Fractional
Differential
Calculus
Volume 9, Number 1 (2019), 125–138 doi:10.7153/fdc-2019-09-09

SOME PERTURBED VERSIONS OF THE GENERALIZED TRAPEZOID TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS

HÜSEYIN BUDAK* AND SAMAD NOEIAGHDAM

(*Communicated by S. S. Dragomir*)

Abstract. In this study, we first obtain an identity for twice differentiable functions. Then we establish some new perturbed trapezoid type integral inequalities for functions whose first derivatives either are of bounded variation or Lipschitzian. Moreover, some perturbed versions of trapezoid type inequalities for mapping whose second derivatives are bounded, of bounded variation or Lipschitzian, respectively.

1. Introduction

In 1938, Ostrowski [32] established a following useful inequality:

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\left\|f'\right\|_\infty, \tag{1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

The following definitions will be frequently used to prove our results.

DEFINITION 1. Let $P: a = x_0 < x_1 < \ldots < x_n = b$ be any partition of $[a,b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Mathematics subject classification (2010): 26D15, 26A45, 26D10.
Keywords and phrases: Function of bounded variation, perturbed trapezoid type inequalities, Riemann-Stieltjes integrals.
* Corresponding author.

Paper FDC-09-09

DEFINITION 2. Let f be of bounded variation on $[a,b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b(f) := \sup\left\{\sum \Delta f(P) : P \in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [21], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

THEOREM 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt - (b-a)f(x)\right| \leqslant \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b(f) \tag{2}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [17], Dragomir proved the trapezoid inequality for functions of bounded variation. Moreover, Cerone et al. established the following generalized trapezoid inequality for mapping of bounded variation in [15]:

THEOREM 3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right| \leqslant \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \bigvee_a^b(f)$$

holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

In the past, many authors have worked on Ostrowski type inequalities for functions of bounded variation, see for example ([1]-[12], [16]-[22], [30], [36]-[38]).

For a function of bounded variation $v : [a,b] \to \mathbb{C}$, we define the C*umulative Variation Function* (CVF) $V : [a,b] \to [0,\infty)$ by

$$V(t) := \bigvee_a^t(v),$$

the total variation of v on the interval $[a,t]$ with $t \in [a,b]$.

It is know that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous at a point $c \in [a,b]$ if and only if the generating function v is continuous in that point. If v is *Lipschitzian* with the constant $L > 0$, i.e.

$$|v(t) - v(s)| \leqslant L|t-s|, \text{ for any } t,s \in [a,b],$$

then V is also Lipschitzian with the same constant.

A simple proof of the following Lemma was given in [22].

LEMMA 1. *Let $f,u:[a,b]\to\mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exists and*

$$\left|\int_a^b f(t)du(t)\right| \leqslant \int_a^b |f(t)|\,d\left(\bigvee_a^t(u)\right) \leqslant \max_{t\in[a,b]}|f(t)|\bigvee_a^b(u). \tag{3}$$

In [23], Dragomir proved the following perturbes version of Ostrowski type inequality for mapping of bounded variation:

THEOREM 4. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have the inequality*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leqslant \frac{1}{b-a}\left[\int_a^x\left(\bigvee_t^x(f-\lambda_1(x)\ell)\right)dt+\int_x^b\left(\bigvee_x^t(f-\lambda_2(x)\ell)\right)dt\right]$$

$$\leqslant \frac{1}{b-a}\left[(x-a)\left(\bigvee_a^x(f-\lambda_1(x)\ell)\right)+(b-x)\left(\bigvee_x^b(f-\lambda_2(x)\ell)\right)\right]$$

$$\leqslant \begin{cases} \max\left\{\bigvee_a^x(f-\lambda_1(x)\ell),\bigvee_x^b(f-\lambda_2(x)\ell)\right\}, \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left(\bigvee_a^x(f-\lambda_1(x)\ell)+\bigvee_x^b(f-\lambda_2(x)\ell\right), \end{cases}$$

where $\ell:[a,b]\to[a,b]$ is the identity function, namely $\ell(t)=t$ for any $t\in[a,b]$.

