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SOME NEW HERMITE–HADAMARD TYPE INEQUALITIES VIA CAPUTO k–FRACTIONAL DERIVATIVES CONCERNING $(n+1)$–DIFFERENTIABLE GENERALIZED RELATIVE SEMI–$(r;m,h_1,h_2)$–PREINVEX MAPPINGS

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Abstract. In this article, we first presented a new identity concerning $(n+1)$-differentiable mappings defined on m-invex set via Caputo k-fractional derivatives. By using the notion of generalized relative semi-$(r;m,h_1,h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via Caputo k-fractional derivatives are established. It is pointed out that some new special cases can be deduced from main results of the article.

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R}=(-\infty,+\infty)$ and I° to denote the interior of I. For any subset $K\subseteq\mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

THEOREM 1. *Let $f: I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a convex function on I and $a,b\in I$ with $a<b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}. \tag{1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions interested readers are referred to [[1]–[8], [11], [12], [14], [17]–[21], [25], [26]].

Let us recall some special functions and evoke some basic definitions as follows.

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DEFINITION 1. The Euler beta function is defined for $a,b>0$ as

$$\beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

DEFINITION 2. The hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

for $c>b>0$ and $|z|<1$, where $\beta(x,y)$ is the Euler beta function for all $x,y>0$.

DEFINITION 3. For $k\in\mathbb{R}^+$ and $x\in\mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x)=\lim_{n\longrightarrow\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{2}$$

Its integral representation is given by

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt. \tag{3}$$

One can note that

$$\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha).$$

For $k=1$, (3) gives integral representation of gamma function.

DEFINITION 4. For $k\in\mathbb{R}^+$ and $x,y\in\mathbb{C}$, the k-beta function with two parameters x and y is defined as

$$\beta_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{4}$$

For $k=1$, (4) gives integral representation of beta function.

THEOREM 2. *Let $x,y>0$, then for k-gamma and k-beta function the following equality holds:*

$$\beta_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{5}$$

DEFINITION 5. [9] Let $\alpha>0$ and $\alpha\notin\{1,2,3,\dots\}$, $n=[\alpha]+1$, $f\in C^n[a,b]$ such that $f^{(n)}$ exists and are continuous on $[a,b]$. The Caputo fractional derivatives of order α are defined as follows:

$${}^cD^\alpha_{a+}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt,\ \ x>a \tag{6}$$

and

$${}^cD^\alpha_{b-}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b\frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt,\ \ x<b. \tag{7}$$

If $\alpha = n \in \{1,2,3,\ldots\}$ and usual derivative of order n exists, then Caputo fractional derivative $\left({}^cD^{\alpha}_{a+}f\right)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$\left({}^cD^{0}_{a+}f\right)(x) = \left({}^cD^{0}_{b-}f\right)(x) = f(x) \tag{8}$$

where $n = 1$ and $\alpha = 0$.

DEFINITION 6. [6] Let $\alpha > 0$, $k \geqslant 1$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$, $f \in C^n[a,b]$. The Caputo k-fractional derivatives of order α are defined as follows:

$$ {}^cD^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt, \quad x > a \tag{9}$$

and

$$ {}^cD^{\alpha,k}_{b-}f(x) = \frac{(-1)^n}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt, \quad x < b. \tag{10}$$

DEFINITION 7. [24] A set $M_\varphi \subseteq \mathbb{R}^n$ is named as a relative convex (φ-convex) set, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \ \forall\, x,y \in \mathbb{R}^n : \varphi(x),\varphi(y) \in M_\varphi, t \in [0,1]. \tag{11}$$

DEFINITION 8. [24] A function f is named as a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leqslant tf(\varphi(x)) + (1-t)f(\varphi(y)), \tag{12}$$

$\forall\, x,y \in \mathbb{R}^n : \varphi(x),\varphi(y) \in M_\varphi, t \in [0,1]$.

