APPROXIMATION OF TIME FRACTIONAL BLACK–SCHOLES EQUATION VIA RADIAL KERNELS AND TRANSFORMATIONS

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Abstract. In the present work, a numerical scheme is constructed for approximation of time fractional Black-Scholes model governing European options. The present numerical scheme has the capability to overcome spurious oscillation in the case of volatility. In the present numerical method, the Laplace transform, radial kernels and quadrature rule are used. The time variable is eliminated by the use of Laplace transform which significantly reduced the computational cost as compared to the time-marching schemes. The spatial operator is discretized using radial kernels in the local setting which results in sparse differentiation matrices. By Laplace transform the solution is represented as integral along a smooth contour in the complex plane which is then evaluated by quadrature. The proposed numerical scheme is used to price several different European options.

1. Introduction

An option is an important financial derivative and to price an option is the most significant problem in the financial market. The Black-Scholes model is used for evaluating European or American call and put options on a paying stock [29]. The Black-Scholes (BS) model was proposed in the year 1973 by the authors Black-Scholes [2]. It is well known that the option prices by the BS model is enough close to the actual prices, yet it has some shortcomings like accurately capturing jumps or movements in the financial markets for narrow time steps [4].

One way to deal high volatility in stokes market, a modeling through stochastic processes of fractional order have been suggested. Although stochastic partial differential equations of fractional order are the generalization of Ito stochastic differential equations, which involves many difficulties to obtain its theoretical solutions. Luckily we have of hand a fractional calculus. This lead the way to define and use a time and space fractional Black-Scholes models for stock exchange dynamics [19]. Differential equations of fractal order have become useful tools for the analysis of fractal dynamics and fractal geometry [15], and with the introduction of fractional order derivatives, the fractional order PDEs have been extensively used in the financial and stochastic models [30]. The use of fractional order PDEs provide stochastic models that better represent

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the statistical nature of financial time series in terms of their statistical self-affine (random fractal) properties, non-Gaussian characteristics and time memory (which are not an inherent characteristic of the classical BS equation).

For the study of an option diffusion process a time fractional Black-Scholes model was investigated in [23]. Wyss studied time fractional BS model to price European call options in [40]. For the study of stock exchange dynamics, time and space fractional Black-Scholes models were investigated in [18, 19]. The exotic options in the markets with the jumps have been studied using a space fractional-order BS models in [5]. In the work [6] the author showed that the European option obeying a partial integro-differential model. Leonenko, et al. [21] used a time-fractional diffusion equation to develop the Black-Scholes formalism. A multi parameter fractional Black-Scholes model was developed by Liang et al. [24] with variable stock price.

In the financial markets with the introduction of fractional order models various types for analytic methods have been developed for the solution of BS model. For example the integral transform methods [40, 19, 24, 8, 9], homotopy analysis and homotopy perturbation methods [14, 20], wavelet methods [16], or separation of variables procedure [9]. The the exact solutions by these methods are mostly in the form of infinite series or convolution of some functions which makes them hard to compute. Some efficient numerical techniques have been developed for solving differential equations of fractional order [25], for example, the finite difference methods [41, 42, 26], the finite element methods [43], the finite volume methods [27], the spectral methods [44, 45], the meshless methods [28, 39, 38], the predictor-corrector methods [10, 5].

In this work we consider the Black-Scholes PDE [46] of fractional order α

$$\partial_t^\alpha u(S,t) + \frac{1}{2}\sigma^2 S^2 \, u_{SS}(S,t) + RS \, u_S(S,t) - R \, u(S,t) = 0, \tag{1}$$

hold for all $(S,t) \in (0,\infty) \times (0,T_e)$.

$$u(0,t) = P(t), \; u(\infty,t) = Q(t), \; u(s,T_e) = v(s), \tag{2}$$

where $\sigma \geqslant 0$ denote the volatility of the stock price S, R is the risk-free rate, T_e is the expiry time, and $0 < \alpha < 1$. It is shown in the work [19], that it is not sufficient to replace time-derivative by a fractional order time derivative in the classical BS model to get the desired results. The fractional Black-Scholes model is an extension of the Black-Scholes model, which displays the long-range dependence observed in empirical data. We will approximate this model by a numerical scheme which is based on Laplace transform, quadrature and radial kernels.