For recent related results for the perturbed inequalities, see ([13],[14], [23]-[29], [31], [33]-[35]).

The aim of this paper is to obtain some new perturbed trapezoid type integral inequalities for functions whose first derivatives either are of bounded variation or Lipschitzian utilizing an identity given for twice differentiable mappings. Moreover, some perturbed versions of trapezoid type inequalities for mapping whose second derivatives are bounded, of bounded variation or Lipschitzian, respectively.

2. An identity for differentiable functions

Before we start our main results, we state and prove following lemma:

LEMMA 2. *Let $f:[a,b]\to\mathbb{C}$ be a twice differentiable function on (a,b) and $x\in[a,b]$. Then for any $\lambda(x)$ complex number the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-x)f(b)+(x-a)f(a)}{b-a}+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)} \tag{4}$$

$$-\frac{1}{2(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\lambda(x)=\frac{1}{2(b-a)}\int_a^b (x-t)^2d\left[f'(t)-\lambda(x)t\right]$$

where the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

Proof. Using the integration by parts for Riemann-Stieltjes, we have

$$\begin{aligned}
&\int_a^b (x-t)^2d\left[f'(t)-\lambda(x)t\right] &(5)\\
&=\int_a^b (x-t)^2df'(t)-\lambda(x)\int_a^b (x-t)^2dt\\
&=(x-t)^2f'(t)\Big|_a^b+2\int_a^b (x-t)f'(t)dt+\frac{\lambda(x)}{3}(x-t)^3\Big|_a^b\\
&=(b-x)^2f'(b)-(x-a)^2f'(a)-2\left[-(x-t)f(t)|_a^b-\int_a^b f(t)dt\right]\\
&\quad-\frac{(b-x)^3+(x-a)^3}{3}\lambda(x)\\
&=(b-x)^2f'(b)-(x-a)^2f'(a)-2(b-x)f(b)\\
&\quad-2(x-a)f(a)+2\int_a^b f(t)dt-\frac{(b-x)^3+(x-a)^3}{3}\lambda(x)
\end{aligned}$$

If we devide (5) by $2(b-a)$, we obtain required identity. □

REMARK 1. If we choose $\lambda(x)=0$ in (4), then we have the following identity

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-x)f(b)+(x-a)f(a)}{b-a}+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)} \tag{6}$$

$$= \frac{1}{2(b-a)} \int_a^b (x-t)^2 \, df'(t)$$

for all $x \in [a,b]$.

COROLLARY 1. *Under assumption of Lemma 2 with $\lambda(x) = \lambda \in \mathbb{C}$, we get*

$$\frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-x)f(b)+(x-a)f(a)}{b-a} + \frac{(b-x)^2 f'(b)-(x-a)^2 f'(a)}{2(b-a)} \tag{7}$$

$$-\frac{1}{2(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\lambda = \frac{1}{2(b-a)}\int_a^b (x-t)^2 \, d\left[f'(t)-\lambda t\right]$$

for all $x \in [a,b]$.

COROLLARY 2. *Under assumption of Lemma 2, we assume that the derivatives $f''_+(a)$, $f''_-(b)$ and $f''(x)$ exist and finite. If we choose $\lambda(x) = \frac{f''_+(a)+f''_-(b)}{2}$ in (4), then we have*

$$\frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-a)f(b)+(x-a)f(a)}{b-a} + \frac{(b-x)^2 f'(b)-(x-a)^2 f'(a)}{2(b-a)} \tag{8}$$

$$-\frac{1}{4(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\left[f''_+(a)+f''_-(b)\right]$$

$$= \frac{1}{2(b-a)}\int_a^b (x-t)^2 \left[f''(t) - \frac{f''_+(a)+f''_-(b)}{2}\right] dt,$$

for all $x \in [a,b]$.

