Fractional
Differential
Calculus
Volume 8, Number 2 (2018), 309–326 doi:10.7153/fdc-2018-08-19

INFINITELY MANY SOLUTIONS FOR A CLASS OF SUPERQUADRATIC FRACTIONAL HAMILTONIAN SYSTEMS

MOHSEN TIMOUMI

(*Communicated by L. Vázquez*)

Abstract. Applying a variant fountain theorem, we prove the existence of infinitely many solutions for a class of fractional Hamiltonian systems

$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t)+L(t)u(t)=\nabla W(t,u(t)),\ t\in\mathbb{R} \\ u\in H^\alpha(\mathbb{R},\ \mathbb{R}^N), \end{cases}$$

where ${}_tD_\infty^\alpha$ and ${}_{-\infty}D_t^\alpha$ are the Liouville-Weyl fractional derivatives of order $\frac{1}{2}<\alpha<1$, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function not required to be either uniformly positive definite nor coercive and $W(t,x)\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ satisfies some weaker superquadratic conditions at infinity in the second variable but does not satisfy the well-known Ambrosetti-Rabinowitz superquadratic growth condition.

1. Introduction

In this paper, we are concerned with the existence of infinitely many solutions for a class of fractional Hamiltonian systems of the following form

$$(\mathscr{F}\mathscr{H}\mathscr{S})\qquad \begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t)+L(t)u(t)=\nabla W(t,u(t)),\ t\in\mathbb{R} \\ u\in H^\alpha(\mathbb{R},\ \mathbb{R}^N), \end{cases}$$

where ${}_{-\infty}D_t^\alpha$ and ${}_tD_\infty^\alpha$ are the Liouville fractional derivatives of order $\frac{1}{2}<\alpha<1$, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $W:\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ is a continuous function, differentiable with respect to the second variable with continuous derivative $\frac{\partial W}{\partial x}(t,x)=\nabla W(t,x)$.

The study of fractional calculus (differentiation and integration) has emerged as an important and popular field in research. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, mechanics, control theory, viscoelasticity, electro chemistry, bioengineering, economics and others [1, 10, 12, 16, 17]. An important characteristic of fractional-order differential operator that distinguishes it from the integer-order differential operator is

Mathematics subject classification (2010): 34C37, 35A15, 35B38.
Keywords and phrases: Fractional Hamiltonian systems, variational methods, variant fountain theorem.

its nonlocal behavior, that is, the future state of a dynamical system or process involving fractional derivative depends on its current state as well as its past states. In other words, differential equations of arbitrary order describe memory and hereditary properties of various materials and process. This is one of the futures that has contributed to the popularity of the subject and has motivated the rechearchers to focus on fractional order models, which are more realistic and practical than the classical integer-order models.

Recently, also equations including both left and right fractional derivatives were investigated and many results were obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of Nonlinear Analysis, such as fixed point theory [4, 24], topological degree theory [5, 8], comparison methods [13, 23], and so on.

In $(\mathscr{FHS})$, if $\alpha = 1$, then it reduces to the following Hamiltonian system

$$(\mathscr{HS}) \qquad \ddot{u}(t) - L(t)u(t) + W(t,u(t)) = 0.$$

It should be noted that critical point theory and variational methods serve as effective tools in the study of integer-order differential equations. The underlying idea in this approach rest on finding critical points for suitable energy functional defined on an appropriate function space. During the last three decades, the critical point theory has developed into a wonderful tool for investigating the existence criteria for the solutions of differential equations with variational structures, for example see [14, 18] and the references cited therein.

Motivated by the classical works in [14,18], for the first time, the author [9] showed that critical point theory and variational methods are an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u)(t) = \nabla W(t,u(t)),\ t \in [0,T] \\ u(0) = u(T), \end{cases}$$

where α and $W(t,x)$ are defined as above, and obtained the existence of at least one nontrivial solution under some suitable conditions on $W(t,x)$. Inspired by this work, Torres [20] considered the fractional Hamiltonian system $(\mathscr{FHS})$ when L satisfies

(1.1) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix-valued function, and there exists an $l \in C(\mathbb{R}, \mathbb{R}_+^*)$ such that $l(t) \longrightarrow +\infty$ as $|t| \longrightarrow \infty$ and

$$L(t)x.x \geqslant l(t)\,|x|^2,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Assuming that W satisfies the well-known Ambrosetti-Rabinowitz superquadratic condition and some other suitable conditions, he showed that the fractional Hamiltonian system $(\mathscr{FHS})$ possesses at least one nontrivial solution using the Mountain Pass Theorem. Since then, the existence and multiplicity of solutions of problem $(\mathscr{FHS})$ via critical point theory have been investigated in many papers [2, 3, 6, 7, 15, 19–22, 24–27]. In [2, 3, 6, 7, 19, 20, 27], the function L satisfies the coercive condition (1.1), but in [21, 22, 26], L satisfies the following boundedness condition

(1.2) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix-valued function and there are constants $0 < \tau_1 < \tau_2 < +\infty$ such that

$$\tau_1\,|x|^2 \leqslant L(t)x.x \leqslant \tau_2\,|x|^2,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

however in [15], the function L is required to satisfy the two following noncoercive conditions:

(1.3) $L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R},\mathbb{R})$ such that

$$\inf_{t\in\mathbb{R}} l(t) > 0 \text{ and } L(t)x.x \geqslant l(t)\,|x|^2,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N;$$

(L) There exists a constant $r_0 > 0$ such that

$$\lim_{|s|\longrightarrow\infty} meas(\{t \in]s-r_0, s+r_0[/L(t) < bI_N\}) = 0,\ \forall b > 0,$$

where *meas* denotes the Lebesgue's measure on $\mathbb{R}$, which guarantee the compactness of Sobolev embedding. Besides, in all the above mentioned papers, the potential W is required to be subquadratic or to satisfy the Ambrosetti-Rabinowitz superquadratic condition (AR) at infinity.

The aim of this paper is to study the existence of infinitely many solutions for $(\mathscr{F}\mathscr{H}\mathscr{S})$, when the function L is unnecessarily positive definite nor coercive, and the potential W satisfies some superquadratic conditions at infinity, weaker than the (AR) condition. More precisely, we make the following hypotheses:

(L_0) The smallest eigenvalue $l(t) = \inf_{|\xi|=1} L(t)\xi.\xi$ of $L(t)$ is bounded from below;

(W_1) $W(t,0) = 0$ and there exist constants $c > 0$ and $\nu > 2$ such that

$$|\nabla W(t,x)| \leqslant c(|x| + |x|^{\nu-1}),\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N;$$

(W_2)
$$\lim_{|x|\longrightarrow\infty} \frac{W(t,x)}{|x|^2} = +\infty, \text{ uniformly for } t \in \mathbb{R};$$

(W_3) There exists a constant $\sigma \geqslant 1$ such that

$$\sigma\widehat{W}(t,x) \geqslant \widehat{W}(t,sx),\ \forall (s,t,x) \in [0,1]\times\mathbb{R}\times\mathbb{R}^N,$$

where $\widehat{W}(t,x) = \nabla W(t,x).x - 2W(t,x)$;

(W_4)
$$W(t,-x) = W(t,x),\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Our main result reads as follows:

THEOREM 1. *Assume (L_0), (L) and (W_1)–(W_4) are satisfied. Then the fractional Hamiltonian system $(\mathscr{F}\mathscr{H}\mathscr{S})$ possesses a sequence of nontrivial solutions (u_k) satisfying*

$$\frac{1}{2}\int_{\mathbb{R}} [|{}_{-\infty}D_t^{\alpha} u_k|^2 + L(t)u_k.u_k]dt - \int_{\mathbb{R}} W(t,u_k)dt \longrightarrow +\infty \text{ as } k \longrightarrow \infty.$$

