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NEW ϕ–FRACTIONAL ESTIMATES OF HERMITE–HADAMARD TYPE INEQUALITIES

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Abstract. Using the approach of ϕ-fractional calculus a new estimate of Hermite-Hadamard type inequality via h-preinvex functions is established. Two new ϕ-fractional integral identities are obtained. With the help of these identities some more integral inequalities of Hermite-Hadamard type are derived via twice differentiable s-Breckner and s-Godunova-Levin-Dragomir preinvex functions. Some special cases for the results obtained in the paper are also discussed.

1. Introduction and preliminaries

One cannot deny the significance of theory of convexity in modern analysis. Several authors recently have shown their keen interest in studying theory of convexity deeply. As a result the classical concept of convexity has been extended in different directions using novel and innovative ideas, see [3, 4, 5, 6, 7, 10, 22]. During his study on mathematical programming Hanson [11] introduced the notion of invex functions. These functions played pivotal role in optimization theory. Weir et al. [24] introduced the concept of the invex set and defined the notion of preinvex functions.

Let K be a nonempty closed set in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$ be a continuous function and let $\eta(.,.) : K \times K \to \mathbb{R}^n$ be a continuous bifunction.

DEFINITION 1. ([24]) A set K is said to be invex set with respect to $\eta(.,.)$, if

$$u + t\eta(v,u) \in K, \quad \forall u, v \in K, \ t \in [0,1]. \tag{1}$$

The invex set K is also called η-connected set.

The class of preinvex functions are defined as:

DEFINITION 2. ([24]) A function f is said to be preinvex with respect to arbitrary bifunction $\eta(.,.)$, if

$$f(u + t\eta(v,u)) \leqslant (1-t)f(u) + tf(v), \quad \forall u, v \in K, \ t \in [0,1]. \tag{2}$$

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It is evident that Definition 2 recaptures the definition of classical convex functions for $\eta(v,u) = v-u$.

It has been noticed by them that differentiable preinvex functions imply invex functions. Under certain suitable conditions, it has been shown that the classes of invex functions and preinvex functions are equivalent. For more details, see [1, 9, 2, 11, 13] and the references therein. Noor et al. [19] introduced the notion of h-preinvex functions as:

DEFINITION 3. ([19]) Let $h: J \to \mathbb{R}$ where $(0,1) \subseteq J$ be an interval in $\mathbb{R}$, and let K be an invex set with respect to $\eta(.,.)$. A function $f: K \to \mathbb{R}$ is called h-preinvex with respect to $\eta(.,.)$, if

$$f(u+t\eta(v,u)) \leqslant h(1-t)f(u)+h(t)f(v), \quad \forall u,v \in K, \ t \in (0,1).$$

REMARK 1. Note that if $h(t) = t$, t^s, t^{-s}, t^{-1} and $h(t) = 1$, then, we have the definitions of preinvex, s-preinvex and s-Godunova-Levin preinvex, Godunova-Levin preinvex and P-preinvex functions respectively. For more details, see [19].

It has been observed that theory of convexity has a close relationship with theory of inequalities. Several inequalities have been obtained via convex functions and for its variant forms. For useful information, interested readers are referred to [8, 14]. We now recall some preliminaries of ϕ-fractional calculus which will be helpful in the establishment of our main results.

Let f be piecewise continuous on $I^* = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then for $t > 0$, we consider ϕ-Riemann-Liouville fractional integrals of f of order α,

$${}_{\phi}\mathscr{I}^{\alpha}_{a^+} f(x) = \frac{1}{\phi\Gamma_{\phi}(\alpha)} \int\limits_a^x (x-t)^{\frac{\alpha}{\phi}-1} f(t)\mathrm{d}t, \quad x > a, \ \phi > 0,$$

and

$${}_{\phi}\mathscr{I}^{\alpha}_{b^-} f(x) = \frac{1}{\phi\Gamma_{\phi}(\alpha)} \int\limits_x^b (t-x)^{\frac{\alpha}{\phi}-1} f(t)\mathrm{d}t, \quad x < b, \ \phi > 0,$$

where $\alpha > 0$ is a parameter.

