

ON AN INTEGRO–DIFFERENTIAL EQUATION OF ARBITRARY (FRACTIONAL) ORDERS WITH NONLOCAL INTEGRAL AND INFINITE–POINT BOUNDARY CONDITIONS

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(Communicated by C. Lizama)

Abstract. In this paper, we study the existence and uniqueness of solutions for an integro–differential equation of arbitrary (fractional) orders with nonlocal integral and infinite-point boundary conditions, continuous dependence of the solution on nonlocal data, on initial condition and on functional equation also will be study. An examples to prove main results.

1. Introduction

Boundary value problems for nonlinear differential equations (fractional differential equations) have attracted great research efforts worldwide, as they arise from the study of many important problems in various discipline areas such as fluid flows, electrical networks, rheology, biology and chemical physics. In practical applications, it is important to establish the conditions for the existence solutions. Hence, many authors have investigated the existence solutions for various functional differential equation (fractional differential equation) boundary value problems, and for details, the reader is referred to [1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 13, 14, 16, 19] and the references therein.

In [15], the authors proved the existence of absolute continuous solutions of the nonlocal first-order boundary value problem (BVP) with the nonlinear function involving the Liouville-Caputo fractional derivative:

$$\frac{dx}{dt} = f(t, D^\alpha x(t)), \quad a.e \quad t \in (0, 1), \quad 0 < \alpha \leq 1,$$

together with either the Riemann-Stieltjes functional integral boundary condition (with the advanced or deviated argument f) given by $\int_0^1 x(\phi(s))dg(s) = x_0$, or the infinite-point boundary conditions given by $\sum_{k=1}^{\infty} a_k x(\phi(\tau_k)) = x_0$.

Motivated by the above mentioned paper, the purpose of this paper is to investigate the existence solutions for a more general problem. Obviously, our work is different from those in [20, 21, 22].

Mathematics subject classification (2010): 26A33, 34B18, 34A30, 34K37, 34A08, 34B10.

Keywords and phrases: Functional equations, existence of solutions, continuous dependence, fractional–order, boundary value problem, infinite point condition, Liouville–Caputo fractional derivative.

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In this paper, we are concerned with the nonlocal problem for the integro-differential equation

$$\frac{dx}{dt} = f(t, x(t), \int_0^t g(s, D^\alpha x(s)) ds), \quad a.e \quad t \in (0, 1), \quad \alpha \in (0, 1], \tag{1}$$

with the nonlocal condition

$$\sum_{k=1}^m a_k x(\tau_k) = x_0, \quad a_k \geq 0, \quad \tau_k \in (0, 1). \tag{2}$$

The existence solution $x \in AC[0, T]$ of the nonlocal problem (1)-(2) we be proved. The continuous dependence of the solution on the nonlocal data a_k and on the function g , will be studied.

As applications, the nonlocal problem of equation (1) with the integral condition

$$\int_0^1 x(s) dg(s) = x_0, \tag{3}$$

will be studied. Also, the nonlocal problem of equation (1) with infinite-point boundary condition

$$\sum_{k=1}^\infty a_k x(\tau_k) = x_0, \tag{4}$$

will be studied. Finally, we give an examples to prove main results.

Main results in this paper are based onto Kolmogorov’s Compactness Criterion [6], Schauder’s Fixed Point Theorem [10] and the following definitions:

DEFINITION 1. [18] The fractional order integral of order α of $f \in L_1$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds.$$

DEFINITION 2. [18] The Caputo fractional-order derivative D_a^α of order $\alpha \in (0, 1]$ of the absolutely continuous function $f(t)$ is given by

$$D_a^\alpha f(t) = I^{1-\alpha} \frac{d}{dt} f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} f(s) ds.$$

2. Integral Representation

Consider the nonlocal problem (1)-(2) with the assumptions:

1. $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies Caratheodory condition i. e f is measurable in t for any $x, y \in \mathbb{R}$ and continuous in x, y for almost all $t \in [0, 1]$. There exist a function $a_1 \in L^1[0, 1]$ and a positive constant $b_1 > 0$, such that

$$|f(t, \eta, \varphi)| \leq a_1(t) + b_1 |\eta| + b_1 |\varphi|.$$

2. $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Caratheodory condition i. e g is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0, 1]$. There exist a function $c_2 \in L^1[0, 1]$ and a positive constant $b_2 > 0$, such that

$$|g(t, \eta)| \leq a_2(t) + b_2|\eta|.$$

3.