3. Perturbed trapezoid type inequalities

In this section, we obtain some perturbed versions of trapezoid type inequalities for twice differentiable functions.

3.1. The case of that f' is of bounded variation

THEOREM 5. *Let $f:[a,b] \to \mathbb{C}$ be a twice differentiable function on I° and $[a,b] \subset I^\circ$ and $x \in [a,b]$. If the first derivative f' is of bounded variation on $[a,b]$, then for any $\lambda(x)$ complex number we have*

(9)

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-x)f(b)+(x-a)f(a)}{b-a} + \frac{(b-x)^2 f'(b)-(x-a)^2 f'(a)}{2(b-a)}\right.$$
$$\left. - \frac{1}{2(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\lambda(x)\right|$$
$$\leqslant \frac{1}{(b-a)}\left[\int_x^b (t-x)\left(\bigvee_x^b (f'-\lambda_1(x)\ell)\right)dt + \int_a^x (x-t)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right)dt\right]$$
$$\leqslant \frac{1}{2(b-a)}\left[(x-a)^2\bigvee_a^x (f'-\lambda(x)\ell) + (b-x)^2\bigvee_x^b (f'-\lambda(x)\ell)\right]$$
$$\leqslant \frac{(b-a)}{2}\begin{cases}\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\left[\frac{1}{2}\bigvee_a^b (f'-\lambda(x)\ell)+\frac{1}{2}\left|\bigvee_a^x (f'-\lambda(x)\ell)-\bigvee_x^b (f'-\lambda(x)\ell)\right|\right], \\ \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]^2\bigvee_a^b (f'-\lambda(x)\ell),\end{cases}$$

where $\ell : [a,b] \to [a,b]$ *denotes the identity function, namely* $\ell(t) = t$ *for any* $t \in [a,b]$.

Proof. Taking modulus (4) and applying Lemma 1, we get

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-x)f(b)+(x-a)f(a)}{b-a}\right. \tag{10}$$
$$\left. + \frac{(b-x)^2 f'(b)-(x-a)^2 f'(a)}{2(b-a)} - \frac{1}{2(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\lambda(x)\right|$$
$$\leqslant \frac{1}{2(b-a)}\left|\int_a^b (x-t)^2 d\left[f'(t)-\lambda(x)t\right]\right| \leqslant \frac{1}{2(b-a)}\int_a^b (x-t)^2 d\left(\bigvee_a^t (f'-\lambda(x)\ell)\right).$$

Integrating by parts in the Riemann-Stieltjes integral, we get

$$\int_a^b (x-t)^2 d\left(\bigvee_a^t (f'-\lambda(x)\ell)\right) \tag{11}$$
$$= (x-t)^2\bigvee_a^t (f'-\lambda(x)\ell)\Big|_a^b + 2\int_a^b (x-t)\left(\bigvee_a^t (f'-\lambda(x)\ell)\right)dt$$
$$= (b-x)^2\bigvee_a^b (f'-\lambda_1(x)\ell) + 2\int_a^b (x-t)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right)dt$$

$$= -2\int_x^b (x-t)\left(\bigvee_a^b (f'-\lambda_1(x)\ell)\right)dt + 2\int_a^b (x-t)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right)dt$$
$$= 2\int_x^b (t-x)\left(\bigvee_x^b (f'-\lambda_1(x)\ell)\right)dt + 2\int_a^x (x-t)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right)dt.$$

This completes the proof of the first inequality in (9). Moreover, we have

$$\int_x^b (t-x)\left(\bigvee_x^b (f'-\lambda_1(x)\ell)\right)dt = \frac{1}{2}(b-x)^2\bigvee_x^b (f'-\lambda_1(x)\ell) \tag{12}$$

and

$$\int_a^x (x-t)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right)dt \leqslant \frac{1}{2}(x-a)^2\bigvee_a^x (f'-\lambda_1(x)\ell). \tag{13}$$

With the inequalities (12) and (13), the proof second inequality in (9) is completed.