Note that $\Gamma_{\phi}(x) = \int\limits_0^{\infty} t^{x-1} e^{-\frac{t^{\phi}}{\phi}} \mathrm{d}t$ is the ϕ-Gamma function. Clearly $\Gamma(x) = \lim\limits_{\phi \to 1} \Gamma_{\phi}(x)$ and $\Gamma_{\phi}(x+\phi) = x\Gamma_{\phi}(x)$. For some more details, see [16, 17, 20]. Note that when $\phi \to 1$ ϕ-Riemann-Liouville fractional integrals become classical Riemann-Liouville fractional integral [12].

To prove some results in the paper, we need the well-known Condition C introduced by Mohan and Neogy in [15].

CONDITION C. Let $K \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(.,.)$. Then for any $x, y \in K$ and $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y),$$
$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

Note that for every $x, y \in K$, t_1, $t_2 \in [0,1]$ and from Condition C, we have

$$\eta(y + t_2\eta(x,y), \quad y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y).$$

Using Condition C, Noor [18] obtained a new variant of Hermite-Hadamard type inequality via preinvex functions

THEOREM 1. ([18]) *Let $f : K \to \mathbb{R}$ be preinvex function, such that $\eta(b,a) > 0$ and $\eta(.,.)$ satisfies the Condition C, then*

$$f\left(\frac{2a + \eta(b,a)}{2}\right) \leqslant \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\mathrm{d}u \leqslant \frac{f(a) + (b)}{2}.$$

The following result will be used in deriving some of our main results.

LEMMA 1. ([23]) *For any $A > B \geqslant 0$ and $q \geqslant 1$, we have*

$$(A - B)^q \leqslant A^q - B^q.$$

From now onwards $I = [a, a + \eta(b,a)]$ will be the interval and $L(.,.)$ be the integrable functions unless otherwise specified.

2. Auxiliary results

In this section, we derive two new ϕ-fractional identities.

LEMMA 2. *Let $f : I \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, a + \eta(b,a)]$, where $a, a + \eta(b,a) \in I$ with $\eta(b,a) > 0$, then*

$$T_f(a,b;\alpha,\phi;\eta) = \frac{\eta(b,a)}{2} \int_0^1 [t^{\frac{\alpha}{\phi}} - (1-t)^{\frac{\alpha}{\phi}}] f'(a + t\eta(b,a))\mathrm{d}t, \tag{3}$$

where

$$\begin{aligned} &T_f(a,b;\alpha,\phi;\eta) \\ &= \frac{f(a) + f(a + \eta(b,a))}{2} \\ &\quad - \frac{\Gamma_\phi(\alpha+1)}{2\eta^{\frac{\alpha}{\phi}}(b,a)} \left[{}_\phi\mathscr{I}^\alpha_{a^+} f(a + \eta(b,a)) + {}_\phi\mathscr{I}^\alpha_{(a+\eta(b,a))^-} f(a + \eta(b,a)) \right]. \end{aligned} \tag{4}$$

Proof. It suffices to show that

$$K = \int_0^1 [t^{\frac{\alpha}{\phi}} - (1-t)^{\frac{\alpha}{\phi}}] f'(a + t\eta(b,a))\,\mathrm{d}t$$
$$= K_1 + K_2. \tag{5}$$