$$\sup_{t \in [0,1]} \int_0^t a_1(s)ds \leq M_1, \quad \sup_{t \in [0,1]} \int_0^t \int_0^s a(\theta)d\theta ds \leq M_2.$$

4. $(2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)}) < 1.$

LEMMA 1. Let $A = \sum_{k=1}^m a_k \neq 0$, the solution of the nonlocal problem (1)-(2), if it exist, then it can be expressed by the integral equation

$$x(t) = A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s)ds] + \int_0^t v(s)ds. \tag{5}$$

where

$$v(t) = f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s)ds] + \int_0^t v(s)ds, \int_0^t g(s, I^{1-\alpha}v(s))ds). \tag{6}$$

Proof. Let $\frac{dx}{dt} = v(t)$ in (1), we obtain

$$v(t) = f(t, x(t), \int_0^t g(s, I^{1-\alpha}v(s))ds), \quad t \in (0, 1), \tag{7}$$

where

$$x(t) = x(0) + \int_0^t v(s)ds, \tag{8}$$

Using the nonlocal condition (2), we get

$$\sum_{k=1}^m a_k x(\tau_k) = x(0) \sum_{k=1}^m a_k + \sum_{k=1}^m a_k \int_0^{\tau_k} v(s)ds,$$

then

$$x(0) = \frac{1}{\sum_{k=1}^m a_k} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s)ds]. \tag{9}$$

We obtain (5)and (6) from (7),(8) and (9). This completes the proof.

3. Existence of solution

DEFINITION 3. By a solution of the functional integral equation (6) we mean a function $v \in L_1[0, 1]$ that satisfies (6).

THEOREM 1. Let the assumptions 1–4 be satisfied, then the functional integral equation (6) has at least one solution $v \in L_1[0, 1]$.

Proof. Define the operator F associated with the $v \in L_1[0, 1]$ by

$$Fv(t) = f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s)ds] + \int_0^t v(s)ds, \int_0^t g(s, I^{1-\alpha}v(s))ds).$$

Let $\Omega_l = \{v \in \mathbb{R} : \|v\| \leq l\}$, where $l = \frac{M_3 + b_1 A^{-1}|x_0| + b_1 M_4}{1 - (2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)})}$.

Then we have, for $v \in \Omega_l$

$$\begin{aligned} \|Fv(t)\|_{L_1} &\leq \int_0^1 |f(s, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(\theta)d\theta] + \int_0^s v(\theta)d\theta, \\ &\quad \int_0^s g(\theta, I^{1-\alpha}v(\theta))d\theta)| ds. \\ &\leq \int_0^1 [a_1(s) + b_1 A^{-1}|x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(\theta)d\theta| \\ &\quad + b_1 |\int_0^s v(\theta)d\theta| + b_1 \int_0^s |g(\theta, I^{1-\alpha}v(\theta))d\theta|] ds \\ &\leq \int_0^1 a_1(s)ds + b_1 A^{-1}|x_0| + b_1 A^{-1} \sum_{k=1}^m a_k \int_0^t \int_0^{\tau_k} |v(\theta)|d\theta ds \\ &\quad + b_1 \int_0^t \int_0^s |v(\theta)|d\theta ds + b_1 \int_0^t \int_0^s a(\theta)d\theta ds \\ &\quad + b_1 b_2 \int_0^t \int_0^s I^{1-\alpha}|v(\theta)|d\theta ds \\ &\leq M_1 + b_1 A^{-1}|x_0| + 2b_1 \|v\|_{L_1} + b_1 M_2 \\ &\quad + b_1 b_2 \int_0^1 \int_0^s \int_0^\theta \frac{(\theta - \lambda)^{-\alpha}}{\Gamma(1-\alpha)} |v(\lambda)|d\lambda d\theta ds \\ &\leq M_1 + b_1 A^{-1}|x_0| + b_1 M_2 + (2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)})l = l \end{aligned}$$

This prove that $F : \Omega_l \rightarrow \Omega_l$ and the class of functions $\{Fv\}$ is uniformly bounded in Ω_l .

Let Ω be bounded subset of Ω_l and $F : \Omega_l \rightarrow \Omega_l$. Then $F(\Omega)$ is also bounded on Ω_l .