DEFINITION 9. [4] A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow [0,+\infty)$ is said to be P-function, if

$$f(tx + (1-t)y) \leqslant f(x) + f(y), \quad \forall x,y \in I, \quad t \in [0,1].$$

DEFINITION 10. [13] Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \leqslant h(1-t)f(x) + h(t)f(y) \tag{13}$$

for each $x,y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 10, f becomes a preinvex function [16]. If the mapping $\eta(y,x) = y - x$ in Definition 10, then the non-negative function f reduces to h-convex mappings [23].

DEFINITION 11. [22] Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, a function $f : K \longrightarrow \mathbb{R}$ is said to be a tgs-convex function on K if the inequality

$$f\big((1-t)x+ty\big) \leqslant t(1-t)[f(x)+f(y)] \tag{14}$$

grips for all $x,y \in K$ and $t \in (0,1)$.

DEFINITION 12. [10] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if it is non-negative and $\forall\, x,y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx+(1-t)y) \leqslant \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{15}$$

DEFINITION 13. [14] A function: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall x,y \in I$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$f\big(tx+m(1-t)y\big) \leqslant \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{16}$$

DEFINITION 14. [15] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m,h_1,h_2)-preinvex with respect to η, if

$$f\big(mx+t\eta(y,x,m)\big) \leqslant mh_1(t)f(x)+h_2(t)f(y) \tag{17}$$

is valid for all $x,y \in K$ and $t \in [0,1]$, with some fixed $m \in (0,1]$.

Motivated by the above literatures, the main objective of this article is to establish some new estimates on Hermite-Hadamard type inequalities via Caputo k-fractional derivatives associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article.

2. Main results

The following definitions will be used in this section.

DEFINITION 15. [5] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx+t\eta(y,x,m) \in K$ holds for each $x,y \in K$ and any $t \in [0,1]$.

REMARK 1. In Definition 15, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m=1$, then the m-invex set degenerates to an invex set on K.

For the simplicities of notations, let

$$\delta(\alpha,\xi)=\int_0^1 |t^\alpha-\xi|dt, \quad \rho(\alpha,\xi,p)=\int_0^1 |t^\alpha-\xi|^p dt. \tag{18}$$

LEMMA 1. *For* $0\leqslant\xi\leqslant 1$, *we have*

(a)

$$\delta(\alpha,\xi):=\begin{cases}\dfrac{1}{\alpha+1}, & \xi=0;\\ \dfrac{2\alpha\xi^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\xi, & 0<\xi<1;\\ \dfrac{\alpha}{\alpha+1}, & \xi=1.\end{cases}$$

(b)

$$\rho(\alpha,\xi,p):=\begin{cases}\dfrac{1}{p\alpha+1}, & \xi=0;\\ \dfrac{\xi^{p+\frac{1}{\alpha}}}{\alpha}\beta\left(\dfrac{1}{\alpha},p+1\right)+\dfrac{(1-\xi)^{p+1}}{\alpha(p+1)} & \\ \times {}_2F_1\left(1-\dfrac{1}{\alpha},1;p+2;1-\xi\right), & 0<\xi<1;\\ \dfrac{1}{\alpha}\beta\left(p+1,\dfrac{1}{\alpha}\right), & \xi=1.\end{cases}$$

Proof. These equalities follow from a straightfoward computation of definite integrals. This completes the proof of the lemma. □

We next introduce a new class called generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings.

DEFINITION 16. Let $K\subseteq\mathbb{R}$ be an open m-invex set with respect to the mapping $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi: I\longrightarrow K$ are continuous. A mapping $f: K\longrightarrow(0,+\infty)$ is said to be generalized relative semi-$(r;m,h_1,h_2)$-preinvex, if

$$f\big(m\varphi(x)+t\eta(\varphi(y),\varphi(x),m)\big)\leqslant M_r(h_1(t),h_2(t);mf(x),f(y)) \tag{19}$$

holds for all $x,y\in I$ and $t\in[0,1]$, with some fixed $m\in(0,1]$, where

$$M_r(h_1(t),h_2(t);mf(x),f(y)):=\begin{cases}\Big[mh_1(t)f^r(x)+h_2(t)f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r\neq 0;\\ \big[mf(x)\big]^{h_1(t)}\big[f(y)\big]^{h_2(t)}, & \text{if } r=0,\end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

REMARK 2. In Definition 16, if we choose $r = 1$ and $\varphi(x) = x$, then we get Definition 14.