2. Preliminaries

DEFINITION 1. The Caputo derivative of fractional order is given by

$$D_\tau^\alpha v(\tau) = \frac{1}{\Gamma(p-\alpha)} \int_0^\tau \frac{1}{(\tau-s)^{\alpha+1-p}} \frac{d^p}{ds^p} v(s)ds, \tag{3}$$

where, $p - 1 < \alpha < p \in Z^+$.

DEFINITION 2. The Laplace transform in the variable τ of a given function v is defined as

$$\widehat{v}(z) = \mathscr{L}\{v(\tau)\} = \int_0^\infty e^{-z\tau} v(\tau) d\tau, \tag{4}$$

if improper integral converges.

LEMMA 1. *Assume the function $v(\tau)$ is continuous over $0 \leqslant \tau \leqslant \tau_n$, if there are some constants C_1, C_2, with the property,*

$$| e^{-C_2 \tau} v(\tau) | < C_1, \forall \, \tau > \tau_n, \tag{5}$$

then the Laplace transform of $v(\tau)$ exists.

LEMMA 2. *If $v(\tau) \in C^p[0, \infty)$, with $\alpha \in (p-1, p) \in Z^+$, then the Caputo fractional derivative has a Laplace transform given by*

$$\mathscr{L}\{D_\tau^\alpha v(\tau)\}(z) = z^\alpha \widehat{v} - \sum_{i=0}^{p-1} z^{\alpha-i-1} v^{(i)}(0). \tag{6}$$

3. Laplace transformed radial kernel method

The more general form of model (1) is given by

$$\partial_t^\alpha u(\mathbf{x}, t) + \mathscr{L} u(\mathbf{x}, t) = f_1(\mathbf{x}, t), \text{ for } \mathbf{x} \in \Omega \subset \mathbb{R}^d, \text{ and } d \geqslant 1, \tag{7}$$

subject to the initial and boundary conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \ \mathbf{x} \in \Omega, \ \mathscr{B} u(\mathbf{x}, t) = f_2(t), \text{ for } \mathbf{x} \in \partial\Omega, \tag{8}$$

where ∂_t^α is the derivative of fractional order α given as

$$\partial_t^\alpha u(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x}, s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \ \alpha \in (0, 1), \tag{9}$$

where $\mathscr{L}$ and $\mathscr{B}$ are linear spatial differential operators. Let $u(t)$ be a smooth function then its Laplace transform is defined by

$$\mathscr{L}\{u(t)\} = \hat{u}(z) = \int_0^\infty u(t) e^{-zt} dt, \tag{10}$$

the Laplace transform of fractional order derivative is given by

$$\mathscr{L}\{\partial_t^\alpha u(t)\} = \hat{u}(z) z^\alpha - \sum_{i=0}^{k-1} z^{\alpha-1-i} u^{(i)}(0), \ k-1 < \alpha < k \in \mathbb{Z}^+. \tag{11}$$

The Laplace transforms of equations (7)-(8) are given by

$$\mathscr{L}\{\hat{u}(\mathbf{x},z)\} + \left[z^{\alpha}\hat{u}(\mathbf{x},z) - z^{\alpha-1}u_0\right] = \hat{f}_1(\mathbf{x},z), \text{ for } \mathbf{x} \in \Omega, \tag{12}$$

$$\mathscr{B}\{\hat{u}(\mathbf{x},z)\} = \hat{f}_2(z), \text{ for } \mathbf{x} \in \partial\Omega. \tag{13}$$

The transformed system can be represented by

$$[\mathscr{L} + Iz^{\alpha}]\{\hat{u}(\mathbf{x},z)\} = \hat{f}_1(\mathbf{x},z) + z^{\alpha-1}u_0, \text{ for } \mathbf{x} \in \Omega, \tag{14}$$

$$\mathscr{B}\{\hat{u}(\mathbf{x},z)\} = \hat{f}_2(z), \text{ for } \mathbf{x} \in \partial\Omega. \tag{15}$$

Then we need to solve the above system for the transformed solution $\hat{u}$ of the original solution $u(t)$. So if u denote the solution of problem (1) then its Laplace transform $\hat{u}$ gives holomorphic extension in a complex Banach space X

$$\hat{u} : \mathbb{C}\backslash\Sigma_\theta \to X, \tag{16}$$

which lies outside to the acute sector defined by

$$\Sigma_\theta = \{z \in \mathbb{C} : |\arg(-z)| \leqslant \theta\}, \ 0 < \theta < \frac{\pi}{2}. \tag{17}$$