Let $v \in \Omega$, then

$$\begin{aligned} \|(Fv)_h - Fv\|_{L_1} &= \int_0^1 \left| \frac{1}{h} \int_{\theta}^{\theta+h} (Fv)(s) ds - (Fv)(\theta) \right| d\theta \\ &\leq \int_0^1 \frac{1}{h} \int_{\theta}^{\theta+h} |f(s, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{s_k} u(\lambda) d\lambda] + \int_0^s v(\lambda) d\lambda, \\ &\quad \int_0^s g(\lambda, I^{1-\alpha} v(\lambda)) d\lambda) - f(\theta, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\theta_k} u(\lambda) d\lambda] \\ &\quad + \int_0^{\theta} v(\lambda) d\lambda, \int_0^{\theta} g(\lambda, I^{1-\alpha} v(\lambda)) d\lambda)| ds d\theta, \end{aligned}$$

since $f \in L_1[0, T]$, It follows that

$$\begin{aligned} &\frac{1}{h} \int_{\theta}^{\theta+h} |f(s, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{s_k} u(\lambda) d\lambda] + \int_0^s v(\lambda) d\lambda, \\ &\quad \int_0^s g(\lambda, I^{1-\alpha} v(\lambda)) d\lambda) - f(\theta, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\theta_k} u(\lambda) d\lambda] + \int_0^{\theta} v(\lambda) d\lambda, \\ &\quad \int_0^{\theta} g(\lambda, I^{1-\alpha} v(\lambda)) d\lambda)| ds \rightarrow 0 \quad \text{as } h \rightarrow 0, \end{aligned}$$

then $(Fv)_h \rightarrow (Fv)$ uniformly. Hence $F(\Omega)$ is relatively compact. Hence F is compact operator.

Let $\{v_n\} \subset Q_l$ and $v_n \rightarrow v$

$$\begin{aligned} \lim_{n \rightarrow \infty} Fv_n &= \lim_{n \rightarrow \infty} f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v_n(s) ds] + \int_0^t v_n(s) ds, \\ &\quad \int_0^t g(s, I^{1-\alpha} v_n(s)) ds) = Fv. \end{aligned}$$

Then $v_n \rightarrow v \Rightarrow Fv_n \rightarrow Fv$ as $n \rightarrow \infty$. This mean that the operator F is continuous operator.

By Schauder fixed point Theorem [10] there exist at least one solution $v \in L_1[0, T]$ of the functional integral equations (6).

Thus, based on the Lemma 1, the nonlocal problem (1)–(2) possess a solution $x \in AC[0, 1]$. Now, from (5), we have

$$x(0) = \lim_{t \rightarrow 0} x(t) = x(t) = A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds],$$

and

$$x(1) = \lim_{t \rightarrow 1} x(t) = x(t) = A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^1 u(s) ds.$$

Therefor the Equation (5) has a solution $x \in AC[0, 1]$. Consequently, the nonlocal problem (1)–(2) has a solution $x \in AC[0, 1]$ given by (5).

4. Uniqueness of the solution

Let f and g satisfy the following assumptions

5. $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is measurable in t for any $x, y \in \mathbb{R}$ and satisfies the lipschitz condition

$$|f(t, x, y) - f(t, u, v)| \leq b_1|x - u| + b_1|y - v|,$$

6. $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable in t for any $x \in \mathbb{R}$ and satisfies the lipschitz condition

$$|g(t, x) - g(t, u)| \leq b_2|x - u|,$$

- 7.

$$\sup_{t \in [0, 1]} \int_0^t |f(s, 0, 0)| ds \leq L_1, \quad \sup_{t \in [0, 1]} \int_0^t \int_0^s |g(\theta, 0)| d\theta ds \leq L_2.$$

THEOREM 2. *Let the assumptions 5–7 be satisfied, then the solution of the functional integral equation (6) is unique $v \in L_1[0, 1]$.*

Proof. From Theorem 1 the solution of the integral equation (6) exists.