REMARK AND EXAMPLE 1. In our result, $L(t)$ is unnecessarily required to be either uniformly positive definite nor coercive. For example, let $L(t) = (t^2\cos^2 t - 1)I_N$, where I_N is the identity matrix of order N, then L satisfies (L_0) and (L) but it does'nt satisfy (1.1), (1.2) nor (1.3). Moreover, let

$$W(t,x) = a(t)[|x|^2\ln(e+|x|) - \frac{1}{2}|x|^2 + e|x| - e^2(\ln(e+|x|)-1)]$$

where a is a continuous bounded function with positive lower bound. Then an easy computation shows that W satisfies (W_1)–(W_4). However, W does not satisfy the (AR)-condition. By Theorem 1, the corresponding fractional Hamiltonian system $(\mathscr{F}\mathscr{H}\mathscr{S})$ possesses a sequence of nontrivial solutions (u_k) satisfying

$$\frac{1}{2}\int_{\mathbb{R}}[|{}_{-\infty}D_t^{\alpha}u_k|^2 + L(t)u_k.u_k]dt - \int_{\mathbb{R}} W(t,u_k)dt \longrightarrow +\infty \text{ as } k \longrightarrow \infty.$$

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. The third Section is devoted to the proof of our main result.

2. Preliminaries

In this Section, for the reader's convenience, first we will recall some facts about the fractional calculus on the whole real axis. On the other hand, we will give some preliminaries lemmas for using in the sequel.

2.1. Liouville-Weyl fractional calculus

The Liouville-Weyl fractional integrals of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ are defined as (see [11, 12, 17])

$${}_{-\infty}I_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-x)^{\alpha-1}u(x)dx, \tag{2.1}$$

and

$${}_{t}I_{\infty}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(x-t)^{\alpha-1}u(x)dx. \tag{2.2}$$

The Liouville-Weyl fractional derivatives of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [11, 12, 17])

$${}_{-\infty}D_t^{\alpha}u(t) = \frac{d}{dt}({}_{-\infty}I_t^{1-\alpha}u)(t), \tag{2.3}$$

and

$${}_{t}D_{\infty}^{\alpha}u(t) = -\frac{d}{dt}({}_{t}I_{\infty}^{1-\alpha}u)(t). \tag{2.4}$$

The definitions of (2.3) and (2.4) may be written in an alternative form as follows

$${}_{-\infty}D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{u(t)-u(t-x)}{x^{\alpha+1}}dx, \tag{2.5}$$

and

$$ {}_tD^{\alpha}_{\infty}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{u(t)-u(t+x)}{x^{\alpha+1}} dx. \tag{2.6} $$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}$ of u is defined by

$$ \widehat{u}(s) = \int_{-\infty}^{\infty} e^{-ist} u(t) dt. $$

Let u be defined on $\mathbb{R}$. Then the Fourier transform of the Liouville-Weyl integrals and differential operators satisfies (see [11, 12])

$$ \widehat{{}_{-\infty}I^{\alpha}_{t}u}(s) = (is)^{-\alpha}\widehat{u}(s), \tag{2.7} $$

$$ \widehat{{}_{t}I^{\alpha}_{\infty}u}(s) = (-is)^{-\alpha}\widehat{u}(s), \tag{2.8} $$

$$ \widehat{{}_{-\infty}D^{\alpha}_{t}u}(s) = (is)^{\alpha}\widehat{u}(s), \tag{2.9} $$

$$ \widehat{{}_{t}D^{\alpha}_{\infty}u}(s) = (-is)^{\alpha}\widehat{u}(s). \tag{2.10} $$

Next, we present some properties for Liouville-Weyl fractional integrals and derivatives on the real axis, which were proved in [11].

Denote by $L^p(\mathbb{R},\mathbb{R}^N)$ $(1 \leqslant p < \infty)$, the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norms

$$ \|u\|_{L^p} = \left(\int_{\mathbb{R}} |u(t)|^p \, dt\right)^{\frac{1}{p}}, $$

and $L^{\infty}(\mathbb{R},\mathbb{R}^N)$ the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm

$$ \|u\|_{\infty} = esssup\left\{|u(t)| \,/\, t \in \mathbb{R}\right\}. $$

PROPOSITION 1. 1) Let $p,q \in [1,\infty]$, $\alpha > 0$. The operators ${}_{-\infty}I^{\alpha}_{t}$ and ${}_{t}I^{\alpha}_{\infty}$ are bounded from $L^p(\mathbb{R},\mathbb{R}^N)$ to $L^q(\mathbb{R},\mathbb{R}^N)$ if and only if

$$ 0 < \alpha < 1, \;\; 1 < p < \frac{1}{\alpha}, \;\; q = \frac{p}{1-\alpha p}, $$

2) If $\alpha > 0$, for "sufficiently good" function f, the relations

$$ ({}_{-\infty}D^{\alpha}_{t}({}_{-\infty}I^{\alpha}_{t}f))(t) = f(t), \; ({}_{t}D^{\alpha}_{\infty}({}_{t}I^{\alpha}_{\infty}f))(t) = f(t) \tag{2.11} $$

are true. In particular, these relations hold for $f \in L^1(\mathbb{R},\mathbb{R}^N)$,

3) Let $\alpha,\beta > 0$ and $p \geqslant 1$ be such that $\alpha+\beta = \frac{1}{p}$. If $f \in L^p(\mathbb{R},\mathbb{R}^N)$, then

$$ ({}_{-\infty}I^{\beta}_{t}({}_{-\infty}I^{\alpha}_{t}u))(t) = {}_{-\infty}I^{\alpha+\beta}_{t}u(t), \; ({}_{t}I^{\beta}_{\infty}({}_{t}I^{\alpha}_{\infty}u))(t) = {}_{t}I^{\alpha+\beta}_{\infty}u(t), \tag{2.12} $$

4) If $\alpha > \beta > 0$, then

$$ ({}_{-\infty}D^{\beta}_{t}({}_{-\infty}I^{\alpha}_{t}u))(t) = {}_{-\infty}I^{\alpha-\beta}_{t}u(t), \; ({}_{t}D^{\beta}_{\infty}({}_{t}I^{\alpha}_{\infty}))(t) = {}_{t}I^{\alpha-\beta}_{\infty}u(t). \tag{2.13} $$

PROPOSITION 2. If $\alpha > 0$, then the relations

$$\int_{\mathbb{R}} \varphi(t)({}_{-\infty}I_t^{\alpha}\psi)(t)dt = \int_{\mathbb{R}} ({}_tI_{\infty}^{\alpha}\varphi)(t)\psi(t)dt, \tag{2.14}$$

$$\int_{\mathbb{R}} u(t)({}_{-\infty}D_t^{\alpha}v)(t)dt = \int_{\mathbb{R}} ({}_tD_{\infty}^{\alpha}u)(t)v(t)dt, \tag{2.15}$$

are valid for "sufficiently good" functions φ, ψ, u, v. In particular, (2.14) holds for functions $\varphi \in L^p(\mathbb{R},\mathbb{R}^N)$ and $\psi \in L^q(\mathbb{R},\mathbb{R}^N)$, while (2.15) holds for $u \in {}_tI_{\infty}^{\alpha}(L^p(\mathbb{R},\mathbb{R}^N))$ and $v \in {}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R},\mathbb{R}^N))$ provided that $p > 1$, $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1+\alpha$, where

$${}_tI_{\infty}^{\alpha}(L^p(\mathbb{R},\mathbb{R}^N)) = \left\{u/\exists\varphi \in L^p(\mathbb{R},\mathbb{R}^N), \quad u = {}_tI_{\infty}^{\alpha}\varphi\right\},$$

similarly, ${}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R},\mathbb{R}^N))$ can be defined.