REMARK 3. For $r = 1$, let us discuss some special cases in Definition 16 as follows.

(I) If taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(m,s)-$ Breckner-preinvex mappings.

(II) If taking $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m,P)-preinvex mappings.

(III) If taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings.

(IV) If taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-(m,h)-preinvex mappings.

(V) If taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex mappings.

(VI) If taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

For establishing our main results regarding some new Hermite-Hadamard type integral inequalities associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvexity via Caputo k-fractional derivatives, we need the following lemma.

LEMMA 2. *Let $\alpha > 0$, $k \geqslant 1$, and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$. Also, let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K \longrightarrow \mathbb{R}$ is a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m) > 0$. Then for any $\lambda,\mu \in [0,1]$ and $r \geqslant 0$, we have the following equality for Caputo k-fractional derivatives:*

$$\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)^{n-\frac{\alpha}{k}} \left\{\lambda\left[f^{(n)}(m\varphi(a)) - f^{(n)}\left(m\varphi(a)+\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)\right]\right.$$
$$+\mu\left[f^{(n)}\left(m\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\varphi(a),m)\right) - f^{(n)}(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right]$$
$$\left.+f^{(n)}\left(m\varphi(a)+\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)+f^{(n)}(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right\}$$

$$-(nk-\alpha)\Gamma_k\left(n-\frac{\alpha}{k}\right)\times\left[{}^cD^{\alpha,k}_{(m\varphi(a))+}f\left(m\varphi(a)+\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)\right.$$
$$\left.+(-1)^n\times{}^cD^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-}f\left(m\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\varphi(a),m)\right)\right]$$
$$=\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)^{n-\frac{\alpha}{k}+1}$$
$$\times\left\{\int_0^1\left(1-\lambda-t^{n-\frac{\alpha}{k}}\right)f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)dt\right.$$
$$\left.+\int_0^1\left(t^{n-\frac{\alpha}{k}}-\mu\right)f^{(n+1)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)dt\right\}. \tag{20}$$

We denote

$$I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r,m,a,b)$$
$$:=\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)^{n-\frac{\alpha}{k}+1}$$
$$\times\left\{\int_0^1\left(1-\lambda-t^{n-\frac{\alpha}{k}}\right)f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)dt\right.$$
$$\left.+\int_0^1\left(t^{n-\frac{\alpha}{k}}-\mu\right)f^{(n+1)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)dt\right\}. \tag{21}$$

Proof. Integrating by parts (21), we get

$$I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r,m,a,b)$$
$$=\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)^{n-\frac{\alpha}{k}+1}$$
$$\times\left\{\left[\frac{-(r+1)\left(1-\lambda-t^{n-\frac{\alpha}{k}}\right)f^{(n)}\left(m\varphi(a)+\left(\frac{1-t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)}{\eta(\varphi(b),\varphi(a),m)}\right|_0^1\right.$$
$$\left.-\frac{(r+1)\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(m\varphi(a)+\left(\frac{1-t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)dt\right]$$
$$+\left[\frac{(r+1)\left(t^{n-\frac{\alpha}{k}}-\mu\right)f^{(n)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)}{\eta(\varphi(b),\varphi(a),m)}\right|_0^1$$
$$\left.\left.-\frac{(r+1)\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(b),\varphi(a),m)\right)dt\right]\right\}$$

