where $c_\alpha=c_\alpha(\varphi)$ are positive constants. From here, by the scheme of [11, Lemma 2.9] we get the continuity of $D^{\bar{\alpha}}(\widehat{\mathscr{G}_0}\varphi)(y,\tau)=D^{\bar{\alpha}}\int\limits_{\tau}^{T}dt\int\limits_{\mathbb{R}^n}\varphi(x,t)G_0(x-y,t-\tau)dx$, $(y,\tau)\in\bar{Q}$ for every multi-index $\bar{\alpha}$, $\varphi\in\mathscr{D}(\bar{Q})$ and that

$$\widehat{\mathscr{G}_0}:\mathscr{D}(\bar{Q})\to\mathscr{X}(\bar{Q}),\ \ \widehat{\mathscr{G}_j}:\mathscr{D}(\bar{Q})\to C^\infty(\mathbb{R}^n),\ \ j=\overline{1,m}. \tag{10}$$

Therefore, under the assumptions of Theorem 1 the right-hand side of (9) exists for any $\varphi\in\mathscr{D}(\bar{Q})$ and the function $u\in\mathscr{D}'(\bar{Q})$ is defined by (9).

For all $\psi\in\mathscr{X}(\overline{Q})$ we have

$$(u,\widehat{L}\psi)=\Big(F_0(y)\cdot g(\tau),(\widehat{\mathscr{G}_0}(\widehat{L}\psi))(y,\tau)\Big)+\sum_{j=1}^{m}\Big(F_j,\widehat{\mathscr{G}_j}(\widehat{L}\psi)\Big). \tag{11}$$

By [11, Lemma 2.4] for any $\psi\in\mathscr{X}(\bar{Q})$ the relations

$$\begin{gathered}(\widehat{\mathscr{G}_0}(\widehat{L}\psi))(y,\tau)=\psi(y,\tau),\ \ (y,\tau)\in\bar{Q},\\(\widehat{\mathscr{G}_j}(\widehat{L}\psi))(y)=\big(f_{j-\beta}(\tau),\psi(y,\tau)\big),\ \ y\in\mathrm{R}^n,\ \ j=\overline{1,m}\end{gathered}$$

hold. Using them from (11) we get (4). By Definition 2 the function (9) is the solution of the problem (1), (2).

If u_1,u_2 are two solutions of the problem (1), (2), $u=u_1-u_2$ then from (4) we obtain

$$(u,\widehat{L}\psi)=0\ \ \forall\psi\in\mathscr{X}(\overline{Q}). \tag{12}$$

By Remark 1, from (12) we get $(u,\varphi)=0$ for all $\varphi\in\mathscr{D}(\bar{Q})$, that is $u=0$ in $\mathscr{D}'(\bar{Q})$. □

We pass to the inverse Cauchy problem (1)–(3) with $m=1,2$.

Let u be the solution of the Cauchy problem (1), (2). It follows from the equation (1) that

$$\big(u_t^{(\beta)}(\cdot,t),\varphi(\cdot)\big)=\big(u(\cdot,t),\Delta\varphi(\cdot)\big)+(F_0,\varphi)g(t)\ \ \forall\varphi\in\mathscr{D}(\mathbb{R}^n),\ \ t\in(0,T],$$

in particular,

$$\big(u_t^{(\beta)}(\cdot,t),\varphi_0(\cdot)\big)=\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big)+(F_0,\varphi_0)g(t),\ \ t\in(0,T].$$

By the over-determination condition (3) we get

$$F^{(\beta)}(t)=\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big)+(F_0,\varphi_0)g(t)$$

and under assumption (A2) find the following expression for $g(t)$ (through u):

$$g(t) = \left[F^{(\beta)}(t) - \big(u(\cdot,t), \Delta\varphi_0(\cdot)\big)\right][(F_0,\varphi_0)]^{-1}, \quad t \in (0,T]. \tag{13}$$