2.2. Fractional derivative spaces

In order to establish the variational structure which enables us to reduce the existence of solutions of $(\mathscr{F}\mathscr{H}\mathscr{S})$ to find critical points of the corresponding functional, it is necessary to construct the appropriate functional spaces.

For $\alpha > 0$, define the semi-norm

$$|u|_{I_{-\infty}^{\alpha}} = \|{}_{-\infty}D_t^{\alpha}u\|_{L^2}$$

and the norm

$$\|u\|_{I_{-\infty}^{\alpha}} = (\|u\|_{L^2} + |u|_{I_{-\infty}^{\alpha}}^2)^{\frac{1}{2}},$$

and let

$$I_{-\infty}^{\alpha} = \overline{C_0^{\infty}(\mathbb{R},\mathbb{R}^N)}^{\|.\|_{I_{-\infty}^{\alpha}}}$$

where $C_0^{\infty}(\mathbb{R},\mathbb{R}^N)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with vanishing property at infinity.

Now, we can define the fractional Sobolev space $H^{\alpha}(\mathbb{R},\mathbb{R}^N)$ in terms of the Fourier transform. Choose $0 < \alpha < 1$, define the semi-norm

$$|u|_{\alpha} = \left\||s|^{\alpha}\,\widehat{u}\right\|_{L^2}$$

and the norm

$$\|u\|_{\alpha} = (\|u\|_{L^2} + |u|_{\alpha}^2)^{\frac{1}{2}},$$

and let

$$H^{\alpha}(\mathbb{R},\mathbb{R}^N) = \overline{C_0^{\infty}(\mathbb{R},\mathbb{R}^N)}^{\|.\|_{\alpha}}.$$

Moreover, we note that a function $u \in L^2(\mathbb{R},\mathbb{R}^N)$ belongs to $I_{-\infty}^{\alpha}$ if and only if

$$|s|^{\alpha}\,\widehat{u} \in L^2(\mathbb{R},\mathbb{R}^N).$$

Especially, we have

$$|u|_{I_{-\infty}^{\alpha}} = \left\||s|^{\alpha}\,\widehat{u}\right\|_{L^2}.$$

Therefore, $I_{-\infty}^{\alpha}$ and $H^{\alpha}(\mathbb{R},\mathbb{R}^N)$ are equivalent with equivalent semi-norms and norms. Analogous to $I_{-\infty}^{\alpha}$, we introduce I_{∞}^{α}. Define the semi-norm

$$|u|_{I_{\infty}^{\alpha}} = \|{}_tD_{\infty}^{\alpha}u\|_{L^2}$$

and the norm

$$\|u\|_{I_{\infty}^{\alpha}} = (\|u\|_{L^2} + |u|_{I_{\infty}^{\alpha}}^2)^{\frac{1}{2}},$$

and let

$$I_{\infty}^{\alpha} = \overline{C_0^{\infty}(\mathbb{R},\mathbb{R}^N)}^{\|.\|_{I_{\infty}^{\alpha}}}$$

Then $I_{-\infty}^{\alpha}$ and I_{∞}^{α} are equivalent with equivalent semi-norms and norms.

Let $C(\mathbb{R},\mathbb{R}^N)$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. Then we obtain the following Sobolev lemma.

LEMMA 1. ([20], Theorem 2.1) *If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R},\mathbb{R}^N) \subset C(\mathbb{R},\mathbb{R}^N)$, and there exists a constant $C = C_{\alpha}$ such that*

$$\|u\|_{\infty} = \sup_{t\in\mathbb{R}} |u(t)| \leqslant C_{\alpha} \|u\|_{\alpha}, \forall u \in H^{\alpha}(\mathbb{R},\mathbb{R}^N). \tag{2.16}$$

REMARK 1. From Lemma 1, we know that if $u \in H^{\alpha}(\mathbb{R},\mathbb{R}^N)$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^p(\mathbb{R},\mathbb{R}^N)$ for all $p \in [2,\infty[$, because

$$\int_{\mathbb{R}} |u(t)|^p\, dt \leqslant \|u\|_{\infty}^{p-2} \|u\|_{L^2}^2.$$

In what follows, we assume that there exists a constant $a_0 > 0$ such that

$$(L_0') \qquad L(t)x.x \geqslant a_0 |x|^2,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N$$

and we introduce the following fractional space

$$X^{\alpha} = \left\{ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^N) / \int_{\mathbb{R}} L(t)u(t).u(t)dt < \infty \right\}.$$

Then X^{α} is a Hilbert space with the inner product

$$\langle u,v\rangle_{X^{\alpha}} = \int_{\mathbb{R}} [{}_{-\infty}D_t^{\alpha}u(t).{}_{-\infty}D_t^{\alpha}v(t) + L(t)u(t).v(t)]dt$$

and the corresponding norm

$$\|u\|_{X^{\alpha}}^2 = \langle u,u\rangle_{X^{\alpha}}.$$

It is easy to see that X^{α} is continuously embedded in $H^{\alpha}(\mathbb{R},\mathbb{R}^N)$. In fact, for $u \in X^{\alpha}$, we have

$$\begin{aligned}
\|u\|_{X^{\alpha}}^2 &= \int_{\mathbb{R}} [|{}_{-\infty}D_t^{\alpha}u(t)|^2 + L(t)u(t).u(t)]dt \\
&\geqslant \int_{\mathbb{R}} [|{}_{-\infty}D_t^{\alpha}u(t)|^2 + a_0|u(t)|^2]dt \\
&\geqslant \inf(1,a_0)\|u\|_{H^{\alpha}}^2 \geqslant \inf(1,a_0)\|u\|_{L^2}^2.
\end{aligned}$$

For $p \in]2,\infty[$, we have by Remark 1

$$\|u\|_{L^p}^p = \int_{\mathbb{R}} |u(t)|^p\, dt \leqslant \|u\|_\infty^{p-2} \|u\|_{L^2}^2 \leqslant \frac{C_\alpha^{p-2}}{\inf(1,a_0)} \|u\|_{X^\alpha}^p = \eta_p(\alpha) \|u\|_{X^\alpha}^p . \quad (2.17)$$

The main difficulty in dealing with the existence of infinitely many solutions for $(\mathscr{F}\mathscr{H}\mathscr{S})$ is the lack of compactness of the Sobolev embedding. To overcome this difficulty under the assumptions of Theorem 1, we employ the following compact embedding lemma.

LEMMA 2. [15] *Assume* (L_0') *and* (L) *are satisfied. Then* X^α *is compactly embedded in* $L^2(\mathbb{R},\mathbb{R}^N)$.

REMARK 2. From Remark 1 and Lemma 2, it is easy to verify that the embedding of X^α in $L^p(\mathbb{R},\mathbb{R}^N)$ is also compact for $p \in]2,\infty[$.

To study the critical points of the variational functional associated with $(\mathscr{F}\mathscr{H}\mathscr{S})$, we need the following variant fountain theorem established by Zou [28].