Now integrating by parts yields:

$$K_1 = \int_0^1 t^{\frac{\alpha}{\phi}} f'(a + t\eta(b,a))\,\mathrm{d}t$$
$$= \frac{1}{2}\Bigg[f(a+\eta(b,a))$$
$$- \frac{\phi\Gamma_\phi(\alpha+1)}{\phi} \frac{1}{\eta^{\frac{\alpha}{\phi}}(b,a)} \frac{1}{\phi\Gamma_\phi(\alpha)} \int_a^{a+\eta(b,a)} \left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{\phi}-1} f(x) \frac{\mathrm{d}x}{\eta(b,a)} \Bigg]$$
$$= \frac{f(a+\eta(b,a))}{\eta(b,a)} - \frac{\Gamma_\phi(\alpha+1)}{\eta^{\frac{\alpha}{\phi}+1}(b,a)} {}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-} f(a). \tag{6}$$

Similarly

$$K_2 = \int_0^1 (1-t)^{\frac{\alpha}{\phi}} f'(a+t\eta(b,a))\,\mathrm{d}t = \frac{f(a)}{\eta(b,a)} - \frac{\Gamma_\phi(\alpha+1)}{\eta^{\frac{\alpha}{\phi}+1}(b,a)} {}_\phi\mathscr{J}^{\alpha}_{a^+} f(a+\eta(b,a)). \tag{7}$$

Combining (5), (6), (7) and multiplying both sides by $\frac{\eta(b,a)}{2}$ completes the proof. □

REMARK 2. Note that if $\phi \to 1$ in Lemma 2, then we have Lemma 2 [21]. Now utilizing Lemma 2 we develop another new ϕ-fractional integral identity for twice differentiable functions.

LEMMA 3. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, a+\eta(b,a)]$, where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$, then*

$$T_f(a,b;\alpha,\phi;\eta) = \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)} \int_0^1 [1 - t^{\frac{\alpha}{\phi}+1} - (1-t)^{\frac{\alpha}{\phi}+1}] f''(a+t\eta(b,a))\,\mathrm{d}t,$$

where $T_f(a,b;\alpha,\phi;\eta)$ is given by (4).

Proof. It is sufficient to verify that

$$\begin{aligned}
&\int_0^1 [t^{\frac{\alpha}{\phi}} - (1-t)^{\frac{\alpha}{\phi}}] f'(a+t\eta(b,a))\mathrm{d}t \\
&= \int_0^1 f'(a+t\eta(b,a))\,\mathrm{d}\frac{t^{\frac{\alpha}{\phi}} + (1-t)^{\frac{\alpha}{\phi}}}{\frac{\alpha}{\phi}+1} \\
&= \frac{\phi}{\alpha+\phi}\left[f'(a+\eta(b,a)) - f'(a)\right] - \frac{\phi\eta(b,a)}{\alpha+\phi}\int_0^1 \left[t^{\frac{\alpha}{\phi}+1} + (1-t)^{\frac{\alpha}{\phi}+1}\right] f''(a+t\eta(b,a))\mathrm{d}t \\
&= \frac{\phi\eta(b,a)}{\alpha+\phi}\left[\int_0^1 \left(1 - t^{\frac{\alpha}{\phi}+1} - (1-t)^{\frac{\alpha}{\phi}+1}\right) f''(a+t\eta(b,a))\mathrm{d}t\right]. \qquad (8)
\end{aligned}$$

Utilizing (8) in (3) completes the proof. □

3. Results and discussions

In this section, we derive our main results.

THEOREM 2. *Let $f : I \to \mathbb{R}$ be h-preinvex function where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$. If $f \in L[a, a+\eta(b,a)]$ and $\eta(.,.)$ satisfies the Condition C, then, for $h\left(\frac{1}{2}\right) \neq 0$, we have*

$$\begin{aligned}
\frac{\phi}{\alpha h\left(\frac{1}{2}\right)} f\left(\frac{2a+\eta(b,a)}{2}\right) &\leqslant \frac{\phi\Gamma_\phi(\alpha)}{\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{ {}_\phi\mathscr{I}^\alpha_{(a+\eta(b,a))^-} f(a) + {}_\phi\mathscr{I}^\alpha_{a^+} f(a+\eta(b,a))\right\} \\
&\leqslant [f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{\phi}-1}[h(1-t)+h(t)]\mathrm{d}t.
\end{aligned}$$