The proof of the last inequality in (9) is obvious from the property $\max\{a,b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$. $\square$

COROLLARY 3. *If we choose $\lambda(x) = 0$ in (9), then we have the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-x)f(b)+(x-a)f(a)}{b-a} + \frac{(b-x)^2 f'(b)-(x-a)^2 f'(a)}{2(b-a)}\right|$$
$$\leqslant \frac{1}{(b-a)}\left[\int_x^b (t-x)\left(\bigvee_x^b (f')\right)dt + \int_a^x (x-t)\left(\bigvee_a^t (f')\right)dt\right]$$
$$\leqslant \frac{1}{2(b-a)}\left[(x-a)^2\bigvee_a^x (f') + (b-x)^2\bigvee_x^b (f'\ell)\right]$$
$$\leqslant \frac{(b-a)}{2}\begin{cases}\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\ \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]^2 \bigvee_a^b (f'),\end{cases}$$

for all $x \in [a,b]$.

COROLLARY 4. *If we choose $\lambda(x) = \lambda$ and $x = \frac{a+b}{2}$ in (9), then we have the* *following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)+f(a)}{2}+\frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right]-\frac{(b-a)^2}{24}\lambda\right|$$

$$\leqslant \frac{1}{(b-a)}\left[\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)\left(\bigvee_{\frac{a+b}{2}}^{b}(f'-\lambda\ell)\right)dt+\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left(\bigvee_a^t(f'-\lambda\ell)\right)dt\right]$$

$$\leqslant \frac{(b-a)}{8}\bigvee_a^b(f'-\lambda\ell).$$

3.2. The case of that f' is Lipschitzian mapping

THEOREM 6. *Let $f:[a,b]\to\mathbb{C}$ be a twice differentiable function on I° and $[a,b]\subset I^\circ$. If there exists the positive number $K(x)$ such that $f'-\lambda(x)\ell$ is Lipschitzian with the constant $K(x)$ on the interval $[a,b]$, then for any $x\in[a,b]$ we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-x)f(b)+(x-a)f(a)}{b-a}+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)}\right. \tag{14}$$

$$\left.-\frac{1}{2(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\lambda(x)\right|\leqslant\frac{K(x)}{6(b-a)}\left[(x-a)^3+(b-x)^3\right].$$

Proof. It is known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$, then the Riemann-Stieltjes integral $\int_c^d g(t)du(t)$ exist and

$$\left|\int_c^d g(t)du(t)\right|\leqslant K\int_c^d|g(t)|dt.$$

Taking the madulus (4), we get

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-x)f(b)+(x-a)f(a)}{b-a}+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)}\right.$$

$$\left.-\frac{1}{2(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\lambda(x)\right|\leqslant\frac{1}{2(b-a)}\left|\int_a^b(x-t)^2d\left[f'(t)-\lambda(x)t\right]\right|$$

$$\leqslant\frac{K(x)}{2(b-a)}\int_a^b\left|(x-t)^2\right|dt=\frac{K(x)}{6(b-a)}\left[(x-a)^3+(b-x)^3\right].$$

This completes the proof. □

COROLLARY 5. *If we choose* $x = \frac{a+b}{2}$ *and* $\lambda(x) = \lambda \in \mathbb{C}$ *in (14), we get the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b)+f(a)}{2} + \frac{(b-a)^2}{8} \left[f'(b) - f'(a) \right] - \frac{(b-a)^2}{24} \lambda \right| \leqslant \frac{(b-a)}{24} K(x).$$

3.3. The case of that f'' is bounded

Recall the sets of complex-valued functions:

$$\overline{U}_{[a,b]}(\gamma, \Gamma)$$

$$:= \left\{ f : [a,b] \to \mathbb{C} | \left[(\Gamma - f(t)) \left(\overline{f(t)} \right) - \overline{\gamma} \right] \geqslant 0 \text{ for almost every } t \in [a,b] \right\}$$

and

$$\overline{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a,b] \to \mathbb{C} \left| \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leqslant \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a,b] \right. \right\}.$$