Let X be a Banach space with the norm $\|.\|$ and $X = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider a functionals $f_\lambda \in C^1(X,\mathbb{R})$ of the type

$$f_\lambda(u) = A(u) - \lambda B(u),\ u \in E,\ \lambda \in [1,2].$$

LEMMA 3. (Variant fountain theorem) [28] *Assume that the functionals* f_λ *satisfy*

a) f_λ *maps bounded sets into bounded sets for* $\lambda \in [1,2]$ *and*

$$f_\lambda(-u) = f_\lambda(u) \ for\ all\ (\lambda,u) \in [1,2]\times X;$$

b) $B(u) \geqslant 0$ *for all* $u \in X$ *and* $A(u) \longrightarrow +\infty$ *or* $B(u) \longrightarrow +\infty$ *as* $\|u\| \longrightarrow \infty$*;*

c) There exist $\rho_k > r_k > 0$ *such that for all* $\lambda \in [1,2]$

$$\alpha_k(\lambda) = \inf_{u\in Z_k, \|u\|=r_k} f_\lambda(u) > \beta_k(\lambda) = \max_{u\in Y_k, \|u\|=\rho_k} f_\lambda(u).$$

Then

$$\alpha_k(\lambda) \leqslant \xi_k(\lambda) = \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} f_\lambda(\gamma(u)),\ \forall\lambda \in [1,2],$$

where

$$B_k = \{u \in Y_k / \|u\| \leqslant \rho_k\}\ and\ \Gamma_k = \left\{\gamma \in C(B_k,X)/\gamma\ \ is\ odd,\ \ \gamma_{|\partial B_k} = id\right\}.$$

Moreover, for almost every $\lambda \in [1,2]$*, there exists a sequence* $(u_m^k(\lambda))_{m\in\mathbb{N}}$ *such that*

$$\sup_{m\in\mathbb{N}} \left\|u_m^k(\lambda)\right\| < \infty,\ f_\lambda'(u_m^k(\lambda)) \longrightarrow 0,\ \ f_\lambda(u_m^k(\lambda)) \longrightarrow \xi_k(\lambda)\ as\ m \longrightarrow \infty.$$

3. Proof of theorem 1

First, note that (L_0), (W_1) and (W_2) imply that there exists a constant $a_0 > 0$ such that $\widetilde{L}(t) = L(t) + 2a_0 I_N \geqslant a_0 I_N$ for all $t \in \mathbb{R}$ and $\widetilde{W}(t,x) = W(t,x) + a_0 |x|^2 \geqslant 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Consider the following fractional system

$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t) + \widetilde{L}(t)u(t) = \nabla \widetilde{W}(t,u(t)),\ t \in \mathbb{R} \\ u \in H^\alpha(\mathbb{R},\ \mathbb{R}^N), \end{cases} \tag{3.1}$$

then (3.1) is equivalent to $(\mathscr{FHS})$. Moreover, it is easy to check that the hypotheses (W_1)–(W_4) still hold for $\widetilde{W}(t,x)$ provided that those hold for $W(t,x)$ and the function $\widetilde{L}$ satisfies (L_0') and (L). Hence in what follows, we always assume without loss of generality that L satisfies (L_0') and (L), $W(t,x) \geqslant 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $W(t,x)$ satisfies (W_1)–(W_4).

Consider the variational function f associated to the fractional system $(\mathscr{FHS})$:

$$f(u) = \frac{1}{2}\int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t).u(t))dt - \int_{\mathbb{R}} W(t,u(t))dt$$

defined on the Hilbert space X^α introduced in Section 2. Set

$$g(u) = \int_{\mathbb{R}} W(t,u)dt,\ \ u \in X^\alpha.$$

LEMMA 4. *Assume (L_0'), (L) and (W_1) are satisfied. Then $g \in C^1(X^\alpha,\mathbb{R})$ and $g' : X^\alpha \longrightarrow (X^\alpha)^*$ is compact, and $f \in C^1(X^\alpha,\mathbb{R})$. Moreover, for all $u,v \in X^\alpha$*

$$g'(u)v = \int_{\mathbb{R}} \nabla W(t,u).vdt \tag{3.2}$$

$$f'(u)v = \langle u,v\rangle - \int_{\mathbb{R}} \nabla W(t,u).vdt. \tag{3.3}$$

Proof. By (W_1), for any $s \in [0,1]$ and $u,v \in X^\alpha$, we have

$$\begin{aligned} |\nabla W(t,u+sv)v| &\leqslant c(|u+sv| + |u+sv|^{\nu-1})\,|v| \\ &\leqslant c[|u| + |v| + 2^{\nu-2}(|u|^{\nu-1} + |v|^{\nu-1})]\,|v| \\ &\leqslant c2^{\nu-2}[|u|\,|v| + |v|^2 + |u|^{\nu-1}\,|v| + |v|^\nu]. \end{aligned} \tag{3.4}$$

The Hölder's inequality implies

$$\int_{\mathbb{R}} (|u|\,|v| + |u|^{\nu-1}\,|v|)dt \leqslant \|u\|_2\,\|v\|_2 + \|u\|_\nu^{\nu-1}\,\|v\|_\nu. \tag{3.5}$$

Hence, by (3.4), (3.5), the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we get for all $u,v \in X^\alpha$

$$\begin{aligned} \lim_{s\longrightarrow 0} \frac{g(u+sv)-g(u)}{s} &= \lim_{s\longrightarrow 0} \int_{\mathbb{R}}\int_0^1 \nabla W(t,u+sv).vdsdt \\ &= \int_{\mathbb{R}} \nabla W(t,u).vdt = J(u,v). \end{aligned} \tag{3.6}$$

Moreover, it follows from (2.17), (W_1) and (3.5) that

$$\begin{aligned}|J(u,v)| &\leqslant \int_{\mathbb{R}} |\nabla W(t,u)|\,|v|\,dt \leqslant c\int_{\mathbb{R}}(|u|\,|v|+|u|^{\nu-1}\,|v|)dt \\ &\leqslant c(\|u\|_2\,\|v\|_2+\|u\|_\nu^{\nu-1}\,\|v\|_2) \leqslant c(\eta_2^2\,\|u\|_{X^\alpha}+\eta_\nu^\nu\,\|u\|_{X^\alpha}^{\nu-1})\,\|v\|_{X^\alpha}\,.\end{aligned}$$

Therefore, $J(u,.)$ is linear and bounded, and $J(u,.)$ is the Gâteaux derivative of g at u.

Next, we prove that $J(u,.)$ is weakly continuous in u. For this end, we first claim that if $u_n \rightharpoonup u$ in X^α, then $\nabla W(t,u_n) \longrightarrow \nabla W(t,u)$ in $L^2(\mathbb{R})$. Arguing indirectly, by Lemma 2, we may assume that there exists a subsequence (u_{n_k}) such that

$$u_{n_k} \longrightarrow u \text{ in both } L^2(\mathbb{R}) \text{ and } L^{2(\nu-1)}(\mathbb{R}) \text{ and } u_{n_k} \longrightarrow u \text{ a.e. in } \mathbb{R} \text{ as } k \longrightarrow \infty \tag{3.7}$$

and

$$\int_{\mathbb{R}} \left|\nabla W(t,u_{n_k})-\nabla W(t,u)\right|^2 dt \geqslant \varepsilon_0,\ \forall k\in\mathbb{N} \tag{3.8}$$

for some positive constant ε_0. By (3.7) and up to a subsequence if necessary, we can assume that $\sum_{k=1}^\infty \left\|u_{n_k}-u\right\|_{L^2} < \infty$ and $\sum_{k=1}^\infty \left\|u_{n_k}-u\right\|_{L^{2(\nu-1)}} < \infty$. Let $w(t) = \sum_{k=1}^\infty \left|u_{n_k}(t)-u(t)\right|$ for all $t\in\mathbb{R}$, then $w\in L^2(\mathbb{R})\cap L^{2(\nu-1)}(\mathbb{R})$. By (W_1), there holds for all $k\in\mathbb{N}$ and $t\in\mathbb{R}$