Proof. Since f is an h-preinvex function and $\eta(.,.)$ satisfies Condition C, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant h\left(\frac{1}{2}\right)[f(a+t\eta(b,a)) + f(a+(1-t)\eta(b,a))].$$

Multiplying both sides of above inequality by $t^{\frac{\alpha}{\phi}-1}$ and integrating it with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\frac{\phi}{\alpha} f\left(\frac{2a+\eta(b,a)}{2}\right)\\
&= f\left(\frac{2a+\eta(b,a)}{2}\right)\int_0^1 t^{\frac{\alpha}{\phi}-1}\mathrm{d}t\\
&\leqslant h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\frac{\alpha}{\phi}-1} f(a+t\eta(b,a))\mathrm{d}t+\int_0^1 t^{\frac{\alpha}{\phi}-1} f(a+(1-t)\eta(b,a))\mathrm{d}t\right]\\
&= h\left(\frac{1}{2}\right)\frac{1}{\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{\int_a^{a+\eta(b,a)}(x-a)^{\frac{\alpha}{\phi}-1} f(x)\,\mathrm{d}x+\int_a^{a+\eta(b,a)}(a+\eta(b,a)-x)^{\frac{\alpha}{\phi}-1} f(x)\,\mathrm{d}x\right\}\\
&= h\left(\frac{1}{2}\right)\frac{1}{\eta^{\frac{\alpha}{\phi}}(b,a)}\phi\Gamma_\phi(\alpha)\left\{{}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-} f(a)+{}_\phi\mathscr{J}^{\alpha}_{a^+} f(a+\eta(b,a))\right\}.
\end{aligned}$$

This implies

$$\frac{\phi}{\alpha h\left(\frac{1}{2}\right)} f\left(\frac{2a+\eta(b,a)}{2}\right)\leqslant\frac{\phi\Gamma_\phi(\alpha)}{\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{{}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-} f(a)+{}_\phi\mathscr{J}^{\alpha}_{a^+} f(a+\eta(b,a))\right\}. \tag{9}$$

Now we prove the right hand side of the inequality. Since f is preinvex with respect to $\eta(b,a)$ and $\eta(b,a)$ satisfies the Condition C, we have

$$f(a+t\eta(b,a))\leqslant h(1-t)f(a)+h(t)f(b),$$

similarly

$$f(a+(1-t)\eta(b,a))\leqslant h(t)f(a)+h(1-t)f(b).$$

Adding above two inequalities and multiplying both sides by $t^{\frac{\alpha}{\phi}-1}$, we have

$$t^{\frac{\alpha}{\phi}-1} f(a+t\eta(b,a))+t^{\frac{\alpha}{\phi}-1} f(a+(1-t)\eta(b,a))\leqslant t^{\frac{\alpha}{\phi}-1}[h(1-t)+h(t)][f(a)+f(b)].$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\frac{\phi\Gamma_\phi(\alpha)}{\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{{}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-} f(a)+{}_\phi\mathscr{J}^{\alpha}_{a^+} f(a+\eta(b,a))\right\}\\
&\leqslant [f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{\phi}-1}[h(1-t)+h(t)]\mathrm{d}t.
\end{aligned} \tag{10}$$

Summing inequalities (9) and (10) completes the proof. □

We now discuss some special cases of Theorem 2.