$$= \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)^{n-\frac{\alpha}{k}} \times \Bigg\{ \lambda \left[f^{(n)}(m\varphi(a)) - f^{(n)}\left(m\varphi(a) + \frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)\right]$$
$$+\mu\left[f^{(n)}\left(m\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\varphi(a),m)\right)-f^{(n)}(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right]$$
$$+f^{(n)}\left(m\varphi(a)+\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)+f^{(n)}(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\Bigg\}$$
$$-(nk-\alpha)\Gamma_k\left(n-\frac{\alpha}{k}\right)\times\left[{}^cD^{\alpha,k}_{(m\varphi(a))+}f\left(m\varphi(a)+\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)\right.$$
$$\left.+(-1)^n\times{}^cD^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-}f\left(m\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\varphi(a),m)\right)\right].$$

This completes the proof of the lemma. □

Using Lemmas 1 and 2, we now state the following theorems for the corresponding version for power of $(n+1)$-derivative.

THEOREM 3. *Let $\alpha > 0$, $k \geqslant 1$, $0 < r \leqslant 1$, and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$. Also, let $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous. Assume that $f : K \longrightarrow (0,+\infty)$ is a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m) > 0$. If $\left(f^{(n+1)}\right)^q$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex functions, $q > 1$, $p^{-1}+q^{-1} = 1$, then for any $\lambda, \mu \in [0,1]$ and $r_1 \geqslant 0$, the following inequality for Caputo k-fractional derivatives holds:*

$$\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|$$
$$\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \Bigg\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)$$
$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1)+\left(f^{(n+1)}(b)\right)^{rq}I^r(h_2(t);r,r_1)\right]^{\frac{1}{rq}}$$
$$+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)$$
$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^r(h_1(t);r,r_1)+\left(f^{(n+1)}(b)\right)^{rq}\overline{I}^r(h_2(t);r,r_1)\right]^{\frac{1}{rq}}\Bigg\}, \qquad (22)$$

where

$$I(h_i(t);r,r_1) := \int_0^1 h_i^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt, \quad \overline{I}(h_i(t);r,r_1) := \int_0^1 h_i^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt, \ \forall i=1,2,$$

and $\rho\left(n-\frac{\alpha}{k},1-\lambda,p\right)$, $\rho\left(n-\frac{\alpha}{k},\mu,p\right)$ are defined as in Lemma 1.

Proof. Suppose that $q>1$, $r_1\geqslant 0$ and $0<r\leqslant 1$. From Lemmas 1 and 2, generalized relative semi-$(r;m,h_1,h_2)$-preinvexity of $\left(f^{(n+1)}\right)^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$
\begin{aligned}
&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b))\right| \\
&\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \\
&\quad\times\left\{\int_0^1\left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left|t^{n-\frac{\alpha}{k}}-\mu\right|\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right|dt\right\} \\
&\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\left(\int_0^1\left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\int_0^1\left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right)^q dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\left|t^{n-\frac{\alpha}{k}}-\mu\right|^p dt\right)^{\frac{1}{p}} \\
&\quad\left.\times\left(\int_0^1\left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right)^q dt\right)^{\frac{1}{q}}\right\} \\
&\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)\right. \\
&\quad\times\left(\int_0^1\left[mh_1\left(\frac{1-t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}+h_2\left(\frac{1-t}{r_1+1}\right)\left(f^{(n+1)}(b)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&\quad+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right) \\
&\quad\left.\times\left(\int_0^1\left[mh_1\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}+h_2\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(b)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}\right\} \\
&\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)\right. \\
&\quad\times\left[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q h_1^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r\right. \\
&\quad\left.+\left(\int_0^1\left(f^{(n+1)}(b)\right)^q h_2^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r\right]^{\frac{1}{rq}}
\end{aligned}
$$

$$+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\times\left[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q h_1^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1\left(f^{(n+1)}(b)\right)^q h_2^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right]^{\frac{1}{rq}}\Bigg\}$$

$$=\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)$$

$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1)+\left(f^{(n+1)}(b)\right)^{rq}I^r(h_2(t);r,r_1)\right]^{\frac{1}{rq}}$$

$$+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)$$

$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^r(h_1(t);r,r_1)+\left(f^{(n+1)}(b)\right)^{rq}\overline{I}^r(h_2(t);r,r_1)\right]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 3.