PROPOSITION 1. For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\overline{U}_{[a,b]}(\gamma, \Gamma)$ and $\overline{\Delta}_{[a,b]}(\gamma, \Gamma)$ are nonempty and closed sets and

$$\overline{U}_{[a,b]}(\gamma, \Gamma) = \overline{\Delta}_{[a,b]}(\gamma, \Gamma).$$

THEOREM 7. *Let* $f : [a,b] \to \mathbb{C}$ *be a twice differentiable function on* (a,b) *and* $x \in (a,b)$. *Suppose that* $\gamma, \Gamma \in \mathbb{C}$, *and* $f'' \in \overline{U}_{[a,b]}(\gamma, \Gamma)$. *Then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{(b-x)f(b) + (x-a)f(a)}{b-a} \right. \tag{15}$$

$$\left. + \frac{(b-x)^2 f'(b) - (x-a)^2 f'(a)}{2(b-a)} - \frac{(\gamma + \Gamma)}{4(b-a)} \left[\frac{(x-a)^3 + (b-x)^3}{3} \right] \right|$$

$$\leqslant \frac{|\Gamma - \gamma|}{12(b-a)} \left[(x-a)^3 + (b-x)^3 \right].$$

Proof. Taking the modulus identity (4) for $\lambda(x) = \frac{\gamma+\Gamma}{2}$, since $f'' \in \overline{U}_{[a,b]}(\gamma, \Gamma)$, we have

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{(b-x)f(b) + (x-a)f(a)}{b-a} + \frac{(b-x)^2 f'(b) - (x-a)^2 f'(a)}{2(b-a)} \right.$$

$$-\frac{(\gamma+\Gamma)}{4(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\Bigg| \leqslant \frac{1}{2(b-a)}\int_a^b (x-t)^2\left|f''(t)-\frac{\gamma+\Gamma}{2}\right|dt$$

$$\leqslant \frac{|\Gamma-\gamma|}{4(b-a)}\int_a^b (x-t)^2\,dt = \frac{|\Gamma-\gamma|}{12(b-a)}\left[(x-a)^3+(b-x)^3\right]$$

which completes the proof of the inequality (15). □

COROLLARY 6. *Under assumption of Theorem 7 with* $x=\frac{a+b}{2}$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)+f(a)}{2}+\frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right]-\frac{(b-a)^2}{48}(\gamma+\Gamma)\right|$$

$$\leqslant \frac{(b-a)^2}{48}\left|\Gamma_1-\gamma_1\right|.$$

3.4. The case of that f'' is bounded variation

Assume that $f:[a,b]\to\mathbb{C}$ be a twice differentiable function on I° (the interior of I) and $[a,b]\subset I^\circ$. Then, as in (8), we have the identity

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-x)f(b)+(x-a)f(a)}{b-a} \tag{16}$$

$$+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)}-\frac{1}{4(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\left[f''_+(a)+f''_-(b)\right]$$

$$=\frac{1}{2\,(b-a)}\int_a^b (x-t)^2\left[f''(t)-\frac{f''_+(a)+f''_-(b)}{2}\right]dt,$$

for any $x\in[a,b]$.

THEOREM 8. *Let* $f:[a,b]\to\mathbb{C}$ *be a twice differentiable function on* I° *and* $[a,b]\subset I^\circ$. *If the second derivative* f'' *is of bounded variation on* $[a,b]$, *then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-x)f(b)+(x-a)f(a)}{b-a}\right. \tag{17}$$

$$\left.+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)}-\frac{1}{4(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\left[f''(a)+f''(b)\right]\right|$$

$$\leqslant \frac{1}{12(b-a)}\left[(x-a)^3+(b-x)^3\right]\bigvee_a^b(f'')$$

for any $x \in [a,b]$.