I. If $h(t)=t$ in Theorem 2, then, we have following new result.

COROLLARY 1. *Let $f: I \to \mathbb{R}$ be preinvex function, where $a, a+\eta(b,a) \in I$ with $\eta(b,a)>0$. If $f \in L[a, a+\eta(b,a)]$, and $\eta(.,.)$ satisfies the Condition C, then, we have*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant \frac{\Gamma_\phi(\alpha+1)}{2\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{{}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-}f(a)+{}_\phi\mathscr{J}^{\alpha}_{a^+}f(a+\eta(b,a))\right\} \leqslant \frac{f(a)+f(b)}{2}.$$

II. If $h(t)=t^s$ in Theorem 2, then, we have following new result.

COROLLARY 2. *Let $f: I \to \mathbb{R}$ be Breckner type of s-preinvex function, where $a, a+\eta(b,a) \in I$ with $\eta(b,a)>0$. If $f \in L[a, a+\eta(b,a)]$ and $\eta(.,.)$ satisfies the Condition C, then, we have*

$$\frac{2^s\phi}{\alpha}f\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant \frac{\phi\Gamma_\phi(\alpha)}{\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{{}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-}f(a)+{}_\phi\mathscr{J}^{\alpha}_{a^+}f(a+\eta(b,a))\right\}$$
$$\leqslant [f(a)+f(b)]\left(\phi B_\phi(\alpha,\phi(s+1))-\frac{\phi}{\alpha+\phi s}\right).$$

III. If $h(t)=t^{-s}$ in Theorem 2, then, we have following new result.

COROLLARY 3. *Let $f: I \to \mathbb{R}$ be Godunova-Levin type of s-preinvex function, where $a, a+\eta(b,a) \in I$ with $\eta(b,a)>0$. If $f \in L[a, a+\eta(b,a)]$ and $\eta(b,a)>0$ satisfies Condition C, then, for $\alpha>\phi s$, we have*

$$\frac{\phi}{2^s\alpha}f\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant \frac{\phi\Gamma_\phi(\alpha)}{\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{{}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-}f(a)+{}_\phi\mathscr{J}^{\alpha}_{a^+}f(a+\eta(b,a))\right\}$$
$$\leqslant [f(a)+f(b)]\left(\phi B_\phi(\alpha,\phi(1-s))-\frac{\phi}{\alpha-\phi s}\right).$$

IV. If $h(t)=1$ in Theorem 2, then, we have following new result.

COROLLARY 4. *Let $f: I \to \mathbb{R}$ be P-preinvex function, where $a, a+\eta(b,a) \in I$ with $\eta(b,a)>0$. If $f \in L[a, a+\eta(b,a)]$ and $\eta(.,.)$ satisfies Condition C, then, we have*

$$\frac{\phi}{\alpha}f\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant \frac{\phi\Gamma_\phi(\alpha)}{\eta^{\frac{\alpha}{\phi}}(b,a)}\left\{{}_\phi\mathscr{J}^{\alpha}_{(a+\eta(b,a))^-}f(a)+{}_\phi\mathscr{J}^{\alpha}_{a^+}f(a+\eta(b,a))\right\} \leqslant \frac{2\phi[f(a)+f(b)]}{\alpha}.$$

We now derive some new ϕ-analogues of fractional Hermite-Hadamard inequalities via twice differentiable s-Breckner preinvex and s-Godunova-Levin-Dragomir preinvex functions.

THEOREM 3. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, a+\eta(b,a)]$, where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$. If $|f''|$ is an s-Breckner preinvex function, where $s \in [0,1]$ then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right| \leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\psi_1(\alpha,\phi)\left[|f''(a)|+|f''(b)|\right],$$

where

$$\psi_1(\alpha,\phi) := \frac{1}{s+1} - \frac{\phi}{\alpha+\phi s+2\phi} - \mathbb{B}\Big(\frac{\alpha}{\phi}+2, s+1\Big). \tag{11}$$