$$\begin{aligned}&\left|\nabla W(t,u_{n_k})-\nabla W(t,u)\right|^2\\ &\leqslant (\left|\nabla W(t,u_{n_k})\right|+|\nabla W(t,u)|)^2\\ &\leqslant 2(\left|\nabla W(t,u_{n_k})\right|^2+|\nabla W(t,u)|^2)\\ &\leqslant 2c^2[(\left|u_{n_k}\right|+\left|u_{n_k}\right|^{\nu-1})^2+(|u|+|u|^{\nu-1})^2]\\ &\leqslant 2^2c^2[\left|u_{n_k}\right|^2+\left|u_{n_k}\right|^{2(\nu-1)}+|u|^2+|u|^{2(\nu-1)}]\\ &\leqslant 2^2c^2[(\left|u_{n_k}-u\right|+|u|)^2+(\left|u_{n_k}-u\right|+|u|)^{2(\nu-1)}+|u|^2+|u|^{2(\nu-1)}]\\ &\leqslant 2^2c^2[2(\left|u_{n_k}-u\right|^2+|u|^2)+2^{2\nu-3}(\left|u_{n_k}-u\right|^{2(\nu-1)}+|u|^{2(\nu-1)})+|u|^2+|u|^{2(\nu-1)}]\\ &\leqslant c_1(|w|^2+|u|^2+|w|^{2(\nu-1)}+|u|^{2(\nu-1)})\end{aligned}$$

where c_1 is a positive constant. Combining this with (3.7), Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k\longrightarrow\infty}\int_{\mathbb{R}} \left|\nabla W(t,u_{n_k})-\nabla W(t,u)\right|^2 dt = 0$$

which contradicts to (3.8). Hence the claim above is true.

Now, suppose $u_n \rightharpoonup u$ in X^α, then $\nabla W(t,u_n) \longrightarrow \nabla W(t,u)$ in $L^2(\mathbb{R})$. By Hölder's inequality and (2.17), we have

$$\begin{aligned}\|J(u_n,.)-J(u,.)\|_{E^*} &= \sup_{\|v\|=1}\int_{\mathbb{R}}(\nabla W(t,u_n)-\nabla W(t,u)).v dt\\ &\leqslant \eta_2(\int_{\mathbb{R}}|\nabla W(t,u_n)-\nabla W(t,u)|^2\,dt)^{\frac12} \longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned}$$

This means that $u \longmapsto J(u,.)$ is weakly continuous and then it is continuous in X^α. Therefore $g \in C^1(X^\alpha,\mathbb{R})$ and (3.2) is verified. Furthermore, g' is compact by the weak continuity of g' since X^α is reflexive. Due to the form of f, (3.3) is also verified and $f \in C^1(X^\alpha,\mathbb{R})$.

Finally, let $u \in X^\alpha$ be a critical point of f. A standard argument shows that $u \in C^2(\mathbb{R},\mathbb{R}^N)$ and satisfies equation $(\mathscr{FHS})$. The proof is completed. $\square$

In order to apply the variant fountain theorem to prove our main result, choose an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of X^α and let $X_j = span\{e_j\}$ for all $j \in \mathbb{N}$. Define Y_k and Z_k by

$$Y_k = \oplus_{j=1}^k X_j,\ Z_k = \overline{\oplus_{j=k}^\infty X_j}$$

and the functionals A,B and f_λ on our working space X^α by

$$A(u) = \frac{1}{2}\|u\|^2_{X^\alpha},\ B(u) = g(u) = \int_{\mathbb{R}} W(t,u)dt,\ f_\lambda(u) = A(u) - \lambda B(u),$$

for all $(\lambda,u) \in [1,2]\times X^\alpha$.

Assumption (W_1) and property (2.17) imply that f_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. Note that $W(t,-x) = W(t,x)$, so we have $f_\lambda(-u) = f_\lambda(u)$ for all $(\lambda,u) \in [1,2]\times X^\alpha$. Thus condition a) of Lemma 3 holds. Since $W(t,x) \geqslant 0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, it is clear that condition b) is also satisfied. To verify condition c), we need to establish the three following lemmas.

LEMMA 5. *Suppose (L_0') and (L) hold. Then for any $p \in [2,\infty]$*

$$l_p(k) = \sup_{u\in Z_k, \|u\|_{X^\alpha}=1} \|u\|_{L^p} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{3.9}$$

Proof. It is clear that $0 < l_p(k+1) \leqslant l_p(k)$, so that $l_p(k) \longrightarrow \overline{l}_p$ as $k \longrightarrow \infty$. For every $k \geqslant 1$, there exists $u_k \in Z_k$ such that $\|u_k\|_{X^\alpha} = 1$ and $\|u_k\|_{L^p} > \frac{1}{2} l_p(k)$. For any $v \in X^\alpha$, let $v = \sum_{i=1}^\infty v_i e_i$. By the Cauchy-Schartz inequality, one has

$$\begin{aligned}
|\langle u_k, v\rangle| &= \left|\langle u_k, \sum_{i=1}^\infty v_i e_i\rangle\right| = \left|\langle u_k, \sum_{i=k+1}^\infty v_i e_i\rangle\right| \\
&\leqslant \|u_k\|_{X^\alpha} \left\|\sum_{i=k+1}^\infty v_i e_i\right\| \leqslant \sum_{i=k+1}^\infty |v_i|\,\|e_i\|_{X^\alpha} \longrightarrow 0 \text{ as } k \longrightarrow \infty
\end{aligned}$$

which implies that $u_k \rightharpoonup 0$. Without loss of generality, Lemma 2 implies that $u_k \longrightarrow 0$ in $L^2(\mathbb{R})$. Thus we have proved that $\overline{l}_p(k) = 0$. The proof is completed. $\square$

LEMMA 6. *Assume (L_0'), (L) and (W_1) are satisfied. Then there exist a positive integer k_0 and a sequence $r_k \longrightarrow +\infty$ as $k \longrightarrow \infty$ such that*

$$\alpha_k(\lambda) = \inf_{u\in Z_k, \|u\|_{X^\alpha}=r_k} f_\lambda(u) > 0,\ \forall k \geqslant k_0. \tag{3.10}$$

Proof. From (W_1) and the fact that $W(t,x) \geqslant 0$, we have

$$f_\lambda(u) \geqslant \frac{1}{2}\|u\|^2_{X^\alpha} - 2\int_{\mathbb{R}} W(t,u)dt$$
$$\geqslant \frac{1}{2}\|u\|^2_{X^\alpha} - 2c(\|u\|^2_{L^2} + \|u\|^\nu_{L^\nu}), \forall(\lambda,u) \in [1,2]\times X^\alpha. \tag{3.11}$$

Combining (3.9) and (3.11) yields

$$f_\lambda(u) \geqslant \frac{1}{2}\|u\|^2_{X^\alpha} - 2cl_2^2(k)\|u\|^2_{X^\alpha} - 2cl_\nu^\nu(k)\|u\|^\nu_{X^\alpha},\ \forall(\lambda,u)\in[1,2]\times X^\alpha. \tag{3.12}$$

In view of (3.9), there exists an integer k_0 such that

$$2cl_2^2(k) \leqslant \frac{1}{4}, \forall k \geqslant k_0. \tag{3.13}$$

For any $k \geqslant k_0$, let us define

$$r_k = (16cl_\nu^\nu(k))^{\frac{1}{2-\nu}}. \tag{3.14}$$

Since $\nu > 2$, then $r_k \longrightarrow +\infty$ as $k \longrightarrow \infty$. From (3.12)–(3.14), we deduce that for all $k \geqslant k_0$

$$\inf_{u\in Z_k, \|u\|_{X^\alpha}=1} f_\lambda(u) \geqslant \frac{1}{2}r_k^2 - \frac{1}{4}r_k^2 - \frac{1}{8}r_k^{2-\nu}r_k^{2-\nu}r_k^\nu = \frac{1}{8}r_k^2 > 0, \tag{3.15}$$

which completes the proof of Lemma 6. $\square$

LEMMA 7. *Assume (L_0'), (L), (W_1) and (W_2) are satisfied. Then for any $k \geqslant k_0$, there exists $\rho_k > r_k$ such that*

$$\beta_k(\lambda) = \max_{u\in Y_k, \|u\|_{X^\alpha}=\rho_k} f_\lambda(u) < 0,$$

where k_0 is the positive integer obtained in Lemma 6.