It follows from Theorem 1 that $u \in \mathscr{D}'(\bar{Q})$ for every $g \in \mathscr{D}'[0,T]$. Therefore, $\big(u(\cdot,t), \Delta\varphi_0(\cdot)\big)$ belongs to $\mathscr{D}'[0,T]$. From the assumption (A2) we have $F^{(\beta)} \in \mathscr{D}'[0,T]$. So, the right-hand side of (13) belongs to $\mathscr{D}'[0,T]$. By substituting it in (9) (instead of $g(t)$) one obtains

$$\begin{aligned}\big(u,\varphi\big) = {} & \frac{1}{(F_0,\varphi_0)}\Big(F^{(\beta)}(\tau) - \big(u(\cdot,\tau),\Delta\varphi_0(\cdot)\big), \big(F_0(y), (\widehat{\mathscr{G}_0}\varphi)(y,\tau)\big)\Big) \\ & + \sum_{j=1}^{m} \big(F_j(y), (\widehat{\mathscr{G}_j}\varphi)(y,\tau)\big) \quad \forall\varphi \in \mathscr{D}(\bar{Q}),\end{aligned}$$

in particular, for every $\eta \in \mathscr{D}[0,T]$

$$\begin{aligned}&\big(u(x,t), \Delta\varphi_0(x)\eta(t)\big) \\ &= \frac{1}{(F_0,\varphi_0)}\Big(F^{(\beta)}(\tau) - \big(u(\cdot,\tau),\Delta\varphi_0(\cdot)\big), \big(F_0(\cdot), \int_\tau^T (\widehat{G}_0\Delta\varphi_0)(\cdot,t-\tau)\eta(t)dt\big)\Big) \\ &\quad + \sum_{j=1}^{m}\Big(F_j(\cdot), \int_0^T (\widehat{G}_j\Delta\varphi_0)(\cdot,t)\eta(t)dt\Big).\end{aligned}$$

Denote

$$H(u,t) = \big(u(\cdot,t), \Delta\varphi_0(\cdot)\big).$$

We have $H(u,\cdot) \in \mathscr{D}'[0,T]$ for every $u \in \mathscr{D}'(\bar{Q})$, and for every $\eta \in \mathscr{D}[0,T]$

$$(H(u,t), \eta(t)) = -\big(H(u,\tau), \int_\tau^T K(t,\tau)\eta(t)dt\big) + (u_0,\eta)$$

that is

$$\Big(H(u,\tau), \eta(\tau) + \int_\tau^T K(t,\tau)\eta(t)dt\Big) = (u_0,\eta)$$

where

$$K(t,\tau) = \frac{\big(F_0(\cdot), (\widehat{G}_0\Delta\varphi_0)(\cdot,t-\tau)\big)}{(F_0,\varphi_0)}, \tag{14}$$

$$(u_0,\eta) = \big(F^{(\beta)}(\tau), \int_\tau^T K(t,\tau)\eta(t)dt\big) + \sum_{j=1}^{m}\big(F_j(\cdot), \int_0^T (\widehat{G}_j\Delta\varphi_0)(\cdot,t)\eta(t)dt\big) \;\; \forall\eta \in \mathscr{D}[0,T]. \tag{15}$$

THEOREM 2. *Assume that (A2) with $m = 1,2$ holds. Then there exists the unique solution $(u,g) \in \mathscr{D}'(\bar{Q}) \times \mathscr{D}'[0,T]$ of the problem (1)–(3) (with $m = 1,2$): u is defined by (9),*

$$g(t) = \left[F^{(\beta)}(t) - g_0(t)\right]\left[(F_0, \varphi_0)\right]^{-1}, \quad t \in (0,T] \tag{16}$$

where $g_0 \in \mathscr{D}'[0,T]$,

$$(g_0, \mu) = (u_0, \eta_\mu) \ \ \forall \mu \in \mathscr{D}[0,T], \tag{17}$$

$\eta_\mu(t)$ is the solution of the equation

$$\eta(t) + \int_t^T K(t,\tau)\eta(t)dt = \mu(t), \ \ t \in (0,T] \tag{18}$$

with the kernel (14), the function u_0 is defined by (15).