In a sector complement to Σ_θ, and for which the function $\hat{u}(z)$ satisfy the condition

$$\|\hat{u}(z)\| = O(1/|z|).$$

In other words, there exists a constant $M > 0$ such that

$$\|\hat{u}(z)\| \leqslant \frac{M}{|z|}, z \in \mathbb{C}\backslash\Sigma_\theta. \tag{18}$$

The inversion formula corresponding to (10) is read as

$$u(t) = \frac{1}{2\pi\iota} \int_{\Theta} e^{tz}\hat{u}(z) \, dz, t > 0, \tag{19}$$

where the path Θ connects the points $c - \iota\infty$ to $c + \iota\infty$, and is chosen in a way which guarantee evaluation of (19) accurately. Let the parametric form of the path Θ be defined by the mapping $\chi : (a,b) \to \mathbb{C}$, then $u(t)$ may be approximated by discretizing the integral

$$u(t) = \frac{1}{2\pi\iota} \int_a^b e^{t\chi(x)}\hat{u}(\chi(x))\chi'(x) \, dx, \ t > 0. \tag{20}$$

using some quadrature rule. This classic approach is followed in [35, 34, 22], which employed the equal width rule. Thus for given $h > 0$, $N \geqslant 1$, and letting $x_k = hk$, $-N \leqslant k \leqslant N$, then approximation to $u(t)$, $t \geqslant 0$, is given by

$$u_N(t) = \frac{h}{2\pi\iota} \sum_{k=-N}^{N} e^{t\chi(x_k)}\hat{u}(\chi(x_k))\chi'(x_k), \ t > 0. \tag{21}$$

Talbot's classic algorithm for inverting $\hat{u}$, the interval $(-\pi, \pi)$ is used in place of (a,b), while in [34, 22] the interval $(-\infty, +\infty)$ is used for (a,b). This setting is much suited to derive the error bounds of the corresponding truncated trapezoidal rule

$$\int_{-\infty}^{+\infty} q(x)\, dx \simeq h \sum_{k=-N}^{N} q(hk), \ N \geqslant 1, h > 0. \tag{22}$$

From equation (17), for given θ, select $d > 0$ and β such that

$$d < \min\{\beta, \pi/2 - \beta\}, \text{ and } \theta + d + \beta < \pi/2. \tag{23}$$

Define $\zeta(w) = -\sin(\beta + \iota w)$, $w = x + \iota y$, it is shown in work of [22] that the mapping ζ transform the horizontal straight line $\{Im\ w = y, \text{ where } y \in (-d,d)\}$, into a hyperbola with a left branch

$$\left(\frac{Re\ z}{\sin(\beta - y)}\right)^2 - \left(\frac{Im\ z}{\cos(\beta - y)}\right)^2 = 1$$

Therefore, ζ maps the stripe S_d contained in the region $y = \pm d$. Let $\lambda > 0$ be a parameter then we have the mapping

$$z = \sigma(w) = \lambda(1 + \zeta(w)). \tag{24}$$

In view of equation (23), it is clear that χ transforms the stripe S_d into a region which laying outside of the sector defined by Σ_θ. In equation (19), contour Θ which is the left branch of hyperbola and the image of real axis under χ. This gives

$$u(t) = \int_{-\infty}^{+\infty} Q_t(x)\, dx, t > 0,$$

where $Q_t : S_d \to X, t > 0$, is the mapping

$$Q_t(w) = \frac{1}{2\pi\iota} \exp(t\chi(w))\hat{u}(\chi(w))\chi'(w).$$

THEOREM 1. [22] *For fixed* d, β *satisfying* (23) *such that* $\lambda > 0$, *when* $N \geqslant 1$, *and* $h = \frac{\ln N}{N}$, $x_k = kh$, *where* $k \in \mathbb{Z}$, *let the approximate solution be denoted by* $u_N(t)$ *and* $u(t)$ *be the exact solution, and*

$$u_N(t) = h \sum_{k=-N}^{N} Q_t(x_k), t > 0, \tag{25}$$

then

$$\|u_N(t) - u(t)\| \leqslant C_L l(\lambda t \sin(\beta - d))e^{\lambda t}\left(\frac{1}{e^{\frac{2\pi nd}{\ln N}} - 1} + \frac{1}{e^{\frac{\sin \beta \lambda t N}{2}}}\right), \tag{26}$$

$C_L = C_1(\beta, d, M) = \frac{2M}{\pi}\sqrt{(\sin(d+\beta)+1)/(1-\sin(\beta+d))}$, $l(x) = (|\ln(1-e^{-x})| + 1)$, $x > 0$. *which shows,* $\|u_N(t) - u(t)\| = O(e^{\frac{-cN}{\ln N}})$.