Let v, y be two the solution of (6), then

$$\begin{aligned} \int_0^1 |v(t) - y(t)| dt &= \int_0^1 |f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^t v(s) ds, \\ &\quad \int_0^t g(s, I^{1-\alpha} v(s)) ds) \\ &\quad - f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} y(s) ds] + \int_0^t y(s) ds, \\ &\quad \int_0^t g(s, I^{1-\alpha} y(s)) ds) | dt \\ &\leq b_1 \int_0^1 A^{-1} \sum_{k=1}^m a_k \int_0^{\tau_k} |v(s) - y(s)| ds dt \\ &\quad + b_1 \int_0^1 \int_0^t |v(s) - y(s)| ds dt \\ &\quad + b_1 b_2 \int_0^1 \int_0^t I^{1-\alpha} |v(s) - y(s)| ds dt \\ &\leq b_1 A^{-1} \sum_{k=1}^m a_k \int_0^1 \int_{\tau_k}^1 |v(s) - y(s)| dt ds \\ &\quad + b_1 \int_0^1 \int_t^1 |v(s) - y(s)| dt ds \\ &\quad + b_1 b_2 \int_0^1 \int_0^t \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} |v(s) - y(s)| d\theta ds dt \end{aligned}$$

$$\leq 2b_1 \|v - y\| + \frac{b_1 b_2}{\Gamma(2 - \alpha)} \|v - y\|.$$

Hence

$$(1 - (2b_1 + \frac{b_1 b_2}{\Gamma(2 - \alpha)})) \|v - y\| \leq 0.$$

since $(2b_1 + \frac{b_1 b_2}{\Gamma(2 - \alpha)}) < 1$, then $v(t) = y(t)$ and the solution of the functional integral equation (6) is unique.

Thus, based on the Lemma 1, the nonlocal problem (1)–(2) possess a unique solution $x \in AC[0, 1]$. Now, from (5), we have

$$x(0) = \lim_{t \rightarrow 0} x(t) = x(t) = A^{-1} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds],$$

and

$$x(1) = \lim_{t \rightarrow 1} x(t) = x(t) = A^{-1} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^1 u(s) ds.$$

Therefore, the Equation (5) has a unique solution $x \in AC[0, 1]$. Consequently, the nonlocal problem (1)–(2) has a unique solution $x \in AC[0, 1]$ given by (5).

5. Nonlocal integral condition

Let $x \in AC[0, 1]$ be the solution of the nonlocal problem (1)–(2). Let $a_k = h(t_k) - h(t_{k-1})$, h is increasing functions, $\tau_k \in (t_{k-1}, t_k)$, $0 = t_0 < t_1 < t_2, \dots < t_m = 1$ then, as $m \rightarrow \infty$ the nonlocal conditions (2) will be

$$\sum_{k=1}^m h(t_j) - h(t_{k-1}) x(\tau_k) = x_0.$$

And

$$\lim_{m \rightarrow \infty} \sum_{k=1}^m h(t_k) - h(t_{k-1}) x(\tau_k) = \int_0^1 x(s) dh(s) = x_0$$

THEOREM 3. *Let the assumptions 1–4 be satisfied, then the nonlocal problem of (1)–(3) has at least one absolutely continuous solution.*

Proof. As $m \rightarrow \infty$, the solution of the nonlocal problem (1)-(2) will be

$$\begin{aligned} x(t) &= \lim_{m \rightarrow \infty} A^{-1} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} u(s) ds] + \int_0^t v(s) ds \\ &= \frac{1}{h(1) - h(0)} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds (h(t_j) \\ &\quad - h(t_{k-1}))] + \int_0^t v(s) ds \\ &= \frac{1}{h(1) - h(0)} [x_0 - \int_0^1 \int_0^{\tau_k} v(s) ds dh(t)] + \int_0^t v(s) ds, \end{aligned}$$

where

$$\begin{aligned} v(t) &= f(t, \frac{1}{h(1) - h(0)} [x_0 - \int_0^1 \int_0^{\tau_k} v(s) ds dh(t)] \\ &\quad + \int_0^t v(s) ds, \int_0^t g(\theta, I^{1-\alpha} v(s)) d\theta) ds. \end{aligned}$$

6. Infinite-point boundary condition

THEOREM 4. *Let the assumptions 1-4 be satisfied, then the nonlocal problem of (1)-(4) has at least one absolutely continuous solution.*

Proof. Let the assumptions of Theorem 1 be satisfied.

