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ON SOME FRACTIONAL INTEGRO–DIFFERENTIAL INCLUSIONS WITH NONLOCAL MULTI–POINT BOUNDARY CONDITIONS

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Abstract. Existence of solutions for two classes of fractional integro-differential inclusions with nonlocal multi-point boundary conditions is investigated in the case when the values of the set-valued map are not convex.

1. Introduction

In the last years one may see a strong development of the theory of differential equations of fractional order ([4, 9, 12, 13, 14] etc.) and of the theory of fractional differential inclusions (e.g., [15]). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena.

In some recent papers [1, 3] etc. the attention was focused on special classes of boundary value problems associated to fractional differential equations; namely, nonlocal multi-point boundary conditions. This is the explanation for the study in the present paper of some fractional integro-differential inclusions with nonlocal multi-point boundary conditions.

We consider first the problem

$$D^q x(t) \in F(t, x(t), I^\gamma x(t)) \quad a.e.\ ([1,T]), \tag{1.1}$$

$$x(1) = 0, \quad D^r x(T) = \sum_{i=1}^{n} \lambda_i D^r x(\mu_i), \tag{1.2}$$

where D^q is the Hadamard fractional derivative of order q, $q \in (1,2]$, $r \in (0,1)$, I^γ is the Hadamard integral of order γ, $\gamma > 0$, $\mu_i \in (1,T)$, $\lambda_i \in \mathbf{R}$, $i = \overline{1,n}$, $n \geqslant 2$ and $F : [1,T] \times \mathbf{R} \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is a set-valued map.

If F is single-valued and does not depend on the last variable, fractional differential inclusion (1.1) reduces to the fractional differential equation

$$D^q x(t) = f(t, x(t)), \tag{1.3}$$

where $f : [1,T] \times \mathbf{R} \to \mathbf{R}$.

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Existence results for problem (1.3)-(1.2) are obtained in [3] and are based on a nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory.

Our goal is to extend the study in [3] to the more general problem (1.1)-(1.2) and to show that Filippov's ideas ([10]) can be suitably adapted in order to obtain the existence of solutions for this problem. Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([10]) consists in proving the existence of a solution starting from a given "quasi" solution. At the same time, the result provides an estimate between the "quasi" solution and the solution obtained.

Secondly, we obtain similar results for problem

$$D_c^q x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,T]) \tag{1.4}$$

$$x(0) = \delta x(\sigma), \quad aD_c^p x(\xi_1) + bD_c^p x(\xi_2) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i), \tag{1.5}$$

where $q \in (1,2]$, $p \in (0,1)$, $\delta, a, b.\alpha_i \in \mathbf{R}$, $\sigma, \xi_1, \xi_2, \beta_i \in (0,T)$, $i = \overline{1, m-2}$, D_c^q is the Caputo fractional derivative of order q, $F : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is a set-valued map, $V : C([0,1], \mathbf{R}) \to C([0,1], \mathbf{R})$ is a nonlinear Volterra operator $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

In the case when F does not depend on the last variable and is single-valued, fractional differential inclusion (1.4) reduces to the fractional differential equation

$$D_c^q x(t) = f(t, x(t)), \tag{1.6}$$

where $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a given mapping.

In [1] fixed point techniques are employed to obtain the existence of solutions for problem (1.6)-(1.5).

We note that existence results of the type provided in the present paper exists in the literature ([6, 7, 8] etc.), but their exposure in the framework of problems (1.1)-(1.2) and (1.4)-(1.5) is new.

The novelty of the present paper concerns several aspects. On one hand, the study in [1, 3] is extended to the set-valued framework. This allows to deduce certain existence results concerning fractional differential equations in [1, 3] as consequences of more general results. On the other hand, we consider problems whose right-hand side contains an integral term and we implement to these integro-differential inclusions Fillipov techniques. For such kind of problems the usual fixed point techniques (e.g., [15]) are difficult to be adapted.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our results.

2. Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $I = [1,T]$, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_1^e |u(t)|dt$.

The Hadamard fractional integral of order $q > 0$ of a Lebesgue integrable function $f : [1,\infty) \to \mathbf{R}$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\ln \frac{t}{s}\right)^{q-1} \frac{f(s)}{s} ds$$

provided the integral exists and Γ is the (Euler's) Gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}dt$.

The Hadamard fractional derivative of order $q > 0$ of a function $f : [1,\infty) \to \mathbf{R}$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-q-1} \frac{f(s)}{s} ds,$$

where $n = [q]+1$, $[q]$ is the integer part of q.

Details and properties of Hadamard fractional derivative may be found in [11, 12].

The fractional integral of order $q > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by

$$I^q f(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

The Caputo fractional derivative of order $q > 0$ of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$\mathrm{D}_c^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{-q+n-1} f^{(n)}(s)\mathrm{d}s,$$

where $n = [q]+1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

The next technical result is proved in [3]. Set $\Lambda := (\ln T)^{q-r-1} - \sum_{i=1}^n \lambda_i (\ln \mu_i)^{q-r-1}$.

LEMMA 1. *([3]) Assume that* $\Lambda \neq 0$. *For a given* $f(.) \in C(I,\mathbf{R})$, *the unique solution* $x(.)$ *of problem* $D^q x(t) = f(t)$ *a.e.* $([1,T])$ *with boundary conditions (1.2) is given by*

$$x(t) = \frac{(\ln t)^{q-1}}{\Gamma(q)\Lambda}\left(\sum_{i=1}^n \lambda_i \int_1^{\mu_i} (\ln \frac{\mu_i}{s})^{q-r-1} \frac{h(s)}{s} ds - \int_1^T (\ln \frac{T}{s})^{q-r-1} \frac{h(s)}{s} ds\right) + \frac{1}{\Gamma(q)} \int_1^t (\ln \frac{t}{s})^{q-1} \frac{h(s)}{s} ds. \tag{2.1}$$

REMARK 1. If we denote

$$G_1(t,s) = \frac{(\ln t)^{q-1}}{\Gamma(q)\Lambda}(\sum_{i=1}^{n}\lambda_i(\ln\frac{\mu_i}{s})^{q-r-1}\frac{1}{s}\chi_{[1,\mu_i]}(s) - (\ln\frac{T}{s})^{q-r-1}\frac{1}{s}) + \frac{1}{\Gamma(q)}(\ln\frac{t}{s})^{q-1}\frac{1}{s}\chi_{[1,t]}(s),$$

where $\chi_S(\cdot)$ is the characteristic function of the set S, then the solution $x(\cdot)$ in Lemma 1 may be written as $x(t) = \int_1^T G_1(t,s)f(s)ds$.

Using the fact that, for fixed t, the function $g(s) = (\ln\frac{t}{s})^{\alpha}\frac{1}{s}$ with $\alpha > 0$ is decreasing we deduce that, if $q-r-1 > 0$, for any $t,s \in I$,

$$|G_1(t,s)| \leqslant \frac{(\ln T)^{q-1}}{\Gamma(q)|\Lambda|}(\sum_{i=1}^{n}|\lambda_i|(\ln\mu_i)^{q-r-1} + (\ln T)^{q-r-1}) + \frac{1}{\Gamma(q)}(\ln T)^{q-1} =: M_1.$$

DEFINITION 1. A function $x(.) \in C(I,\mathbf{R})$ with its Hadamard derivative of order q existing on $[1,T]$ is called a solution of problem (1.1)-(1.2) if there exists a function $f(.) \in L^1(I,\mathbf{R})$ that satisfies

$$f(t) \in F(t,x(t),I^{\gamma}x(t)) \quad a.e.\ (I)$$

and $x(.)$ is given by (2.1).