Proof. Utilizing Lemma 3, property of modulus and the fact that $|f''|$ is an s-Breckner preinvex function, then we have

$$\begin{aligned}
&\left|T_f(a,b;\alpha,\phi;\eta)\right| \\
&= \left|\frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\int_0^1 [1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1}]f''(a+t\eta(b,a))\,\mathrm{d}t\right| \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\int_0^1 [1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1}]\left|f''(a+t\eta(b,a))\right|\mathrm{d}t \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\int_0^1 [1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1}]\left[(1-t)^s|f''(a)|+t^s|f''(b)|\right]\mathrm{d}t \\
&= \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left[\frac{1}{s+1}-\frac{\phi}{\alpha+\phi s+2\phi}-\mathbb{B}\Big(\frac{\alpha}{\phi}+2,s+1\Big)\right]\left[|f''(a)|+|f''(b)|\right].
\end{aligned}$$

This completes the proof. □

THEOREM 4. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, a+\eta(b,a)]$, where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$. If $|f''|$ is an s-Godunova-Levin-Dragomir preinvex function, where $s \in (0,1)$, then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right| \leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\psi_2(\alpha,\phi)\left[|f''(a)|+|f''(b)|\right],$$

where

$$\psi_2(\alpha,\phi) := \frac{1}{1-s} - \frac{\phi}{\alpha-\phi s+2\phi} - \mathbb{B}\Big(\frac{\alpha}{\phi}+2, 1-s\Big). \tag{12}$$

Proof. The proof directly follows from the proof of Theorem 3. □

THEOREM 5. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, a+\eta(b,a)]$, where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$. If $|f''|^q$ is an s-Breckner preinvex function, where $s \in [0,1]$ and $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right| \leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(1-\frac{2\phi}{p(\alpha+\phi)+\phi}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Proof. Utilizing Lemma 3, property of modulus, Hölders inequality, and the fact that $|f''|^p$ is an s-Breckner preinvex function, then we have

$$\begin{aligned}
&\left|T_f(a,b;\alpha,\phi;\eta)\right| \\
&= \left|\frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\int_0^1 [1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1}]f''(a+t\eta(b,a))\,\mathrm{d}t\right| \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\int_0^1 (1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1})^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(a+t\eta(b,a))\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left\{\int_0^1 \left(1-t^{p(\frac{\alpha}{\phi}+1)}-(1-t)^{p(\frac{\alpha}{\phi}+1)}\right)\mathrm{d}t\right\}^{\frac{1}{p}}\left(\int_0^1 \left|f''(a+t\eta(b,a))\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(1-\frac{2\phi}{p(\alpha+\phi)+\phi}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

THEOREM 6. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, a+\eta(b,a)]$, where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$. If $|f''|^p$ is an s-Godunova-Levin-Dragomir preinvex function, where $s \in (0,1)$ and $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right| \leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(1-\frac{2\phi}{p(\alpha+\phi)+\phi}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{1-s}\right)^{\frac{1}{q}}.$$

Proof. The proof directly follows from the proof of Theorem 5. □

THEOREM 7. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, a+\eta(b,a)]$, where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$. If $|f''|^q$ is an s-Breckner preinvex function, where $s \in [0,1]$ and $q \geqslant 1$, then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right| \leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\frac{\alpha}{\alpha+\phi}\right)^{1-\frac{1}{q}}\psi_1^{\frac{1}{q}}(\alpha,\phi)\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}},$$

where $\psi_1(\alpha,\phi)$ is given by (11).

Proof. Utilizing Lemma 3, property of modulus, power mean's inequality, and the fact that $|f'|^q$ is an s-Breckner preinvex function, then we have

$$\begin{aligned}
&\left|T_f(a,b;\alpha,\phi;\eta)\right| \\
&=\left|\frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\int_0^1[1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1}]f''(a+t\eta(b,a))\,\mathrm{d}t\right| \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\int_0^1(1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1})\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1(1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1})\left|f''(a+t\eta(b,a))\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\frac{\alpha}{\alpha+\phi}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1})\left[(1-t)^s|f''(a)|^q+t^s|f''(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\
&=\frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\frac{\alpha}{\alpha+\phi}\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\left[\frac{1}{s+1}-\frac{\phi}{\alpha+\phi s+2\phi}-\mathbb{B}\left(\frac{\alpha}{\phi}+2,s+1\right)\right]\left[|f''(a)|^q+|f''(b)|^q\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