In addition to approximate the solution $u(t)$ at different time levels $t_0 \leqslant t \leqslant T$, $T = \wedge t_0$, $t_0 \geqslant 0$, $\wedge \geqslant 1$, the transformed values $U(x_k)$ for $-N \leqslant k \leqslant N$ can be computed in parallel. By selecting the optimal values of parameters in χ, the uniform error can be maintained in interval $[t_0, \wedge t_0]$, with moderate $\wedge \geqslant 1$.

4. Localized kernel approximation

To discretize the transform problem (14)-(15) we used the localized meshless method [36, 37]. Consider the nodal points $\{u(x_i), i = 1, 2, ..., m\}$ corresponding to smooth function $u(x)$, such that $\{x_1, ..., x_m\} \subset \Omega \subset \mathbb{R}^d, d \geqslant 1$. The function $u(x)$ is approximated by local kernel method at $x_i \in \Omega$,

$$u(x_i) = \sum_{x_j \in \Omega_i} a_j^i \psi^i(\|x_i - x_j\|), \tag{27}$$

where, $\mathbf{a}^i = [a_1^i, a_2^i, ..., a_n^i]$ is vector of unknown coefficients, and $r_{ij} = \|x_i - x_j\|$ is the norm between notes x_i and x_j, $\psi(r)$, $r \geqslant 0$ is a radial kernel (radial basis function) and $\Omega_i \subset \Omega$ is a local domain for around each x_i, contains n neighboring nodes around the node x_i. So we have m small size linear systems each of order $n \times n$ given by

$$\begin{pmatrix} u_1^i \\ u_2^i \\ \vdots \\ u_n^i \end{pmatrix} = \begin{pmatrix} \psi_{11}^i & \psi_{12}^i & \cdots & \psi_{1n}^i \\ \psi_{21}^i & \psi_{22}^i & \cdots & \psi_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1}^i & \psi_{n2}^i & \cdots & \psi_{nn}^i \end{pmatrix} \begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_n^i \end{pmatrix}, \; i = 1, 2, ..., m,$$

which can be denoted by

$$\mathbf{u}^i = \Psi^i \mathbf{a}^i, i = 1, 2, ..., m, \tag{28}$$

where $\psi_{kj}^i = \psi^i(\|x_k - x_j\|), x_k, x_j \in \Omega_i$, the matrix Ψ^i is called system matrix.

Similarly apply the operator $\mathscr{L}$ we get

$$\mathscr{L}u(x_i) = \sum_{x_j \in \Omega_i} a_j^i \mathscr{L}\psi^i(\|x_i - x_j\|), \tag{29}$$

In vector form we have,

$$\mathscr{L}u(x_i) = \mathbf{v}^i \cdot \mathbf{a}^i, \tag{30}$$

where $\mathbf{v}^i$ is given by

$$\mathbf{v}^i = \mathscr{L}\psi^i(\|x_i - x_j\|), x_j \in \Omega_i, \tag{31}$$

the unknown coefficients can be eliminated from equation (28)

$$\mathbf{a}^i = (\Psi^i)^{-1}\mathbf{u}^i, \tag{32}$$

putting the values of $\mathbf{a}^i$ in (30) to have,

$$\mathscr{L}u(x_i) = \mathbf{v}^i(\Psi^i)^{-1}\mathbf{u}^i = \mathbf{w}^i\mathbf{u}^i \qquad (33)$$

where,

$$\mathbf{w}^i = \mathbf{v}^i(\Psi^i)^{-1}. \qquad (34)$$

This gives the localized discretized form of the linear operator

$$\mathscr{L}u \equiv \Lambda\mathbf{u}, \qquad (35)$$

here Λ is $m \times m$ differentiation matrix, with n non-zeros values while $m - n$ zeros values in each row, where n denote the nodes local sub-domain Ω_i for each i.