COROLLARY 1. *In Theorem 3 if we choose $\lambda=\mu=m=r=1$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$, we get the following inequality for Caputo k-fractional derivatives:*

$$\left|I_f(1,1;\alpha,k,n,r_1,1,a,b)\right|$$

$$\leqslant\left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^q I(h_1(t);1,r_1)+\left(f^{(n+1)}(b)\right)^q I(h_2(t);1,r_1)\right]^{\frac{1}{q}}$$

$$+\left[\frac{1}{\left(n-\frac{\alpha}{k}\right)}\beta\left(p+1,\frac{1}{n-\frac{\alpha}{k}}\right)\right]^{\frac{1}{p}}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^q\overline{I}(h_1(t);1,r_1)+\left(f^{(n+1)}(b)\right)^q\overline{I}(h_2(t);1,r_1)\right]^{\frac{1}{q}}\Bigg\}. \tag{23}$$

COROLLARY 2. *In Theorem 3 if we choose $\lambda=\mu=0$, $m=r=1$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$, we get the following inequality for Caputo k-fractional derivatives:*

$$\left|I_f(0,0;\alpha,k,n,r_1,1,a,b)\right|$$

$$\leqslant\left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\left[\frac{1}{\left(n-\frac{\alpha}{k}\right)}\beta\left(p+1,\frac{1}{n-\frac{\alpha}{k}}\right)\right]^{\frac{1}{p}}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^q I(h_1(t);1,r_1)+\left(f^{(n+1)}(b)\right)^q I(h_2(t);1,r_1)\right]^{\frac{1}{q}}$$
$$+\left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(f^{(n+1)}(a)\right)^q \overline{I}(h_1(t);1,r_1)+\left(f^{(n+1)}(b)\right)^q \overline{I}(h_2(t);1,r_1)\right]^{\frac{1}{q}}\Bigg\}. \tag{24}$$

COROLLARY 3. *In Theorem 3 for* $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ *and* $f^{(n+1)}(x)\leqslant K$, $\forall x\in I$, *we get the following inequality for generalized relative semi-*$(r;m,h)$*-preinvex mappings via Caputo* k*-fractional derivatives:*

$$\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|$$
$$\leqslant K(m+1)^{\frac{1}{rq}}\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}$$
$$\times\left\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)I^{\frac{1}{q}}(h(t);r,r_1)+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\overline{I}^{\frac{1}{q}}(h(t);r,r_1)\right\}. \tag{25}$$

COROLLARY 4. *In Corollary 3 for* $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, *we get the following inequality for generalized relative semi-*$(r;m,s)$*-Breckner-preinvex mappings via Caputo* k*-fractional derivatives:*

$$\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|$$
$$\leqslant K(m+1)^{\frac{1}{rq}}\left(\frac{r}{r+s}\right)^{\frac{1}{q}}\left(\frac{1}{r_1+1}\right)^{\frac{s}{rq}}\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}$$
$$\times\left\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\left[(r_1+1)^{\frac{s}{r}+1}-r_1^{\frac{s}{r}+1}\right]^{\frac{1}{q}}\right\}. \tag{26}$$

COROLLARY 5. *In Corollary 3 for* $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ *and* $0<s<r$, *we get the following inequality for generalized relative semi-*$(r;m,s)$*-Godunova-Levin-Dragomir-preinvex mappings via Caputo* k*-fractional derivatives:*

$$\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|$$
$$\leqslant K(m+1)^{\frac{1}{rq}}\left(\frac{r}{r-s}\right)^{\frac{1}{q}}\left(\frac{1}{r_1+1}\right)^{-\frac{s}{rq}}\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}$$
$$\times\left\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\left[(r_1+1)^{1-\frac{s}{r}}-r_1^{1-\frac{s}{r}}\right]^{\frac{1}{q}}\right\}. \tag{27}$$