Let $\sum_{k=1}^m a_k$ be convergent, then

$$x_m(t) = A^{-1} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v_m(s) ds] + \int_0^t v_m(s) ds. \tag{10}$$

Where

$$\begin{aligned} v_m(t) &= f(t, A^{-1} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v_m(\theta) d\theta] \\ &\quad + \int_0^t v_m(s) ds, \int_0^t g(s, I^{1-\alpha} v_m(s)) ds). \end{aligned}$$

Take the limit to (10), as $m \rightarrow \infty$, we have

$$\lim_{m \rightarrow \infty} x_m(t) = \lim_{m \rightarrow \infty} \frac{1}{\sum_{k=1}^m a_k} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v_m(s) ds] + \int_0^t v_m(s) ds. \tag{11}$$

Where

$$\begin{aligned} \lim_{m \rightarrow \infty} v_m(t) &= \lim_{m \rightarrow \infty} f(t, \frac{1}{\sum_{k=1}^m a_k} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v_m(\theta) d\theta] \\ &\quad + \int_0^t v_m(s) ds, \int_0^t g(s, I^{1-\alpha} v_m(s)) ds) \end{aligned}$$

Now

$$|a_k x(\tau_k)| \leq |a_k| \|x\|, \quad |a_k \int_0^{\tau_k} v(s) ds| \leq |a_k| \|v\|,$$

then by comparison test $\sum_{k=1}^{\infty} a_k x(\tau_k)$, $\sum_{k=1}^{\infty} a_k \int_0^{\tau_k} v(s) ds$ are convergent.

Using assumptions 1-2 and Lebesgue Dominated convergence Theorem [17], from (11) we obtain

$$x(t) = \frac{1}{\sum_{k=1}^{\infty} a_k} \left[x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} v(s) ds \right] + \int_0^t v(s) ds.$$

Where

$$v(t) = f(t, \frac{1}{\sum_{k=1}^{\infty} a_k} \left[x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} v(s) ds \right] + \int_0^t v(s) ds, \int_0^t g(s, I^{1-\alpha} v(s)) ds)$$

The Theorem proved.

7. Continuous dependence

7.1. Continuous dependence on x_0

DEFINITION 4. The solution $x \in AC[0, 1]$ of the nonlocal problem (1)-(2) depends continuously on x_0 , if

$$\forall \varepsilon > 0, \quad \exists \quad \delta(\varepsilon) \quad s.t \quad |x_0 - x_0^*| < \delta \Rightarrow \|x - x^*\|_C < \varepsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{dx^*}{dt} = f(t, x^*(t), \int_0^t g(s, D^\alpha x^*(s)) ds), \quad a.e \quad t \in (0, 1), \tag{12}$$

with the nonlocal condition

$$\sum_{k=1}^n a_k x^*(\tau_k) = x_0^*, \quad a_k \geq 0, \quad \tau_k \in (0, 1). \tag{13}$$

THEOREM 5. Let the assumptions of Theorem 2 be satisfied, then the solution of the nonlocal problem (1)-(2) depends continuously on x_0 .

Proof. Let x, x^* be two solutions of the nonlocal problem (1)-(2) and (12)-(13) respectively. Then

$$\begin{aligned}
 \int_0^1 |v(t) - v^*(t)| dt &= \int_0^1 |f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^t v(s) ds, \\
 &\quad \int_0^t g(s, I^{1-\alpha} v(s)) ds \\
 &\quad - f(t, A^{-1}[x_0^* - \sum_{k=1}^m a_k \int_0^{\tau_k} v^*(s) ds] + \int_0^t v^*(s) ds, \\
 &\quad \int_0^t g(s, I^{1-\alpha} v^*(s)) ds)| dt \\
 &\leq b_1 A^{-1} |x_0 - x_0^*| + b_1 \int_0^1 A^{-1} \sum_{k=1}^m a_k \int_0^{\tau_k} |v(s) - v^*(s)| ds dt \\
 &\quad + b_1 \int_0^1 \int_0^t |v(s) - v^*(s)| ds dt + b_1 b_2 \int_0^1 \int_0^t I^{1-\alpha} |v(s) - v^*(s)| ds dt \\
 &\leq b_1 A^{-1} \delta + b_1 A^{-1} \sum_{k=1}^m a_k \int_0^1 \int_{\tau_k}^1 |v(s) - v^*(s)| dt ds \\
 &\quad + b_1 \int_0^1 \int_t^1 |v(s) - v^*(s)| dt ds \\
 &\quad + b_1 b_2 \int_0^1 \int_0^t \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} |v(s) - v^*(s)| d\theta ds dt \\
 &\leq b_1 A^{-1} \delta + 2b_1 \|v - v^*\|_{L_1} + \frac{b_1 b_2}{\Gamma(2-\alpha)} \|v - v^*\|_{L_1}.
 \end{aligned}$$