Next $I = [0,T]$. The proof of the following lemma may be found in [1]. Define

$$A := (1-\delta)(\frac{a\xi_1^{1-p} + b\xi_2^{1-p}}{\Gamma(2-p)} - \sum_{i=1}^{m-2}\alpha_i\beta_i) - \delta\sigma\sum_{i=1}^{m-2}\alpha_i.$$

LEMMA 2. *([1]) Assume that* $A \neq 0$. *For a given* $f(.) \in C(I,\mathbf{R})$, *the unique solution* $x(.)$ *of problem* $D_c^q x(t) = f(t) \quad a.e.\ ([0,T])$ *with boundary conditions (1.5) is given by*

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds + \frac{\delta}{1-\delta}\int_0^{\sigma}\frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s)ds + [\frac{\delta\sigma}{A(1-\delta)} + \frac{t}{A}][(1-\delta) \cdot (\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}f(s)ds - a\int_0^{\xi_1}\frac{(\xi_1-s)^{q-p-1}}{\Gamma(q-p)}f(s)ds - b\int_0^{\xi_2}\frac{(\xi_2-s)^{q-p-1}}{\Gamma(q-p)}f(s)ds) + \delta\sum_{i=1}^{m-2}\alpha_i\int_0^{\sigma}\frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s)ds.] \tag{2.2}$$

If we denote

$$G_2(t,s)=\frac{(t-s)^{q-1}}{\Gamma(q)}\chi_{[0,t]}(s)+\frac{\delta}{1-\delta}\frac{(\sigma-s)^{q-1}}{\Gamma(q)}\chi_{[0,\sigma]}(s)+[\frac{\delta\sigma}{A(1-\delta)}+\frac{t}{A}][(1-\delta)$$
$$\cdot(\sum_{i=1}^{m-2}\alpha_i\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}\chi_{[0,\beta_i]}(s)-a\frac{(\xi_1-s)^{q-p-1}}{\Gamma(q-p)}\chi_{[0,\xi_1]}(s)$$
$$-b\frac{(\xi_2-s)^{q-p-1}}{\Gamma(q-p)}\chi_{[0,\xi_2]}(s))+\delta\sum_{i=1}^{m-2}\alpha_i\frac{(\sigma-s)^{q-1}}{\Gamma(q)}\chi_{[0,\sigma]}(s)],$$

then solution $x(\cdot)$ in Lemma 2 may be written as $x(t)=\int_0^T G_2(t,s)f(s)ds$.

Moreover, if $q-p-1>0$ for any $t,s\in I$ we have

$$|G_2(t,s)|\leqslant\frac{T^{q-1}}{\Gamma(q)}+\frac{|\delta|}{|1-\delta|}\frac{\sigma^{q-1}}{\Gamma(q)}+\frac{|\delta|T+T|1-\delta|}{|A|)}(\sum_{i=1}^{m-2}|\alpha_i|\frac{\beta_i^{q-1}}{\Gamma(q)}+|a|\frac{\xi_1^{q-p-1}}{\Gamma(q-p)}$$
$$+|b|\frac{\xi_2^{q-p-1}}{\Gamma(q-p)}+|\delta|\sum_{i=1}^{m-2}|\alpha_i|\frac{\sigma^{q-1}}{\Gamma(q)})=:M_2.$$

DEFINITION 2. A function $x(.)\in C(I,\mathbf{R})$ with its Caputo derivative of order q existing on $[0,T]$ is called a solution of problem (1.4)-(1.5) if there exists a function $f(.)\in L^1(I,\mathbf{R})$ that satisfies

$$f(t)\in F(t,x(t),V(x)(t))\quad a.e.\ (I)$$

and $x(.)$ is given by (2.2).

Finally, we recall a selection result ([2]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

LEMMA 3. *([2]) Consider X a separable Banach space, B is the closed unit ball in X, $G:I\to\mathscr{P}(X)$ is a set-valued map with nonempty closed values and $c:I\to X, r:I\to\mathbf{R}_+$ are measurable functions. If*

$$G(t)\cap(c(t)+r(t)B)\neq\emptyset\quad a.e.(I),$$

then the set-valued map $t\to G(t)\cap(c(t)+r(t)B)$ has a measurable selection.