Proof. Firstly, we claim that for any finite-dimensional subspace $F \subset X^\alpha$, there exists a constant $\varepsilon_0 > 0$ such that

$$meas(\{t\in\mathbb{R}/\,|u(t)| \geqslant \varepsilon_0\|u\|\}) \geqslant \varepsilon_0,\ \forall u \in F\setminus\{0\}. \tag{3.16}$$

In not, for any $n \in \mathbb{N}$, there exists $u_n \in F\setminus\{0\}$ such that

$$meas\left(\left\{t\in\mathbb{R}/\,|u_n(t)| \geqslant \frac{1}{n}\|u_n\|_{X^\alpha}\right\}\right) < \frac{1}{n}.$$

Let $v_n = \frac{u_n}{\|u_n\|} \in F$, then $\|v_n\| = 1$ and

$$meas\left(\left\{t\in\mathbb{R}/\,|v_n(t)| \geqslant \frac{1}{n}\right\}\right) < \frac{1}{n}, \forall n\in\mathbb{N}. \tag{3.17}$$

Since F is finite-dimensional, then up to a subsequence if necessary, we may assume $v_n \longrightarrow v_0$ in X^α for some $v_0 \in X^\alpha$. Evidently, $\|v_0\|_{X^\alpha} = 1$. Note that, since any two norms on F are equivalent, we have

$$\int_{\mathbb{R}} |v_n - v_0|\, dt \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.18}$$

The fact that $\|v_0\|_{X^\alpha} = 1$ implies $\|v_0\|_{L^\infty} > 0$. By the definition of $\|.\|_{L^\infty}$, there exists a constant $\delta_0 > 0$ such that

$$meas(\{t \in \mathbb{R}/\, |v_0(t)| \geqslant \delta_0\}) \geqslant \delta_0. \tag{3.19}$$

Otherwise, for each fixed $n \in \mathbb{N}$ and $m > n$, we have

$$meas\left(\left\{t \in \mathbb{R}/\, |v_0(t)| \geqslant \frac{1}{n}\right\}\right) \leqslant meas\left(\left\{t \in \mathbb{R}/\, |v_0(t)| \geqslant \frac{1}{m}\right\}\right) \leqslant \frac{1}{m}.$$

Letting $m \longrightarrow \infty$, we obtain $meas(\left\{t \in \mathbb{R}/\, |v_0(t)| \geqslant \frac{1}{n}\right\}) = 0$. Consequently

$$\begin{aligned} 0 &\leqslant meas(\{t \in \mathbb{R}/\, |v_0(t)| \neq 0\}) \\ &= meas\left(\cup_{n=1}^{\infty} \left\{t \in \mathbb{R}/\, |v_0(t)| \geqslant \frac{1}{n}\right\}\right) \\ &\leqslant \sum_{n=1}^{\infty} meas\left(\left\{t \in \mathbb{R}/\, |v_0(t)| \geqslant \frac{1}{n}\right\}\right) = 0 \end{aligned}$$

which yields $v_0 = 0$ and contradicts $\|v_0\|_{X^\alpha} = 1$. Then (3.19) holds.

For any $n \in \mathbb{N}$, let

$$\Lambda_n = \left\{t \in \mathbb{R}/\, |v_n(t)| < \frac{1}{n}\right\}, \quad \Lambda_0 = \{t \in \mathbb{R}/\, |v_0(t)| \geqslant \delta_0\}.$$

Then for n large enough, by (3.17) and (3.19), we have

$$meas(\Lambda_n \cap \Lambda_0) \geqslant meas(\Lambda_0) - meas(\Lambda_n^c) \geqslant \delta_0 - \frac{1}{n} \geqslant \frac{\delta_0}{2}.$$

Consequently, for n large enough, there holds

$$\begin{aligned} \int_{\mathbb{R}} |v_n - v_0|\, dt &\geqslant \int_{\Lambda_n \cap \Lambda_0} |v_n - v_0|\, dt \geqslant \int_{\Lambda_n \cap \Lambda_0} (|v_0| - |v_n|) dt \\ &\geqslant \left(\delta_0 - \frac{1}{n}\right) meas(\Lambda_n \cap \Lambda_0) \geqslant \frac{\delta_0}{4} > 0. \end{aligned}$$

This contradicts (3.18). Therefore (3.16) holds. Now, note that for any $k \in \mathbb{N}$, Y_k is finite-dimensional, so there exists a constant $\varepsilon_k > 0$ such that

$$meas(\Lambda_u^k) \geqslant \varepsilon_k,\ \forall u \in Y_k \setminus \{0\}, \tag{3.20}$$

where

$$\Lambda_u^k = \{t \in \mathbb{R} / |u(t)| \geqslant \varepsilon_k \|u\|_{X^\alpha}\}$$

for all $k \in \mathbb{N}$ and $u \in Y_k \backslash \{0\}$. By (W_2), for any $k \in \mathbb{N}$, there exists a constant $R_k > 0$ such that

$$W(t,x) \geqslant \frac{|x|^2}{\varepsilon_k^3}, \ \forall t \in \mathbb{R} \text{ and } |x| \geqslant R_k. \tag{3.21}$$

Combining (3.20) with (3.21), for any $k \in \mathbb{N}$ and $\lambda \in [1,2]$, we have

$$\begin{aligned} f_\lambda(u) &\leqslant \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,u)dt \leqslant \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} \frac{|u|^2}{\varepsilon_k^3} dt \\ &\leqslant \frac{1}{2}\|u\|_{X^\alpha}^2 - \varepsilon_k^2 \|u\|_{X^\alpha}^2 \, meas(\Lambda_u^k) \frac{1}{\varepsilon_k^3} \leqslant \frac{1}{2}\|u\|_{X^\alpha}^2 - \|u\|_{x^\alpha}^2 = -\frac{1}{2}\|u\|_{X^\alpha}^2 \end{aligned} \tag{3.22}$$

for all $u \in Y_k$ with $\|u\|_{X^\alpha} \geqslant \frac{R_k}{\varepsilon_k}$. For any $k \geqslant k_0$, choose $\rho_k > \max\left\{r_k, \frac{R_k}{\varepsilon_k}\right\}$, then (3.22) implies

$$\rho_k(\lambda) = \max_{u \in Y_k, \|u\|_{X^\alpha} = \rho_k} f_\lambda(u) \leqslant -\frac{1}{2}\rho_k^2.$$