Proof. By (10) for any $\varphi \in \mathscr{D}(\bar{Q})$ we have $\widehat{\mathscr{G}_0}\varphi \in \mathscr{X}(\bar{Q}) \subset C^{\infty,(0)}(\bar{Q})$. Therefore

$$\begin{aligned}
\int_\tau^T K(t,\tau)\eta(t)dt &= \frac{1}{(F_0,\varphi_0)} \int_\tau^T \Big(F_0(y), \big(\widehat{G}_0(\Delta\varphi_0)\big)(y,t-\tau)\Big)\eta(t)dt \\
&= \frac{1}{(F_0,\varphi_0)} \Big(F_0(y), \int_\tau^T \big(\widehat{G}_0(\Delta\varphi_0)\big)(y,t-\tau)\eta(t)dt\Big) \\
&= \frac{1}{(F_0,\varphi_0)} \Big(F_0(y), \big(\widehat{\mathscr{G}_0}(\Delta\varphi_0\eta)\big)(y,\tau)\Big)
\end{aligned}$$

and belongs to $\mathscr{D}[0,T]$ for any $\eta \in \mathscr{D}[0,T]$. Also by (10)

$$\int_0^T (\widehat{G}_j\Delta\varphi_0)(\cdot,t)\eta(t)dt = \widehat{\mathscr{G}_j}(\Delta\varphi_0\eta) \in C^\infty(\mathrm{R}^n) \ \ \forall \eta \in \mathscr{D}[0,T], \ \ j = \overline{1,m}.$$

So, the right-hand side of (15) exists and defines $u_0 \in \mathscr{D}'[0,T]$.

For every $\mu \in \mathscr{D}[0,T]$ the equation (18) has the unique solution $\eta_\mu \in \mathscr{D}[0,T]$ and by mapping (17) we find $g_0 \in \mathscr{D}'[0,T]$. Then from (16) follows that $g \in \mathscr{D}'[0,T]$. Note that

$$\begin{aligned}
&\Big(g_0(\tau), \eta(\tau) + \int_\tau^T K(t,\tau)\eta(t)dt\Big) \\
&= (u_0,\eta) = \Big(H(u,\tau), \eta(\tau) + \int_\tau^T K(t,\tau)\eta(t)dt\Big) \ \ \forall \eta \in \mathscr{D}[0,T].
\end{aligned}$$

Taking it and the unique solvability of the equation (18) into account we obtain

$$(H(u,\tau),\mu(\tau)) = (g_0(\tau),\mu(\tau)) \ \ \forall \mu \in \mathscr{D}[0,T]$$

and for every solution u of the Cauchy problem (1), (2). So, (13) is the same as (16).

By Theorem 1 the function (9) with any $g \in \mathscr{D}'[0,T]$ satisfies the problem (1), (2). In particular, it satisfies this problem with g, which is defined by (16)–(18). Show that the function (9) with such g that is

$$(u,\varphi) = \Big(\frac{F^{(\beta)}(t) - g_0(t)}{(F_0,\varphi_0)} F_0(y), (\widehat{\mathscr{G}_0}\varphi)(y,t)\Big) + \sum_{j=1}^{m} \Big(F_j(y), (\widehat{\mathscr{G}_j}\varphi)(y,t)\Big), \ \forall \varphi \in \mathscr{D}(\bar{Q}). \tag{19}$$

with g_0 defined by (17), (18) satisfies the condition (3).