3. The main results

In order to prove our results we need the following hypotheses.

HYPOTHESIS H1. i) *$F(.,.,.):I\times\mathbf{R}\times\mathbf{R}\to\mathscr{P}(\mathbf{R})$ has nonempty closed values and is $\mathscr{L}(I)\otimes\mathscr{B}(\mathbf{R}\times\mathbf{R})$ measurable.*

ii) *There exists $l(.)\in L^1(I,(0,\infty))$ such that, for almost all $t\in I$, $F(t,.,.)$ is $l(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2))\leqslant l(t)(|x_1-x_2|+|y_1-y_2|)\ \forall\, x_1,x_2,y_1,y_2\in\mathbf{R}.$$

We use next the following notations

$$L(t) := l(t)(1+\frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}ds) = l(t)(1+\frac{(\ln t)^{\gamma}}{\Gamma(\gamma+1)}), \tag{3.1}$$

$$L_0 = \int_1^T L(t)dt. \tag{3.2}$$

THEOREM 1. *Assume that Hypothesis H1 is satisfied,* $q-r-1>0$, $\Lambda \neq 0$ *and* $M_1L_0<1$. *Consider* $y(.)\in C(I,\mathbf{R})$ *with its Hadamard derivative of order* q *existing on* $[1,T]$ *such that* $y(1)=0$, $D^r y(T)=\sum_{i=1}^n \lambda_i D^r y(\mu_i)$ *and there exists* $p(.)\in L^1(I,\mathbf{R}_+)$ *verifying* $d(D^q y(t),F(t,y(t),I^{\gamma}y(t)))\leqslant p(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem (1.1)-(1.2) satisfying for all* $t\in I$

$$|x(t)-y(t)| \leqslant \frac{M_1}{1-M_1L_0}\int_1^T p(t)dt. \tag{3.3}$$

Proof. The multifunction $t\to F(t,y(t),I^{\gamma}y(t))$ has closed values, is measurable and from hypothesis of theorem one has

$$F(t,y(t),I^{\gamma}y(t))\cap\{D^q y(t)+p(t)[-1,1]\}\neq\emptyset \quad a.e.\ (I).$$

We apply Lemma 3 to find a measurable function $f_1(t)\in F(t,y(t),I^{\gamma}y(t))$ *a.e.* (I) such that

$$|f_1(t)-D^q y(t)|\leqslant p(t) \quad a.e.\ (I) \tag{3.4}$$

Define $x_1(t)=\int_1^T G_1(t,s)f_1(s)ds$ and one has $|x_1(t)-y(t)|\leqslant M_1\int_1^T p(t)dt$.

We point out that it is enough to construct the sequences $x_n(.)\in C(I,\mathbf{R})$, $f_n(.)\in L^1(I,\mathbf{R})$, $n\geqslant 1$, with the following properties

$$x_n(t)=\int_1^T G_1(t,s)f_n(s)ds, \quad t\in I, \tag{3.5}$$

$$f_n(t)\in F(t,x_{n-1}(t),I^{\gamma}x_{n-1}(t)) \quad a.e.\ (I), \tag{3.6}$$

$$|f_{n+1}(t)-f_n(t)|\leqslant L(t)(|x_n(t)-x_{n-1}(t)|+\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_n(s)-x_{n-1}(s)|ds) \tag{3.7}$$

for almost all $t\in I$.