Hence

$$\|v - v^*\|_{L_1} \leq \frac{b_1 A^{-1} \delta}{1 - (2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)})}.$$

And

$$\begin{aligned}
 |x(t) - x^*(t)| &= |A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^t v(s) ds \\
 &\quad - A^{-1}[x_0^* - \sum_{k=1}^m a_k \int_0^{\tau_k} v^*(s) ds] + \int_0^t v^*(s) ds| \\
 &\leq A^{-1} |x_0 - x_0^*| + 2\|v - v^*\|_{L_1}.
 \end{aligned}$$

Hence

$$\|x - x^*\|_C \leq A^{-1} \delta + \frac{2b_1 A^{-1} \delta}{1 - (2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)})} = \varepsilon.$$

Therefore the solution of the nonlocal problem (1)-(2) depends continuously on x_0 .

7.2. Continuous dependence on a_k

DEFINITION 5. The solution $x \in AC[0, 1]$ of the nonlocal problem (1)-(2) depends continuously on a_k , if

$$\forall \varepsilon > 0, \quad \exists \quad \delta(\varepsilon) \quad s.t \quad |a_k - a_k^*| < \delta \Rightarrow \|x - x^*\|_C < \varepsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{dx^*}{dt} = f(t, x^*(t), \int_0^t g(s, D^\alpha x^*(s)) ds), \quad a.e \quad t \in (0, 1), \tag{14}$$

with the nonlocal condition

$$\sum_{k=1}^n a_k^* x^*(\tau_k) = x_0, \quad a_k \geq 0, \quad \tau_k \in (0, 1). \tag{15}$$

THEOREM 6. Let the assumptions of Theorem 2 be satisfied, then the solution of the nonlocal problem (1)-(2) depends continuously on a_k .

Proof. Let $B^* = \sum_{k=1}^n a_k^* \neq 0$ and x, x^* be two solutions of the nonlocal problem (1)-(2) and (14)-(15) respectively. Then

$$\begin{aligned} \int_0^1 |v(t) - v^*(t)| dt &= \int_0^1 |f(t, \frac{1}{\sum_{k=1}^m a_k} [x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^t v(s) ds, \\ &\quad \int_0^t g(s, I^{1-\alpha} v(s)) ds \\ &\quad - f(t, \frac{1}{\sum_{k=1}^m a_k^*} [x_0 - \sum_{k=1}^m a_k^* \int_0^{\tau_k} v^*(s) ds] + \int_0^t v^*(s) ds, \\ &\quad \int_0^t g(s, I^{1-\alpha} v^*(s)) ds)| dt \\ &\leq \frac{m\delta|x_0|b_1}{A A^*} + b_1 \int_0^1 |A^{-1} \sum_{k=1}^m a_k \int_0^{\tau_k} |v(s) \\ &\quad - A^{-1} \sum_{k=1}^m a_k^* \int_0^{\tau_k} v^*(s)| ds dt + b_1 \int_0^1 \int_0^t |v(s) - v^*(s)| ds dt \\ &\quad + b_1 b_2 \int_0^1 \int_0^t I^{1-\alpha} |v(s) - v^*(s)| ds dt \\ &\leq \frac{m\delta|x_0|b_1}{A A^*} + 2b_1 A^{-1} l m \delta + 2b_1 \|v - v^*\|_{L_1} + \frac{b_1 b_2}{\Gamma(2-\alpha)} \|v - v^*\|_{L_1}. \end{aligned}$$

Hence

$$\|v - v^*\|_{L_1} \leq \frac{m\delta|x_0|b_1 A^{-1} A^{-1*} + 2b_1 A^{-1} l m \delta}{1 - (2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)})}$$

And

$$\begin{aligned} |x(t) - x^*(t)| &= |A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s)ds] + \int_0^t v(s)ds \\ &\quad - A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v^*(s)ds] + \int_0^t v^*(s)ds| \\ &\leq 2\|v - v^*\|_{L_1}. \end{aligned}$$

Hence

$$\|x - x^*\|_C \leq \frac{2m\delta|x_0|b_1A^{-1}A^{-1*} + 4b_1A^{-1}lm\delta}{1 - (2b_1 + \frac{b_1b_2}{\Gamma(2-\alpha)})} = \varepsilon.$$

This mean that the solution of the nonlocal problem (1)-(2) depends continuously on a_k . The proof is completed.