Proof. Taking modulus (16), we get

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-a)f(b)+(x-a)f(a)}{b-a}\right. \tag{18}$$
$$\left.+\frac{(b-x)^2 f'(b)-(x-a)^2 f'(a)}{2(b-a)} - \frac{1}{4(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\left[f''_+(a)+f''_-(b)\right]\right|$$
$$\leqslant \frac{1}{2(b-a)}\int_a^b (x-t)^2\left|f''(t)-\frac{f''(a)+f''(b)}{2}\right|dt.$$

Since f'' is of bounded variation on $[a,b]$, we get

$$\left|f''(t)-\frac{f''(a)+f''(b)}{2}\right| \leqslant \frac{|2f''(t)-f''(a)-f''(b)|}{2}$$
$$\leqslant \frac{|f''(t)-f''(a)|+|f''(b)-f''(t)|}{2} \leqslant \frac{1}{2}\bigvee_a^b(f'').$$

Thus,

$$\int_a^x (x-t)^2\left|f''(t)-\frac{f''(a)+f''(x)}{2}\right|dt \leqslant \frac{1}{2}\bigvee_a^b(f'')\int_a^b (x-t)^2\,dt \tag{19}$$
$$= \frac{(x-a)^3+(b-x)^3}{6}\bigvee_a^b(f'').$$

If we substitute the inequality (19) in (18), we obtain the required inequality (17). □

COROLLARY 7. *Under assumptions of Theorem 8 with* $x = \frac{a+b}{2}$, *we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)+f(a)}{2} + \frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right] - \frac{(b-a)^2}{48}\left[f''(a)+f''(b)\right]\right|$$
$$\leqslant \frac{(b-a)^2}{48}\bigvee_a^b(f'').$$

3.5. The case of that f'' is Lipschitzian mapping

THEOREM 9. *Let $f:[a,b]\to\mathbb{C}$ be a twice differentiable function on I° and $[a,b]\subset I^\circ$. If the second derivative f'' is Lipschitzian with the constant $L(x)$ on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-x)f(b)+(x-a)f(a)}{b-a}+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)}\right.$$
$$\left.-\frac{1}{4(b-a)}\left[\frac{(x-a)^3+(b-x)^3}{3}\right]\left[f''(a)+f''(b)\right]\right|\leqslant\frac{(x-a)^3+(b-x)^3}{12}L(x)$$

for any $x\in[a,b]$.

Proof. Since f'' is Lipschitzian with the constant $L_1(x)$ on $[a,b]$, we get

$$\begin{aligned}\left|f''(t)-\frac{f''(a)+f''(b)}{2}\right| &\leqslant \frac{|2f''(t)-f''(a)-f''(b)|}{2}\\ &\leqslant \frac{|f''(t)-f''(a)|+|f''(b)-f''(t)|}{2}\\ &\leqslant \frac{1}{2}L(x)\left[|t-a|+|b-t|\right]=\frac{1}{2}L(x)(b-a).\end{aligned}$$

Thus,

$$\int_a^b (x-t)^2\left|f''(t)-\frac{f''(a)+f''(n)}{2}\right|dt\leqslant\frac{1}{2}L(x)(b-a)\int_a^b(x-t)^2\,dt \tag{20}$$
$$=\frac{L(x)}{6}(b-a)\left[(x-a)^3+(b-x)^3\right].$$

If we substitute the inequalities (20) in (18), we obtain the desired result. □

COROLLARY 8. *Under assumptions of Theorem 9 with $x=\frac{a+b}{2}$, we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)+f(a)}{2}+\frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right]-\frac{(b-a)^2}{48}\left[f''(a)+f''(b)\right]\right|$$
$$\leqslant\frac{(b-a)^3}{48}L(x).$$

REFERENCES

[1] M. W. ALOMARI, *A Generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.

[2] M. W. ALOMARI AND M. A. LATIF, *Weighted companion for the Ostrowski and the generalized trapezoid inequalities for mappings of vounded variation*, RGMIA Research Report Collection, 14(2011),Article 92, 10 pp.