5. Stability

The discretized form of the system (13)-(14) can be represented as

$$\Pi\hat{\mathbf{u}} = \mathbf{g}, \qquad (36)$$

where Π is $m \times m$ differentiation matrix, the stability constant of system (36) is defined by

$$c_s = \sup_{\hat{u}\neq 0} \frac{\|\hat{u}\|}{\|\Pi\hat{u}\|}. \qquad (37)$$

The value c_s is bounded for every discrete norms $\|.\|$ on $\mathbb{R}^m$. Hence we have

$$\|\Pi\|^{-1} \leqslant \frac{\|\hat{u}\|}{\|\Pi\hat{u}\|} \leqslant c_s, \qquad (38)$$

Similarly in case of Pseudoinverse $\Pi^{\dagger}$ of Π, we get

$$\|\Pi^{\dagger}\| = \sup_{v\neq 0} \frac{\|\Pi^{\dagger}v\|}{\|v\|}. \qquad (39)$$

writing

$$\|\Pi^{\dagger}\| \geqslant \sup_{v=\Pi\hat{u}\neq 0} \frac{\|\Pi^{\dagger}\Pi\hat{u}\|}{\|\Pi\hat{u}\|} = \sup_{\hat{u}\neq 0} \frac{\|\hat{u}\|}{\|\Pi\hat{u}\|} = c_s. \qquad (40)$$

Hence equations (38) and (40) give the bounds for stability constant c_s.

6. Numerical examples and application

In this section, the proposed method is applied to approximate time fractional Black-Scholes model. The present numerical scheme is used to price various European options governed by Black-Scholes model of fractional order. Three model problems which are the most interesting problems in the financial markets are considered

to demonstrate the accuracy and convergence of the proposed numerical scheme. We
used the hyperbolic contour as discussed above in the form

$$\sigma(x_k) = \eta + \lambda(1 - \sin(\beta - \iota x_k)) \tag{41}$$

with the following values of optimal parameters which determined the contour σ of
integration.

$$\eta = 2, \beta = 0.3812, \lambda = \frac{\delta r_b N}{\varepsilon T}, x_k = hk,$$

with $\delta = 0.1$, $r_b = 2\pi r$, $r = 0.3431$, $h = \varepsilon/N$, $\varepsilon = \cosh^{-1}(\frac{1}{\delta \tau \sin(\beta)})$, $\tau = t_0/T$,
$t_0 = 0.1$, $T = 5$.

Problem 1. In the first problem we consider fractional order model with the fol-
lowing boundary and initial conditions

$$\partial_t^\alpha u(x,t) = A \frac{\partial^2 u(x,t)}{\partial x^2} + B \frac{\partial u(x,t)}{\partial x} - Cu(x,t) + f(x,t), \tag{42}$$

$$u(x,0) = x^2(1-x), \ u(0,t) = 0, \ u(1,t) = 0, \tag{43}$$

where the source term

$$f = (1-x)x^2 \left(\frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} \right) - \left[A(2-6x) + B(2x - 3x^2) - Cx^2(1-x) \right](t+1)^2,$$