THEOREM 8. *Let $f: I\to\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L[a,a+\eta(b,a)]$, where $a,a+\eta(b,a)\in I$ with $\eta(b,a)>0$. If $|f''|^q$ is an s-Godunova-Levin-Dragomir preinvex function, where $s\in(0,1)$ and $q\geqslant 1$, then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right|\leqslant\frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\frac{\alpha}{\alpha+\phi}\right)^{1-\frac{1}{q}}\psi_2^{\frac{1}{q}}(\alpha,\phi)\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}},$$

where $\psi_2(\alpha,\phi)$ is given by (12).

Proof. The proof directly follows from the proof of Theorem 7. □

THEOREM 9. *Let $f: I\to\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L[a,a+\eta(b,a)]$, where $a,a+\eta(b,a)\in I$ with $\eta(b,a)>0$. If $|f''|^q$ is an s-Breckner preinvex function, where $s\in[0,1]$ and $q\geqslant 1$, then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right|\leqslant\frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\psi_3^{\frac{1}{q}}(\alpha,\phi)\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}},$$

where

$$\psi_3(\alpha,\phi):=\frac{1}{s+1}-\mathbb{B}\left(q\left(\frac{\alpha}{\phi}+1\right)+1,s+1\right)-\frac{\phi}{q\alpha+q\phi+s\phi+\phi}. \tag{13}$$

Proof. Utilizing Lemma 3, property of modulus, Hölder's inequality, and the fact that $|f''|^q$ is an s-Breckner preinvex function, then we have

$$\begin{aligned}
&\left|T_f(a,b;\alpha,\phi;\eta)\right| \\
&= \left|\frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\int_0^1 [1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1}] f''(a+t\eta(b,a))\,\mathrm{d}t\right| \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\int_0^1 1\mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 (1-t^{\frac{\alpha}{\phi}+1}-(1-t)^{\frac{\alpha}{\phi}+1})^q \left|f''(a+t\eta(b,a))\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\int_0^1 (1-t^{q(\frac{\alpha}{\phi}+1)}-(1-t)^{q(\frac{\alpha}{\phi}+1)})\left[(1-t)^s|f''(a)|^q+t^s|f''(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\left(\left[\frac{1}{s+1}-\mathbb{B}\left(q\left(\frac{\alpha}{\phi}+1\right)+1,s+1\right)-\frac{\phi}{q\alpha+q\phi+s\phi+\phi}\right]\right. \\
&\quad \left.\times\left[|f''(a)|^q+|f''(b)|^q\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

THEOREM 10. *Let $f: I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a,a+\eta(b,a)]$, where $a, a+\eta(b,a) \in I$ with $\eta(b,a) > 0$. If $|f''|^q$ is an s-Godunova-Levin-Dragomir preinvex function, where $s \in (0,1)$ and $q \geqslant 1$, then*

$$\left|T_f(a,b;\alpha,\phi;\eta)\right| \leqslant \frac{\phi\eta^2(b,a)}{2(\alpha+\phi)}\psi_3^{\frac{1}{q}}(\alpha,\phi)\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}},$$

where

$$\psi_4(\alpha,\phi) := \frac{1}{1-s}-\mathbb{B}\left(q\left(\frac{\alpha}{\phi}+1\right)+1,1-s\right)-\frac{\phi}{q\alpha+q\phi-s\phi+\phi}. \tag{14}$$

Proof. The proof directly follows from the proof of Theorem 9. □

Conclusion. First, we have obtained some new ϕ-fractional estimates for the Hermite-Hadamard type inequality via different classes of preinvex functions. Then, we have derived two new ϕ-analogues of fractional integral identities via once and twice differentiable functions. Utilizing the identity for twice differentiable functions we then established some new inequalities of Hermite-Hadamard type via twice differentiable

s-Breckner and s-Godunova-Levin-Dragomir preinvex functions. It is expected that this paper may stimulate further research in this area.