[3] M. W. ALOMARI AND S. S. DRAGOMIR, *Mercer–Trapezoid rule for the Riemann–Stieltjes integral with applications*, Journal of Advances in Mathematics, 2(2)(2013), 67–85.

[4] M. W. ALOMARI, *A companion of Ostrowski's inequality with applications*, TJMM, vol. 3, no. 1, pp. 9–14, 2011.

[5] M. W. ALOMARI, *A companion of the generalized trapezoid inequality and applications*, Journal of Mathematics and Applications, vol. 36, pp. 5–15, 2013.

[6] H. BUDAK AND M. Z. SARIKAYA, *On generalization of Dragomir's inequalities*, Turkish Journal of Analysis and Number Theory, 2017, Vol. 5, No. 5, 191–196.

[7] H. BUDAK AND M. Z. SARIKAYA, *New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation*, Transylvanian Journal of Mathematics and Mechanics (TJMM), 8 (2016), No. 1, 21–27.

[8] H. BUDAK AND M. Z. SARIKAYA, *A new generalization of Ostrowski type inequalities for mappings of bounded variation*, Lobachevskii Journal of Mathematics, 39(9), 2018, pp. 1320–1326.

[9] H. BUDAK AND M. Z. SARIKAYA, *On generalization of weighted Ostrowski type inequalities for functions of bounded variation*, Asian-European Journal of Mathematics, 11(4), 2018, 1850049 (11 pages).

[10] H. BUDAK AND M. Z. SARIKAYA, *A new Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Moroccan Journal of Pure and Applied Analysis2(1)(2016), 1–11.

[11] H. BUDAK AND M. Z. SARIKAYA, *A companion of Ostrowski type inequalities for mappings of bounded variation and some applications*, Transactions of A. Razmadze Mathematical Institute, 171(2), 2017, 136–143.

[12] H. BUDAK, M. Z. SARIKAYA AND A. QAYYUM, *Improvement in companion of Ostrowski type inequalities for mappings whose first derivatives are of bounded variation and application*, Filomat, 31:16 (2017), 5305–5314.

[13] H. BUDAK AND M. Z. SARIKAYA, *Some perturbed Ostrowski type inequalities for functions whose first derivatives are of bounded variation*, International Journal of Analysis and Applications, 11(2), 2016, 146–156.

[14] H. BUDAK, M. Z. SARIKAYA, A. AKKURT AND H. YILDIRIM, *Perturbed companion of Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Konuralp Journal of Mathematics, 5(1), pp:161–175, 2017.

[15] P. CERONE, S. S. DRAGOMIR, AND C. E. M. PEARCE, *A generalized trapezoid inequality for functions of bounded variation*, Turk J Math, 24 (2000), 147–163.

[16] S. S. DRAGOMIR, *Approximating real functions which possess nth derivatives of bounded variation and applications*, Computers and Mathematics with Applications56(2008), 2268–2278.

[17] S. S. DRAGOMIR, *On trapezoid quadrature formula and applications*, Kragujevac J. Math., vol. 23, pp. 25–36, 2001.

[18] S. S. DRAGOMIR, *On the midpoint quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math., vol. 22, pp. 13–19, 2000.

[19] S. S. DRAGOMIR, *The Ostrowski integral inequality for mappings of bounded variation*, Bulletin of the Australian Mathematical Society, 60(1)(1999),495–508.

[20] S. S. DRAGOMIR, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Mathematical Inequalities & Applications, 4(2001),no. 1, 59–66.

[21] S. S. DRAGOMIR, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, International Journal of Nonlinear Analysis and Applications, 5(2014) No. 1, 89–97 pp.

[22] S. S. DRAGOMIR, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation*, Arch. Math. (Basel) 91 (2008), no. 5, 450–460.

[23] S. S. DRAGOMIR, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Asian-European Journal of Mathematics, 8(4), 2015, 14 pages.