is selected such that exact solution of (42) becomes $u(x,t) = x^2(1-x)(t+1)^2$. Here
we take the parameters values as $\sigma = 0.25$, $R = 0.05$, $\alpha = 0.7$, $A = \frac{1}{2}\sigma^2$, $C = R$ $B = R - A$. The results in the form of L_∞ error norm and estimate error $L_{est} = (e^{\frac{-cN}{\ln N}}, c = 1)$
of the present numerical scheme are compared with the available results shown in Table
1. Here m denotes the number of nodes in Ω, while n is the number of nodes in Ω_i,
ε is a scale factor of the kernel $\psi(r, \varepsilon) = \sqrt{1 + (\varepsilon r)^2}$ and κ is system matrix condi-
tion number. Figure 1 shows error versus quadrature nodes, and numerical solution at
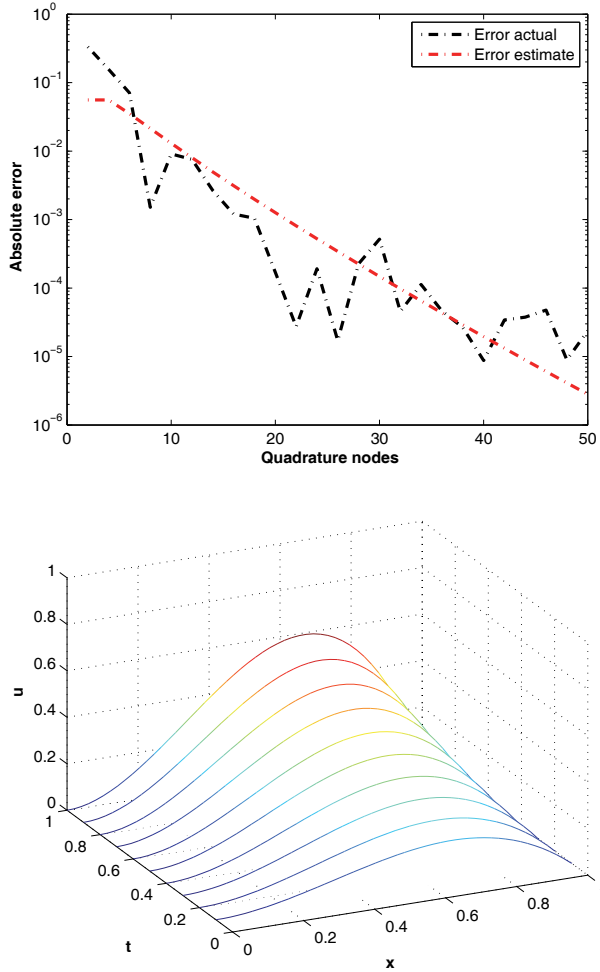different times in $[0,1]$.

Figure 1: Error versus quadrature nodes and numerical solutions at different times of the present numerical method, corresponding to problem 1.

Table 1: Numerical results using Laplace transform-based local kernel method corresponding to model problem 1.

m	n	N	L_∞	L_{est}	ε	κ
50	10	10	0.0091	0.0130	4.5000	1.3509e+012
		30	5.1786e-004	1.4769e-004	4.5000	1.3509e+012
		50	2.3613e-005	2.8134e-006	4.5000	1.3509e+012
20	10	50	4.4704e-005	2.8134e-006	1.7000	2.0443e+012
30			2.6867e-005	2.8134e-006	2.7000	1.0877e+012
60			2.4746e-005	2.8134e-006	5.5000	1.0663e+012
50	10	50	2.3613e-005	2.8134e-006	4.5000	1.3509e+012
	20		4.7345e-005	2.8134e-006	6.2000	1.1457e+012
	30		6.4062e-005	1.8134e-006	6.6000	1.2867e+012
[46](δt=1/160, N=100)			1.0064e-004			

Example 2. In the second problem we consider an other fractional model having the following boundary and initial conditions

$$\partial_t^\alpha u(x,t) = A\frac{\partial^2 u(x,t)}{\partial x^2} + B\frac{\partial u(x,t)}{\partial x} - Cu(x,t) + f(x,t), \tag{44}$$

$$u(x,0) = x^3 + x^2 + 1, \; u(0,t) = (t+1)^2, \; u(1,t) = 3(t+1)^2, \tag{45}$$

where the source term

$$f = (x^3 + x^2 + 1)\left(\frac{2t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right) - \left[A(2+6x) + B(2x+3x^2)\right.$$
$$\left. - C(x^3 + x^2 + 1)\right](t+1)^2,$$