Assume that this construction is done; then from (3.4)-(3.7) we have for almost all $t\in I$

$$|x_{n+1}(t)-x_n(t)|\leqslant M_1(M_1L_0)^n\int_1^T p(t)dt \quad \forall n\in\mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$|x_{n+1}(t)-x_n(t)| \leqslant \int_1^T |G_1(t,t_1)|.|f_{n+1}(t_1)-f_n(t_1)|dt_1$$
$$\leqslant M_1\int_1^T l(t_1)[|x_n(t_1)-x_{n-1}(t_1)|+\frac{1}{\Gamma(\gamma)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\gamma-1}\frac{1}{s}|x_n(s)-x_{n-1}(s)|ds]$$
$$\leqslant M_1\int_1^T l(t_1)(1+\frac{1}{\Gamma(\gamma)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\gamma-1}\frac{1}{s}ds)dt_1.M_1^nL_0^{n-1}\int_1^T p(t)dt$$
$$=M_1(M_1L_0)^n\int_1^T p(t)dt.$$

Thus, $\{x_n(.)\}$ is Cauchy in the Banach space $C(I,\mathbf{R})$, therefore, converging uniformly to some $x(.)\in C(I,\mathbf{R})$. Hence, by (3.7), for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Denote $f(.)$ the pointwise limit of $f_n(.)$.

At the same time, one has

$$|x_n(t)-y(t)| \leqslant |x_1(t)-y(t)|+\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)|$$
$$\leqslant M_1\int_1^T p(t)dt+\sum_{i=1}^{n-1}(M_1\int_1^T p(t)dt)(M_1L_0)^i=\frac{M_1\int_1^T p(t)dt}{1-M_1L_0}. \tag{3.8}$$

Moreover, from (3.4), (3.7) and (3.8) we obtain for almost all $t\in I$

$$|f_n(t)-D^qy(t)| \leqslant \sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-D^qy(t)| \leqslant L(t)\frac{M_1\int_1^T p(t)dt}{1-M_1L_0}+p(t).$$

In particular, the sequence $f_n(.)$ is integrably bounded and thus $f(.)\in L^1(I,\mathbf{R})$.

From Lebesgue's dominated convergence theorem and passing the limit in (3.5), (3.6) we obtain that $x(.)$ is a solution of (1.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(.)$.

In order to finish the proof it remains to realize the construction of the sequences $x_n(.),f_n(.)$ with the properties in (3.5)-(3.7). This will be done by induction.

Since the first step is already realized, assume that for some $N\geqslant 1$ we already constructed $x_n(.)\in C(I,\mathbf{R})$ and $f_n(.)\in L^1(I,\mathbf{R})$, $n=1,2,...N$ satisfying (3.5), (3.7) for $n=1,2,...N$ and (3.6) for $n=1,2,...N-1$. The set-valued map $t\to F(t,x_N(t),I^\gamma x_N(t))$ is measurable; as well as the map $t\to L(t)(|x_N(t)-x_{N-1}(t)|+\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_N(s)-x_{N-1}(s)|ds)$ is measurable. Since $F(t,.,.)$ is Lipschitz we have that for almost all $t\in I$

$$F(t,x_N(t),I^\gamma x_N(t))\cap\{f_N(t)+L(t)(|x_N(t)-x_{N-1}(t)|+\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_N(s)-x_{N-1}(s)|ds)[-1,1]\}\neq\emptyset.$$

Lemma 3 allows to find a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.),I^\gamma x_N(.))$ such that for almost all $t\in I$

$$|f_{N+1}(t)-f_N(t)|\leqslant L(t)(|x_N(t)-x_{N-1}(t)|+\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_N(s)-x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (3.5) with $n=N+1$. Thus $f_{N+1}(.)$ satisfies (3.6) and (3.7) and the proof is complete. □

The assumptions in Theorem 1 are satisfied, in particular, for $y(.)=0$ and therefore with $p(.)=l(.)$. We obtain the following consequence of Theorem 1.