The proof is completed. □

Consequently, Lemmas 6,7 show that condition c) of Lemma 3 is satisfied for all $k \geqslant k_0$. By the above, all the conditions Lemma 3 hold for all $k \geqslant k_0$. Therefore, for any $k \geqslant k_0$ and $\lambda \in [1,2]$, there exists a sequence $(u_n^k)_{n \in \mathbb{N}} \subset X^\alpha$ such that

$$\sup_{n \in \mathbb{N}} \left\| u_n^k(\lambda) \right\| < \infty, \ f_\lambda'(u_n^k(\lambda)) \longrightarrow 0 \text{ and } f_\lambda(u_n^k(\lambda)) \longrightarrow \xi_k(\lambda) \text{ as } n \longrightarrow \infty, \tag{2.23}$$

where

$$\xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \sup_{u \in B_k} f_\lambda(\gamma(u)), \ \forall \lambda \in [1,2]$$

with $B_k = \{u \in Y_k / \|u\| \leqslant \rho_k\}$ and $\Gamma_k = \left\{\gamma \in C(B_k, X^\alpha) / \gamma \text{ is odd}, \ \gamma_{|\partial B_k} = id\right\}$. From (3.15) and Lemma 3, we infer that

$$\xi_k(\lambda) \in [\overline{\alpha}_k, \overline{\xi}_k], \ \forall k \geqslant k_0, \ \lambda \in [1,2], \tag{3.24}$$

where $\overline{\xi}_k = \max_{u \in B_k} f_1(u)$ and $\overline{\alpha}_k = \frac{r_k^2}{8} \longrightarrow \infty$ as $k \longrightarrow \infty$.

In view of (3.23), for any $k \geqslant k_0$, we can choose a sequence $\lambda_n \longrightarrow 1$ and the corresponding sequences $(u_m^k(\lambda_n))$ satisfying

$$\sup_{m \in \mathbb{N}} \left\| u_m^k(\lambda_n) \right\|_{X^\alpha} < \infty \text{ and } f_{\lambda_n}'(u_m^k(\lambda_n)) \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{3.25}$$

LEMMA 8. *For any $n \in \mathbb{N}$ and $k \geqslant k_0$, there exists $u_n^k \in X^\alpha$ such that*

$$\lim_{m \longrightarrow \infty} u_m^k(\lambda_n) = u_n^k \text{ in } X^\alpha. \tag{3.26}$$

Proof. Throughout this proof and for the sake of simplicity, we shall let $u_m = u_m^k(\lambda_n)$ for $m \in \mathbb{N}$. Without loss of generality, we may assume by (3.23) that

$$u_m \rightharpoonup u \text{ as } m \longrightarrow \infty \tag{3.27}$$

for some $u \in X^\alpha$. Using (3.3), we get

$$\begin{aligned}\|u_m - u\|^2 = f'_{\lambda_n}(u_m).(u_m - u) - f'_{\lambda_n}(u).(u_m - u)\\ + \lambda_n \int_{\mathbb{R}} (\nabla W(t,u_m) - \nabla W(t,u)).(u_m - u)dt.\end{aligned} \tag{3.28}$$

By (3.25), one has

$$f'_{\lambda_n}(u_m).(u_m - u) \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{3.29}$$

Moreover (3.27) yields

$$f'_{\lambda_n}(u).(u_m - u) \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{3.30}$$

Now, by (2.17) and Holder's inequality, we have

$$\begin{aligned}&\left| \int_{\mathbb{R}} (\nabla W(t,u_m) - \nabla W(t,u)).(u_m - u)dt \right|\\ &\leqslant \left(\int_{\mathbb{R}} |\nabla W(t,u_m) - \nabla W(t,u)|^2 dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |u_m - u|^2 dt\right)^{\frac{1}{2}}\\ &\leqslant \eta_2 \left(\int_{\mathbb{R}} |\nabla W(t,u_m) - \nabla W(t,u)|^2 dt\right)^{\frac{1}{2}} \|u_m - u\|_{X^\alpha}.\end{aligned} \tag{3.31}$$

As in the proof of Lemma 4, by passing to a subsequence if necessary, we may assume that $\int_{\mathbb{R}} |\nabla W(t,u_m) - \nabla W(t,u)|^2 dt \longrightarrow 0$ as $m \longrightarrow \infty$. Hence (3.31) implies

$$\int_{\mathbb{R}} (\nabla W(t,u_m) - \nabla W(t,u)).(u_m - u)dt \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{3.32}$$

Combining (3.28)–(3.30) and (3.32) yields $u_m \longrightarrow \infty$ as $m \longrightarrow \infty$ in X^α. The proof is completed. □

Note that (3.23) and (3.24) imply

$$f'_{\lambda_n}(u_n^k) = 0, \;\; f_{\lambda_n}(u_n^k) \in [\overline{\alpha}_k, \overline{\xi}_k], \; \forall n \in \mathbb{N} \text{ and } k \geqslant k_0. \tag{3.33}$$

LEMMA 9. *For any $k \geqslant k_0$, the sequence $(u_n^k)_{n\in\mathbb{N}}$ obtained above is bounded.*

Proof. For notational simplicity, we set $u_n = u_n^k$ for all $n \in \mathbb{N}$. Assuming indirectly that (u_n) is unbounded. By going to a subsequence if necessary, we may assume

$$\|u_n\| \longrightarrow \infty \text{ and } v_n = \frac{u_n}{\|u_n\|} \rightharpoonup v \text{ as } n \longrightarrow \infty. \tag{3.34}$$

By Lemma 2 and (3.34), without loss of generality, we have

$$v_n \longrightarrow v \text{ both in } L^2(\mathbb{R}) \text{ and } L^\nu(\mathbb{R}) \text{ and } v_n(t) \longrightarrow v(t) \text{ a.e. } t \in \mathbb{R} \text{ as } n \longrightarrow \infty. \tag{3.35}$$

First case. When $v=0$ occurs. Let (s_n) be a sequence such that

$$f_{\lambda_n}(s_n u_n) = \max_{s\in[0,1]} f_{\lambda_n}(su_n),\ \forall n \in \mathbb{N}. \tag{3.36}$$

For $R>0$, let $w_n = 2\sqrt{R}v_n$. By (3.35), we have

$$w_n \longrightarrow 2\sqrt{R}v = 0 \text{ both in } L^2(\mathbb{R}) \text{ and } L^{\nu}(\mathbb{R}) \tag{3.37}$$

which with (W_1) implies

$$\left|\int_{\mathbb{R}} W(t,w_n)dt\right| \leqslant c\int_{\mathbb{R}}(|w_n|^2+|w_n|^{\nu})dt \longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.38}$$

Note that (3.34) implies that $0<\frac{2\sqrt{R}}{\|u_n\|}<1$ for n large enough. This together with (3.36) and (3.38) implies

$$\begin{aligned} f_{\lambda_n}(s_n u_n) \geqslant f_{\lambda_n}(w_n) &= \frac{1}{2}\|u_n\|^2_{X^\alpha} - \lambda_n\int_{\mathbb{R}} W(t,w_n)dt \\ &\geqslant 2R - 2\int_{\mathbb{R}} W(t,w_n)dt \geqslant R \end{aligned}$$

for n large enough. Since R is arbitrarily, it follows that

$$\lim_{n\longrightarrow\infty} f_{\lambda_n}(s_n u_n) = +\infty. \tag{3.39}$$

Note that, since $f_{\lambda_n}(0)=0$ and $f_{\lambda_n}(u_n)\in[\overline{\alpha}_k,\overline{\xi}_k]$, then $s_n\in]0,1[$ in (3.36) for n large enough. Therefore

$$0 = s_n\frac{d}{ds}(f_{\lambda_n})(su_n)_{|s=s_n} = f'_{\lambda_n}(s_n u_n).s_n u_n. \tag{3.40}$$

Combining (3.33), (3.39) and (W_3) yields

$$\begin{aligned} f_{\lambda_n}(u_n) - \frac{1}{2}f'_{\lambda_n}(u_n).u_n &= \frac{\lambda_n}{2}\int_{\mathbb{R}}\widehat{W}(t,w_n)dt \geqslant \frac{\lambda_n}{2\sigma}\int_{\mathbb{R}}\widehat{W}(t,s_n w_n)dt \\ &= \frac{1}{\sigma}[f_{\lambda_n}(s_n u_n) - \frac{1}{2}f'_{\lambda_n}(s_n u_n).s_n u_n] \\ &= \frac{1}{\sigma}f_{\lambda_n}(s_n u_n) \longrightarrow +\infty \text{ as } n\longrightarrow\infty, \end{aligned}$$

a contradiction with (3.33).