is chosen so that the exact solution of (44) is $u(x,t) = (x^3 + x^2 + 1)(t+1)^2$. Here we take the parameters values as $\alpha = 0.7$, $R = 0.5$, $A = 1$, $C = R$, $B = R - A$. The results in the form of L_∞ error norm and estimated error $L_{est} = (e^{\frac{-cN}{\ln N}}, \; c = 1)$ of the present numerical scheme are compared with the available results shown in Table 2. Here m denotes the number of nodes in the global domain Ω, while n is the number of nodes in the local sub-domain Ω_i, ε is the shape parameter of radial kernel $\psi(r,\varepsilon) = \sqrt{1 + (\varepsilon r)^2}$ and κ is the matrix condition number. Figure 2 shows error versus quadrature nodes, and numerical solution at different times in $[0,1]$.
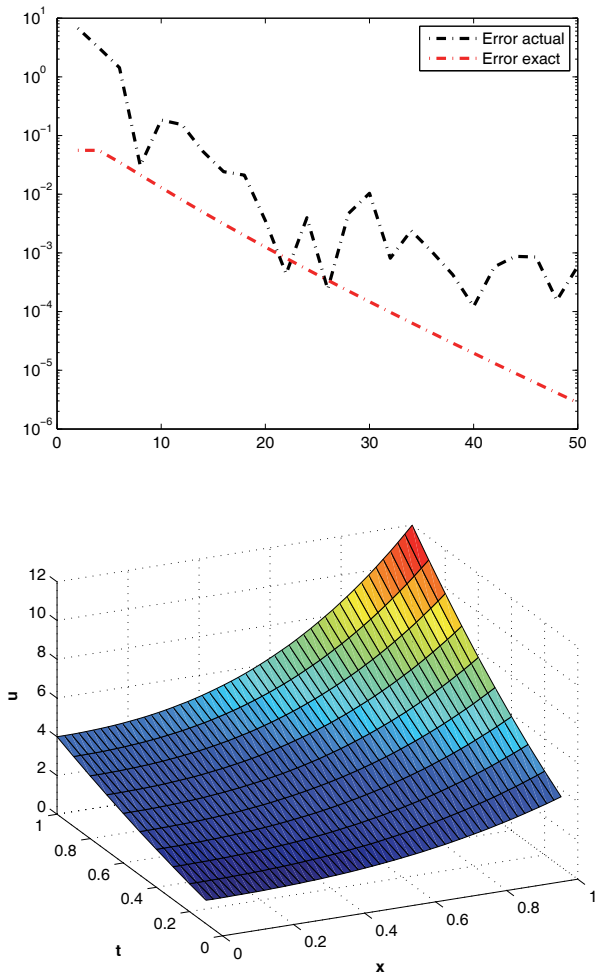
Figure 2: Error versus quadrature nodes and numerical solutions at different times of the present numerical method, corresponding to problem 2.

Table 2: Numerical results using Laplace transform-based local kernel method corresponding to model problem 2.

m	n	N	L_∞	L_{est}	ε	κ
50	10	10	0.1834	0.0130	4.5000	1.3509e+012
		30	0.0104	1.4769e-004	4.5000	1.3509e+012
		50	5.9369e-004	2.8134e-006	4.5000	1.3509e+012
20	10	50	8.1328e-004	2.8134e-006	1.7000	2.0443e+012
30			4.5944e-004	2.8134e-006	2.7000	1.0877e+012
60			5.8654e-004	2.8134e-006	5.5000	1.0663e+012
50	10	50	5.9369e-004	2.8134e-006	4.5000	1.3509e+012
	20		6.4655e-004	2.8134e-006	6.2000	1.1457e+012
	30		6.3545e-004	2.8134e-006	6.6000	1.2867e+012
[46](δt=1/160, N=100)			1.3065e-004			

Example 3. In the last problem we consider the BS model of fractional order which governing the European option

$$\frac{\partial^\alpha u(S,t)}{\partial t^\alpha} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u(S,t)}{\partial S^2} + (R-D)S\frac{\partial u(S,t)}{\partial S} - Ru(S,t), \ (S,t) \in (0,\infty) \times (0,T_e), \tag{46}$$

$$u(b_1,t) = P(t), \ u(b_2,t) = Q(t), \ u(s,T_e) = v(s), \tag{47}$$

Case 1. We applied the proposed transform-based localized kernel method to solve the BS equation governing the European model. Here we take the initial $v(s) = \max\{0, S - K\}$ and boundary values $P(t) = 0$ and $Q(t) = 0$ of the B-S model (46). This represent a time fractional B-S model governing European double barrier knockout call option.

The solution curves corresponding to double barrier option price at different values of fractional order α are shown in Figure 3. The parameters $\sigma = 0.45$, $R = 0.03$ $T_e = 1$ (year), $K = 10$ $b_1 = 3$, $b_2 = 15$ and the dividend is $D = 0.01$. Similar types of parameters values are considered which are used in [46]. The plots in Figure 3 are well consistent with the results in [46]. It is shown in Figure 3 that when S is less than a critical value K i.e strike price, lower prices values are obtained. while for fat tails, the higher prices are obtained when $(S > K)$. It is concluded that the present numerical scheme successfully capture jump or large movement in the process.

Case 2. For the purpose of comparison while computing the European call option, the same boundary and initial conditions $P(t) = 0$, and $Q(t) = b_2 K \exp(-RT_e t)$ and $v(S) = \max\{0, S - K\}$. The parameters $\sigma = 0.25$, $R = 0.05$, $b_1 = 0.1$, $b_2 = 100$, $T_e = 1$ (year) and $K = 50$.