COROLLARY 1. *Assume that Hypothesis H1 is satisfied,* $d(0,F(t,0,0)\leqslant L(t)$ *a.e.* (I), $q-r-1>0$, $\Lambda\neq 0$ *and* $M_1L_0<1$. *Then there exists* $x(.)$ *a solution of problem (1.1)-(1.2) satisfying for all* $t\in I$

$$|x(t)|\leqslant\frac{M_1}{1-M_1L_0}\int_1^T l(t)dt.$$

If F does not depend on the last variable, Hypothesis H1 becomes

HYPOTHESIS H2. i) $F(.,.):I\times\mathbf{R}\to\mathscr{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathscr{L}(I)\otimes\mathscr{B}(\mathbf{R})$ *measurable.*

ii) *There exists* $l(.)\in L^1(I,(0,\infty))$ *such that, for almost all* $t\in I$, $F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x_1),F(t,x_2))\leqslant l(t)|x_1-x_2|\quad\forall\,x_1,x_2\in\mathbf{R}.$$

Denote $M_0=\int_1^T l(t)dt$ and consider the fractional differential inclusion

$$D^qx(t)\in F(t,x(t))\quad a.e.\ ([1,T]),\tag{3.9}$$

COROLLARY 2. *Assume that Hypothesis H2 is satisfied,* $d(0,F(t,0)\leqslant L(t)$ *a.e.* (I), $q-r-1>0$, $\Lambda\neq 0$ *and* $M_1M_0<1$. *Then there exists* $x(.)$ *a solution of problem (3.9)-(1.2) satisfying for all* $t\in I$

$$|x(t)|\leqslant\frac{M_1M_0}{1-M_1M_0}.$$

REMARK 2. If in (3.9) F is single-valued, then a similar result to the one in Corollary 2 may be found in [3]; namely, Theorem 3.3.

We are concern next with problem (1.4)-(1.5). In what follows $I=[0,T]$ and we make the following notations

$$N(t):=l(t)(1+\int_0^t l(u)du),\quad t\in I,\quad N_0=\int_0^T N(t)dt.$$

THEOREM 2. *Assume that Hypothesis H1 is satisfied,* $q-p-1>0$, $A\neq 0$ *and* $M_2N_0<1$. *Consider* $y(.)\in C(I,\mathbf{R})$ *with its Caputo derivative of order* q *existing on* $[0,T]$ *such that* $y(0)=\delta y(\sigma)$, $aD_c^p y(\xi_1)+bD_c^p y(\xi_2)=\sum_{i=1}^{m-2}\alpha_i y(\beta_i)$ *and there exists* $q(.)\in L^1(I,\mathbf{R}_+)$ *verifying* $d(D_c^q y(t),F(t,y(t),V(y)(t)))\leqslant q(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem (1.4)-(1.5) satisfying for all* $t\in I$

$$|x(t)-y(t)|\leqslant \frac{M_2}{1-M_2N_0}\int_0^T q(t)dt.$$

Proof. The proof is similar to the proof of Theorem 1. □

If in Theorem 2, $y(.)=0$ and $q(.)=l(.)$ we get the following consequence of Theorem 2.

COROLLARY 3. *Assume that Hypothesis H1 is satisfied,* $d(0,F(t,0,0)\leqslant L(t)$ *a.e.* (I), $q-p-1>0$, $A\neq 0$ *and* $M_2N_0<1$. *Then there exists* $x(.)$ *a solution of problem (1.4)-(1.5) satisfying for all* $t\in I$

$$|x(t)|\leqslant \frac{M_2}{1-M_2N_0}\int_0^T l(t)dt.$$

Next F does not depend on the last variable. Set $K_0=\int_0^T l(t)dt$ and consider the fractional differential inclusion

$$D_c^q x(t)\in F(t,x(t)) \quad a.e.\ ([0,T]), \tag{3.10}$$

COROLLARY 4. *Assume that Hypothesis H2 is satisfied,* $d(0,F(t,0)\leqslant L(t)$ *a.e.* (I), $q-p-1>0$, $A\neq 0$ *and* $M_2K_0<1$. *Then there exists* $x(.)$ *a solution of problem (3.10)-(1.2) satisfying for all* $t\in I$

$$|x(t)|\leqslant \frac{M_2K_0}{1-M_2K_0}.$$

REMARK 3. If in (3.10), F is single-valued, then a similar result to the one in Corollary 4 is Theorem 1 in [1].

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