If $F^*(t) = (u(\cdot,t), \varphi_0(\cdot))$, $t \in (0,T]$, that is

$$(F^*,\eta) = \big(u(x,t), \varphi_0(x)\eta(t)\big) \ \forall \eta \in \mathscr{D}[0,T],$$

$$(u_0^*,\eta) = \big(F^{*(\beta)}(\tau), \int_\tau^T K(t,\tau)\eta(t)dt\big) + \sum_{j=1}^{m} \big(F_j(\cdot), \int_0^T (\widehat{G}_j\Delta\varphi_0)(\cdot,t)\eta(t)dt\big) \ \forall \eta \in \mathscr{D}[0,T]$$

and

$$(g_0^*,\mu) = (u_0^*,\eta_\mu) \ \ \forall \mu \in \mathscr{D}[0,T],$$

where $\eta_\mu(t)$ is the solution of the equation (18), then, as before, from the equation (1) we obtain

$$g(t) = \big[F^{*(\beta)}(t) - g_0^*(t)\big]\big[(F_0,\varphi_0)\big]^{-1}, \quad t \in (0,T]. \tag{20}$$

Now from (16) and (20) we get

$$F^{*(\beta)}(t) - g_0^*(t) = F^{(\beta)}(t) - g_0(t) \iff F^{*(\beta)}(t) - F^{(\beta)}(t) = g_0^*(t) - g_0(t).$$

Therefore

$$\begin{aligned} &\big(F^{*(\beta)}(\tau) - F^{(\beta)}(\tau), \eta(\tau) + \int_\tau^T K(t,\tau)\eta(t)dt\big) \\ &= \big(g_0^*(\tau) - g_0(\tau), \eta(\tau) + \int_\tau^T K(t,\tau)\eta(t)dt\big) = (u_0^* - u_0, \eta) \\ &= \big(F^{*(\beta)}(\tau) - F^{(\beta)}(\tau), \int_\tau^T K(t,\tau)\eta(t)dt\big) \ \forall \eta \in \mathscr{D}[0,T] \end{aligned}$$

and we get

$$\big(F^{*(\beta)}(\tau) - F^{(\beta)}(\tau), \eta(\tau)\big) = 0 \ \forall \eta \in \mathscr{D}[0,T].$$

So, $F^{*(\beta)} = F^{(\beta)}$ in $\mathscr{D}'[0,T]$. Therefore, $F^* = F$ in $\mathscr{D}'[0,T]$, the function (19) satisfies the condition (3) and the pair (u,g), which is defined by (9), (16)–(18), is the solution of the inverse problem (1)–(3).

If (u_1,g_1), (u_2,g_2) are two solutions of the problem (1)–(3) then for $u=u_1-u_2$, $g=g_1-g_2$ we obtain the problem

$$\begin{gathered} Lu(x,t)=F_0(x)g(t), \quad (x,t)\in Q, \\ \frac{\partial^{j-1}}{\partial t^{j-1}}u(x,0)=0,\ x\in\mathbb{R}^n,\ \ j=\overline{1,m}, \\ \big(u(\cdot,t),\varphi_0(\cdot)\big)=0,\ \ t\in(0,T]. \end{gathered}$$

As before, we find its solution

$$\begin{gathered} \big(u,\varphi\big)=-\frac{1}{(F_0,\varphi_0)}\Big(g_0(\tau),\big(F_0(\cdot),\int\limits_{\tau}^{T}(\widehat{G}_0\varphi)(\cdot,t-\tau)dt\big)\Big) \ \ \forall\varphi\in\mathscr{D}(\bar{Q}), \\ g(t)=-\frac{g_0(t)}{(F_0,\varphi_0)}, \quad t\in(0,T], \end{gathered} \tag{21}$$

where

$$\big(g_0(\tau),\eta(\tau)+\int\limits_{\tau}^{T}K(t,\tau)\eta(t)dt\big)=0 \ \ \forall\eta\in\mathscr{D}[0,T].$$

From the last equality by uniqueness of a solution of the equation (18) we obtain $(g_0,\mu)=0$ for all $\mu\in\mathscr{D}[0,T]$, that is $g_0=0$ in $\mathscr{D}'[0,T]$. Then, from (21), we obtain $g=0$ in $\mathscr{D}'[0,T]$ and $u=0$ in $\mathscr{D}'(\bar{Q})$. □

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