Case 3, Again for European put option the initial and boundary values $v(S) = \max\{0, S - K\}$, $P(t) = K \exp(-RT_e t)$ and $Q(t) = D = 0$ and used with other parameters $\sigma = 0.25$, $R = 0.05$, $b_1 = 0.1$, $b_2 = 100$, $T_e = 1$ (year) and $K = 50$ as considered

in [46]. The solution curves corresponding to call option price as well as put option price are shown in Figures 4 and 5, respectively.
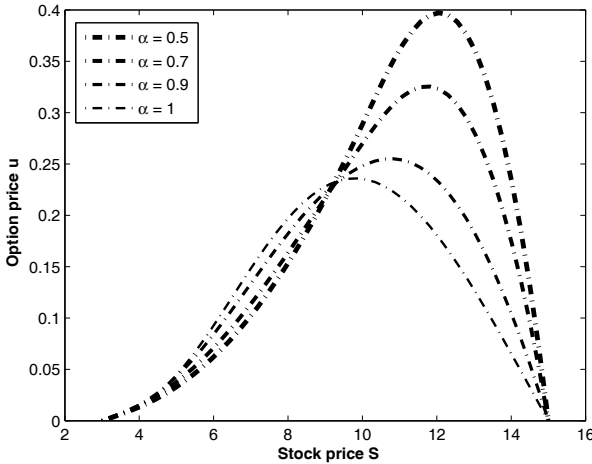

Figure 3: Double barrier option prices obtained by the Laplace transform based local kernel method for different values of α.

7. Conclusion

In this paper the Laplace transform is combined with the localized radial kernel method to approximate the solution of time fractional Black-Scholes equation. The stability conditions and the error bounds of the proposed scheme is discussed. As the time fractional models are the generalization of integral order models. Approximation of such types of PDEs either by analytical methods or by numerical methods is a difficult task as compared to integral order PDEs. The present numerical scheme successfully and very accurately approximate the time fractional BS equation. Similar time fractional order PDEs can be solved by the proposed numerical scheme very efficiently and accurately.

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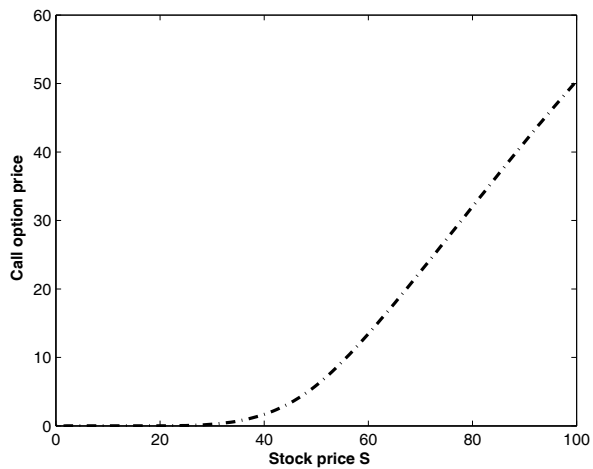
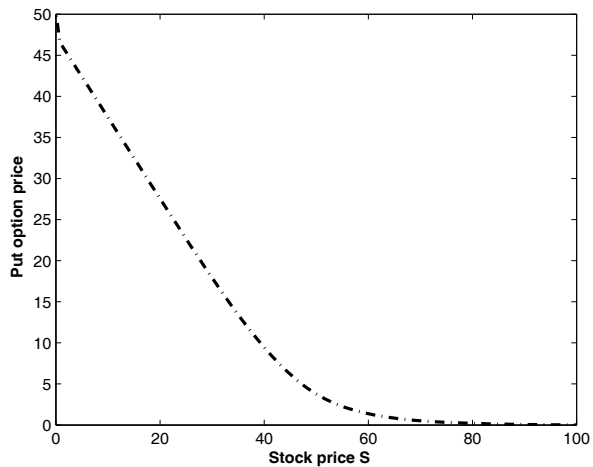Figure 4:   Call option prices obtained by the Laplace transform based local kernel method for $\alpha = 0.7$.

Figure 5:   Put option prices obtained by the Laplace transform based local kernel method for $\alpha = 0.7$.

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