COROLLARY 6. *In Theorem 3 for $r=1$, $h_1(t)=h_2(t)=t(1-t)$ and $f^{(n+1)}(x)\leqslant K$, $\forall x\in I$, we get the following inequality for generalized relative semi-(m,tgs)-preinvex mappings via Caputo k-fractional derivatives:*

$$\begin{aligned}&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|\\ &\leqslant K(m+1)^{\frac{1}{q}}\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\left(\frac{3r_1+1}{6(r_1+1)^2}\right)^{\frac{1}{q}}\\ &\times\left\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\right\}.\end{aligned}\tag{28}$$

COROLLARY 7. *In Theorem 3 for $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $f^{(n+1)}(x)\leqslant K$, $\forall x\in I$, we get the following inequality for generalized relative semi-$(r;m)$-MT-preinvex mappings via Caputo k-fractional derivatives:*

$$\begin{aligned}&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|\\ &\leqslant K(m+1)^{\frac{1}{rq}}\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\\ &\times\left\{\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)I^{\frac{1}{q}}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1\right)\right.\\ &\left.+\rho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\overline{I}^{\frac{1}{q}}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1\right)\right\}.\end{aligned}\tag{29}$$

THEOREM 4. *Let $\alpha>0$, $k\geqslant 1$, $0<r\leqslant 1$, and $\alpha\notin\{1,2,3,\dots\}$, $n=[\alpha]+1$. Suppose $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$. Also, let $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous. Assume that $f:K\longrightarrow(0,+\infty)$ is a function on K° such that $f\in C^{n+1}[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m)>0$. If $\left(f^{(n+1)}\right)^q$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex functions, $q\geqslant 1$, then for any $\lambda,\mu\in[0,1]$ and $r_1\geqslant 0$, the following inequality for Caputo k-fractional derivatives holds:*

$$\begin{aligned}&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|\\ &\leqslant\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right.\\ &\times\left[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1,\lambda,\alpha,k,n)+\left(f^{(n+1)}(b)\right)^{rq}I^r(h_2(t);r,r_1,\lambda,\alpha,k,n)\right]^{\frac{1}{rq}}\\ &+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\left[m\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^r(h_1(t);r,r_1,\mu,\alpha,k,n)\right.\\ &\left.\left.+\left(f^{(n+1)}(b)\right)^{rq}\overline{I}^r(h_2(t);r,r_1,\mu,\alpha,k,n)\right]^{\frac{1}{rq}}\right\},\end{aligned}\tag{30}$$

where

$$I(h_i(t);r,r_1,\lambda,\alpha,k,n) := \int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right| h_i^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt,\ \ \forall i=1,2,$$
$$\overline{I}(h_i(t);r,r_1,\mu,\alpha,k,n) := \int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right| h_i^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt,\ \ \forall i=1,2,$$

and $\delta\left(n-\frac{\alpha}{k},1-\lambda\right),\quad \delta\left(n-\frac{\alpha}{k},\mu\right)$ *are defined as in Lemma 1.*

Proof. Suppose that $q \geqslant 1$, $r_1 \geqslant 0$ and $0 < r \leqslant 1$. From Lemmas 1 and 2, generalized relative semi-$(r;m,h_1,h_2)$-preinvexity of $\left(f^{(n+1)}\right)^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b))\right| \\
&\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \\
&\times\left\{\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right|dt\right. \\
&\left.+\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right|\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right|dt\right\} \\
&\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\left(\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|\left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right)^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right|dt\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right|\left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(b),\varphi(a),m)\right)\right)^q dt\right)^{\frac{1}{q}}\right\} \\
&\leqslant \left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right. \\
&\times\left(\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|\left[mh_1\left(\frac{1-t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}\right.\right. \\
&\left.\left.+h_2\left(\frac{1-t}{r_1+1}\right)\left(f^{(n+1)}(b)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)
\end{aligned}$$