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REFERENCES

[1] T. ANTCZAK, *Mean value in invexity analysis*, Nonl. Anal. **60** (2005), 1473–1484.

[2] A. BARANI, A. G. GHAZANFARI, S. S. DRAGOMIR, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. **2012**, Article Number 247, 9 pages, (2012).

[3] W. W. BRECKNER, *Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen raumen*, Pupl. Inst. Math. **23**, 13–20, (1978).

[4] G. CRISTESCU AND L. LUPŞA, *Non-connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[5] G. CRISTESCU, M. A. NOOR, M. U. AWAN, *Bounds of the second degree cumulative frontier gaps of functions with generalized convexity*, Carpathian J. Math. **31** (2), 173–180, (2015).

[6] S. S. DRAGOMIR, *Inequalities of Hermite–Hadamard type for h-convex functions on linear spaces*, Proyecciones (Antofagasta) **34** (4), 323–341, (2015).

[7] S. S. DRAGOMIR, J. PEČARIĆ, L. E. PERSSON, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (3), 335–341, (1995).

[8] S. S. DRAGOMIR, C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Victoria University Australia, (2000).

[9] T.-S. DU, J. G. LIAO, L. Z. CHEN, AND M. U. AWAN, *Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized (α,m)-preinvex functions*, J. Inequal. Appl. **2016**, Article Number 306, 24 pages, (2016).

[10] E. K. GODUNOVA, V. I. LEVIN, *Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii*, Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva, (1985) 138–142, (in Russian).

[11] M. A. HANSON, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80**, 545–550, (1981).

[12] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V, Amsterdam, Netherlands, 2006.

[13] M. MATLOKA, *On some Hadamard-type inequalities for (h_1,h_2)-preinvex functions on the coordinates*, J. Inequal. Appl. **2013**, Article Number 227, 12 pages, (2013).

[14] M. V. MIHAI, M. A. NOOR, K. I. NOOR, M. U. AWAN, *Some integral inequalities for harmonic h-convex functions involving hypergeometric functions*, Appl. Math. Comput. **252** (1) (2015), 257–262.

[15] S. R. MOHAN, S. K. NEOGY, *On invex sets and preinvex functions*, J. Math. Anal. Appl. **189** (1995), 901–908.

[16] S. MUBEEN, G. M. HABIBULLAH, *k-fractional integrals and applications*, Inter. J. Contemp. Math. Sci. **7** (2), (2012), 89–94.

[17] S. MUBEEN, S. IQBAL AND G. RAHMAN, *Contiguous function relations and an integral reprezentation for Appell k-series $F_{1,k}$*, Inter. J. Math. Research **4** (2) (2015), 53–63.

[18] M. A. NOOR, *Hermite-Hadamard integral inequalities for $\log$-preinvex functions*, J. Math. Anal. Approx. Theory **2** (2007), 126–131.

[19] M. A. NOOR, K. I. NOOR, M. U. AWAN, J. LI, *On Hermite-Hadamard Inequalities for h-preinvex functions*, Filomat **28** (7), 1463–1474, (2014).

[20] M. Z. SARIKAYA, A. KARACA, *On the k-Riemann-Liouville fractional integral and applications*, Inter. J. Stat. Math. **1** (3), (2014), 033–043.

[21] M. Z. SARIKAYA, E. SET, H. YALDIZ, N. BASAK, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model. **57**, (2013), 2403–2407.

[22] S. VAROŠANEC, *On h-convexity*, J. Math. Anal. Appl. **326**, 303–311, (2007).

[23] J.-R WANG, X. LIA, M. FECKAN Y. ZHOU, *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal. **92** (11), 2241–2253, (2013).

[24] T. WEIR, AND B. MOND, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl. **136**, 29–38, (1988).

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