[24] S. S. DRAGOMIR, *Some perturbed Ostrowski type inequalities for absolutely continuous functions (I)*, Acta Universitatis Matthiae Belii, series Mathematics 23 2015, 71–86.

[25] S. S. DRAGOMIR, *Some perturbed Ostrowski type inequalities for absolutely continuous functions (II)*, Acta Universitatis Apulensis, 43, 2015, 209–228.

[26] S. S. DRAGOMIR, *Some perturbed Ostrowski type inequalities for absolutely continuous functions (III)*, TJMM, 7(1), 2015,31–43.

[27] S. S. DRAGOMIR, *Perturbed companions of Ostrowski's inequality for functions of bounded variation*, RGMIA Research Report Collection, 17(2014), Article 1, 16 pp.

[28] S. S. DRAGOMIR, *Perturbed companions of Ostrowski's inequality for absolutely continuous functions (I)*, Annals of West University of Timisoara - Mathematics and Computer Science, LIV, 1, (2016), 119–138.

[29] S. S. DRAGOMIR, *Perturbed companions of Ostrowski's inequality for absolutely continuous functions (II)*, GMIA Research Report Collection, 17, 2014, Article 19, 11 pp.

[30] W. LIU AND Y. SUN, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, 2012.

[31] W. LIU AND J. PARK, *Some perturbed versions of the generalized trapezoid inequality for function of bounded variation*, J. Comp. Analy. App. 22(1), 2017, pp.11–18.

[32] A. M. OSTROWSKI, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert*, Comment. Math. Helv. 10(1938),226–227.

[33] M. Z.SARIKAYA, H. BUDAK, T. TUNC, S. ERDEN AND H. YALDIZ, *Perturbed companion of Ostrowski type inequality for twice differentiable functions*, Facta Universitatis, Series: Mathematics and Informatics, 31(3), 2016, 593–607.

[34] M. Z. SARIKAYA, H. BUDAK AND A.QAYYUM, *An improved version of perturbed companion of Ostrowski type inequalities*, Journal of Inequalities and Special Functions, 7(3), 2016, 12–25.

[35] M. Z. SARIKAYA, H. BUDAK, S. ERDEN AND A. QAYYUM, *A generalized and refined perturbed version of Ostrowski type inequalities*, International Journal of Analysis and Applications, 13(1), (2017), 70–81.

[36] K-L TSENG, G-S YANG, AND S. S. DRAGOMIR, *Generalizations of weighted trapezoidal inequality for mappings of bounded variation and their applications*, Mathematical and Computer Modelling 40 (2004) 77–84.

[37] K-L TSENG, *Improvements of some inequalites of Ostrowski type and their applications*, Taiwan. J. Math. 12 (9) (2008) 2427–2441.

[38] K-L TSENG, S-R HWANG AND S. S. DRAGOMIR, *Generalizations of weighted Ostrowski type inequalities for mappings of bounded variation and applications*, Computers and Mathematics with Applications, 55(2008), 1785–1793.

[39] K-L TSENG, S-R HWANG, G-S YANG AND Y-M CHOU, *Improvements of the Ostrowski integral inequality for mappings of bounded variation I*, Applied Mathematics and Computation 217 (2010) 2348–2355.

[40] K-L TSENG, S-R HWANG, G-S YANG AND Y-M CHOU, *Weighted Ostrowski integral inequality for mappings of bounded variation*, Taiwanese J. of Math., Vol. 15, No. 2, pp. 573–585, April 2011.

[41] K-L TSENG, *Improvements of the Ostrowski integral inequality for mappings of bounded variation II*, Applied Mathematics and Computation 218 (2012) 5841–5847.

(Received December 9, 2018)

Hüseyin Budak
Department of Mathematics
Faculty of Science and Arts, Düzce University
Düzce,Turkey
e-mail: hsyn.budak@gmail.com

Samad Noeiaghdam
Department of Mathematics
Central Tehran Branch, Islamic Azad University
Tehran,Iran
e-mail: samadnoeiaghdam@gmail.com