$$\times\left(\int_0^1\left|t^{n-\frac{\alpha}{k}}-\mu\right|\left[mh_1\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}+h_2\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(b)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leqslant\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)$$

$$\times\Bigg[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q\left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|h_1^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1\left(f^{(n+1)}(b)\right)^q\left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|h_2^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\Bigg[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q\left|t^{n-\frac{\alpha}{k}}-\mu\right|h_1^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1\left(f^{(n+1)}(b)\right)^q\left|t^{n-\frac{\alpha}{k}}-\mu\right|h_2^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}$$

$$=\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)$$

$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1,\lambda,\alpha,k,n)+\left(f^{(n+1)}(b)\right)^{rq}I^r(h_2(t);r,r_1,\lambda,\alpha,k,n)\right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\Bigg[m\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^r(h_1(t);r,r_1,\mu,\alpha,k,n)$$

$$+\left(f^{(n+1)}(b)\right)^{rq}\overline{I}^r(h_2(t);r,r_1,\mu,\alpha,k,n)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 4.

COROLLARY 8. *In Theorem 4 if we choose $\lambda=\mu=m=r=1$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$, we get the following inequality for Caputo k-fractional derivatives:*

$$\left|I_f(1,1;\alpha,k,n,r_1,1,a,b)\right|$$

$$\leqslant\left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^qI(h_1(t);1,r_1,1,\alpha,k,n)+\left(f^{(n+1)}(b)\right)^qI(h_2(t);1,r_1,1,\alpha,k,n)\right]^{\frac{1}{q}}$$

$$+\left(\frac{n-\frac{\alpha}{k}}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\times\left[\left(f^{(n+1)}(a)\right)^{q}\overline{I}(h_1(t);1,r_1,1,\alpha,k,n)\right.$$
$$\left.+\left(f^{(n+1)}(b)\right)^{q}\overline{I}(h_2(t);1,r_1,1,\alpha,k,n)\right]^{\frac{1}{q}}\Bigg\}. \tag{31}$$

COROLLARY 9. *In Theorem 4 if we choose $\lambda=\mu=0$, $m=r=1$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$, we get the following inequality for Caputo k-fractional derivatives:*

$$\left|I_f(0,0;\alpha,k,n,r_1,1,a,b)\right|$$
$$\leqslant\left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\left(\frac{n-\frac{\alpha}{k}}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(f^{(n+1)}(a)\right)^{q}I(h_1(t);1,r_1,0,\alpha,k,n)+\left(f^{(n+1)}(b)\right)^{q}I(h_2(t);1,r_1,0,\alpha,k,n)\right]^{\frac{1}{q}}$$
$$+\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\times\left[\left(f^{(n+1)}(a)\right)^{q}\overline{I}(h_1(t);1,r_1,0,\alpha,k,n)\right.$$
$$\left.+\left(f^{(n+1)}(b)\right)^{q}\overline{I}(h_2(t);1,r_1,0,\alpha,k,n)\right]^{\frac{1}{q}}\Bigg\}. \tag{32}$$

COROLLARY 10. *In Theorem 4 for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $f^{(n+1)}(x)\leqslant K$, $\forall x\in I$, we get the following inequality for generalized relative semi-$(r;m,h)$-preinvex mappings via Caputo k-fractional derivatives:*

$$\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right|$$
$$\leqslant K\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)$$
$$\times\left[mI^r(h(1-t);r,r_1,\lambda,\alpha,k,n)+I^r(h(t);r,r_1,\lambda,\alpha,k,n)\right]^{\frac{1}{rq}}$$
$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)$$
$$\times\left[m\overline{I}^r(h(1-t);r,r_1,\mu,\alpha,k,n)+\overline{I}^r(h(t);r,r_1,\mu,\alpha,k,n)\right]^{\frac{1}{rq}}\Bigg\}. \tag{33}$$