Second case. When $v\neq 0$ occurs. The set $\Lambda = \{t\in\mathbb{R}/v(t)\neq 0\}$ has a positive measure. By (3.34), it holds that

$$|u_n(t)| \longrightarrow \infty \text{ as } n\longrightarrow\infty,\ \forall t\in\Lambda. \tag{3.41}$$

Combining (3.35), (3.41) and (W_2), Fatou's lemma implies

$$\begin{aligned}\frac{1}{2}-\frac{f_{\lambda_n}(u_n)}{\|u_n\|^2}&=\lambda_n\int_{\mathbb{R}}\frac{W(t,u_n)}{\|u_n\|^2}dt\\&\geqslant\int_{\Lambda}|v_n|^2\frac{W(t,u_n)}{|u_n|^2}dt\longrightarrow\infty \text{ as } n\longrightarrow\infty,\end{aligned}$$

which provides a contradiction with (3.33) and (3.34). The proof is completed. □

Finally, using the similar arguments in the proof of Lemma 8 and in view of Lemma 9 and (3.33), we can show that for any $k\geqslant k_0$, the sequence $(u_n^k)_{n\in\mathbb{N}}$ possesses a strong convergent subsequence with the limit u^k being a critical point of $f=f_1$. Since $\overline{\alpha}_k\longrightarrow+\infty$ as $k\longrightarrow\infty$ and $f(u^k)\in[\overline{\alpha}_k,\overline{\xi}_k]$ for all $k\geqslant k_0$, then f has infinitely many critical points. Consequently, $(\mathscr{F}\mathscr{H}\mathscr{S})$ has infinitely many nontrivial solutions.

Acknowledgements. The author thanks the referee and the editor for their helpful comments and suggestions.

REFERENCES

[1] O. AGRAWAL, J. TENREIRO MACHADO, J. SABATIER, *Fractional derivatives and their applications; Nonlinear dynamics*, Springer-Verlag, Berlin, 2004.

[2] N. NYAMORADI, A. ALSAEDI, B. AHMAD, Y. ZOU, *Multiplicity of homoclinic solutions for fractional Hamiltonian systems with subquadratic potential*, Entropy **2017**, 19, 50, 1–24.

[3] N. NYAMORADI, A. ALSAEDI, B. AHMAD, Y. ZOU, *Variational approach to homoclinic solutions for fractional Hamiltonian systems*, J. Optim. Theory Appl. **2017**.

[4] Z. BAI, H. LÜ, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. (2005), **311**, 495–505.

[5] Z. BAI, Y. ZHANG, *The existence of solutions for a fractional multi-point boundary value problem*, Computers and Mathematics with Applications **2010**, 69, 2364–2372.

[6] P. CHEN, X. HE, X. H. TANG, *Infinitely many solutions for a class of Hamiltonian systems via critical point theory*, Math. Meth. Appl. Sci. **2016**, 39, 1005–1019.

[7] Y. LI, B. DAI, *Existence and multiplicity of nontrivial solutions for Liouville-Weyl fractional nonlinear Schrödinger equation*, RA SAM (2017).

[8] W. JIANG, *The existence of solutions for boundary value problems of fractional differential equations at resonance*, Nonlinear Analysis (2011), **74**, 1987–1994.

[9] F. JIAO, Y. ZHOU, *Existence results for fractional boundary value problem via critical point theory*, Intern. Journal of Bif. and Chaos **22**, no. 4 (2012), 1–17.

[10] R. HIFFER, *Applications of fractional calculus in physics*, World Science, Singapore, 2000.

[11] S. G. SAMKO, A. A KILBAS, O. I. MARICHEV, *Fractional integrals and derivatives, Theory and applications*, Gordon and Breach, Switzerland 1993.

[12] A. A. KILBAS, H. M. SRIVASTAWA, J. J. TRUJILLO, *Theory and applications of fractional differential equations*, North-Holland Mathematical Studies, vol. 204, Singapore 2006.

[13] S. LIANG, J. ZHANG, *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Nonlinear Analysis, 2009, 71, 5545-5550.

[14] J. MAWHIN, M. WILLEM, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, Springer, Berlin, 1989:

[15] A. MÈNDEZ, C. TORRES, *Multiplicity of solutionsfor a fractional Hamiltonian systems with Liouville-Weyl fractional derivative*, `arXiv:1409.0765v1` [math-ph] 2 Sep. 2014.

[16] K. MILLER, B. ROSS, *An introduction to differential equations*, Wiley and Sons, New York, 1993.

[17] I. POLLUBNY, *Fractional differential equations*, Academic Press, 1999.

[18] P. H. RABINOWITZ, *Minimax methods in critical point theory with applications to differential equations*, CBMS Amer. Math. Soc., no. 65, 1986.
[19] K. TANG, *Multiple homoclinic vsolutions for a class of fractional Hamiltonian systems* Progr. Fract. Diff. Appl. **2**, no. 4 (2016), 265–276.
[20] C. TORRES, *Existence of solutionsfor fractional Hamiltonian systems*, Electr. J. DIff. Eq., vol. 2013 (2013), no. 259, 1–12.
[21] C. TORRES LEDESMA, *Existence of solutions for fractional Hamiltonian systems with nonlinear derivative dependence in* $\mathbb{R}$ J. Fractional Calculus and Applications, vol. 7 (2) (2016) 74–87.
[22] X. WU, Z. ZHANG, *Solutions for perturbed fractional Hamiltonian systems without coercive conditions*, Boundary Value Problems (2015) 2015: 149, 1–12.
[23] S. ZHANG, *Existence of solutions for the fractional equations with nonlinear boundary conditions*, Computers and Mathematics with Applications (2011), **61**, 1202–1208.
[24] S. ZHANG, *Existence of solutions for a boundary value problems of fractional differential equations at resonance*, Nonlinear Analysis (2011), **74**, 1987–1994.
[25] Z. ZHANG, R. YUAN, *Existence of solutions to fractional Hamiltonian systems with combined nonlinearities*, Electr. J. Diff. Eq., vol. 2016 (2016) no. 40, 1–13.
[26] Z. ZHANG, R. YUAN, *Solutions for subquadratic fractional Hamiltonian systems without coercive conditions*, Math. Meth. Appl. Sci. (2014) **37**, 2934–2945.
[27] Z. ZHANG, R. YUAN, *Variational approach to solutions for a class of fractional Hamiltonian systems*, Math. Meth. Appl. Sci. (2014) **37**, 1873–1883.
[28] W. ZOU, *Variant fountain theorems and their applications*, Manuscripta Math., vol. 104, issue 3 (2001) 343–358.

(Received October 27, 2017)

Mohsen Timoumi
Faculty of Sciences
5000 Monastir, Tunisia
e-mail: m_timoumi@yahoo.com