COROLLARY 11. *In Corollary 10 for $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, we get the following inequality for generalized relative semi-$(r;m,s)$-Breckner-preinvex mappings* *via Caputo k-fractional derivatives:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right| \\
&\leqslant K\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right. \\
&\quad\times\left[mI^r((1-t)^s;r,r_1,\lambda,\alpha,k,n)+I^r(t^s;r,r_1,\lambda,\alpha,k,n)\right]^{\frac{1}{rq}} \\
&\quad+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right) \\
&\quad\left.\times\left[m\overline{I}^r((1-t)^s;r,r_1,\mu,\alpha,k,n)+\overline{I}^r(t^s;r,r_1,\mu,\alpha,k,n)\right]^{\frac{1}{rq}}\right\}. \qquad (34)
\end{aligned}$$

COROLLARY 12. *In Corollary 10 for $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$, we get the following inequality for generalized relative semi-$(r;m,s)$-Godunova-Levin-Dragomir-preinvex mappings via Caputo k-fractional derivatives:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right| \\
&\leqslant K\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right. \\
&\quad\times\left[mI^r((1-t)^{-s};r,r_1,\lambda,\alpha,k,n)+I^r(t^{-s};r,r_1,\lambda,\alpha,k,n)\right]^{\frac{1}{rq}} \\
&\quad+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right) \\
&\quad\left.\times\left[m\overline{I}^r((1-t)^{-s};r,r_1,\mu,\alpha,k,n)+\overline{I}^r(t^{-s};r,r_1,\mu,\alpha,k,n)\right]^{\frac{1}{rq}}\right\}. \qquad (35)
\end{aligned}$$

COROLLARY 13. *In Theorem 4 for $h_1(t)=h_2(t)=t(1-t)$ and $f^{(n+1)}(x)\leqslant K$, $\forall x\in I$, we get the following inequality for generalized relative semi-$(r;m,tgs)$-preinvex mappings via Caputo k-fractional derivatives:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right| \\
&\leqslant K(m+1)^{\frac{1}{rq}}\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \\
&\quad\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)I^{\frac{1}{q}}(t(1-t);r,r_1,\lambda,\alpha,k,n)\right. \\
&\quad\left.+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\overline{I}^{\frac{1}{q}}(t(1-t);r,r_1,\mu,\alpha,k,n)\right\}. \qquad (36)
\end{aligned}$$

COROLLARY 14. *In Theorem 4 for* $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $f^{(n+1)}(x) \leqslant K$, $\forall x \in I$, *we get the following inequality for generalized relative semi-*$(r;m)$*-MT-preinvex mappings via Caputo* k*-fractional derivatives:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi}(\lambda,\mu;\alpha,k,n,r_1,m,a,b)\right| \\
&\leqslant K\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right. \\
&\times\left[mI^r\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,r_1,\lambda,\alpha,k,n\right)+I^r\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1,\lambda,\alpha,k,n\right)\right]^{\frac{1}{rq}} \\
&+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right) \\
&\left.\times\left[m\overline{I}^r\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,r_1,\mu,\alpha,k,n\right)+\overline{I}^r\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1,\mu,\alpha,k,n\right)\right]^{\frac{1}{rq}}\right\}. \quad (37)
\end{aligned}$$

REMARK 4. For $k = 1$, by our Theorems 3 and 4, we can get some new special Hermite-Hadamard type inequalities associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings via Caputo fractional derivatives of order α.

REMARK 5. Also, applying our Theorems 3 and 4, we can deduce some new inequalities using special means associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings.

3. Conclusions

In this article, we first presented a new identity concerning $(n+1)$-differentiable mappings defined on m-invex set via Caputo k-fractional derivatives is derived. By using the notion of generalized relative semi-$(r;m,h_1,h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via Caputo k-fractional derivatives are established. It is pointed out that some new special cases are deduced from main results of the article. Motivated by this new interesting class of generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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