7.3. Continuous dependence on the functional g

DEFINITION 6. The solution $x \in AC[0, 1]$ of the nonlocal problem (1)-(2) depends continuously on the functional g , if

$$\forall \varepsilon > 0, \quad \exists \quad \delta(\varepsilon) \quad s.t \quad |g - g^*| < \delta \Rightarrow \|x - x^*\|_C < \varepsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{dx^*}{dt} = f(t, x^*(t), \int_0^t g^*(s, x^*(s))ds), \quad a.e \quad t \in (0, 1), \quad (16)$$

with the nonlocal condition

$$\sum_{k=1}^n a_k x^*(\tau_k) = x_0, \quad a_k \geq 0, \quad \tau_k \in (0, 1). \quad (17)$$

THEOREM 7. Let the assumptions of Theorem 2 be satisfied, then the solution of the nonlocal problem (1)-(2) depends continuously on the functional g .

Proof. Let x, x^* be two solutions of the nonlocal problem (1)-(2) and (16)-(17) respectively. Then

$$\begin{aligned}
 \int_0^1 |v(t) - v^*(t)| dt &= \left| \int_0^1 [f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^t v(s) ds, \right. \\
 &\quad \left. \int_0^t g(s, I^{1-\alpha} v(s)) ds \right. \\
 &\quad \left. - f(t, A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v^*(s) ds] + \int_0^t v^*(s) ds, \right. \\
 &\quad \left. \int_0^t g^*(s, I^{1-\alpha} v^*(s)) ds \right| dt \\
 &\leq b_1 \int_0^1 A^{-1} \sum_{k=1}^m a_k \int_0^{\tau_k} |v(s) - v^*(s)| ds dt \\
 &\quad + b_1 \int_0^1 \int_0^t |v(s) - v^*(s)| ds dt \\
 &\quad + b_1 \int_0^1 \int_0^t (g(s, I^{1-\alpha} v(s)) - g^*(s, I^{1-\alpha} v^*(s))) ds dt \\
 &\leq b_1 A^{-1} \sum_{k=1}^m a_k \int_0^1 \int_{\tau_k}^1 |v(s) - v^*(s)| dt ds \\
 &\quad + b_1 \int_0^1 \int_t^1 |v(s) - v^*(s)| dt ds \\
 &\quad + b_1 \delta + b_1 b_2 \int_0^1 \int_0^t \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} |v(s) - v^*(s)| d\theta ds dt \\
 &\leq b_1 \delta + 2b_1 \|v - v^*\|_{L_1} + \frac{b_1 b_2}{\Gamma(2-\alpha)} \|v - v^*\|_{L_1}
 \end{aligned}$$

Hence

$$\|v - v^*\|_{L_1} \leq \frac{b_1 \delta}{1 - (2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)})}.$$

And

$$\begin{aligned}
 |x(t) - x^*(t)| &= |A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v(s) ds] + \int_0^t v(s) ds \\
 &\quad - A^{-1}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} v^*(s) ds] + \int_0^t v^*(s) ds| \\
 &\leq 2 \|v - v^*\|_{L_1}.
 \end{aligned}$$

Hence

$$\|x - x^*\|_C \leq \frac{2b_1 \delta}{1 - (2b_1 + \frac{b_1 b_2}{\Gamma(2-\alpha)})} = \varepsilon.$$

This mean that the solution of the nonlocal problem (1)-(2) depends continuously on the functional g . The proof is completed.

8. Examples:

In this section we offer some examples to illustrate our results

EXAMPLE 1. Consider the following nonlinear integro-differential equation

$$\frac{dx}{dt} = t^3 e^t + \frac{\ln(1+x(t))}{3+t^2} + \int_0^t \frac{1}{9} (\cos(3s+3) + e^s D^{\frac{1}{2}} x(s)) dt, \quad a.e \quad t \in (0, 1), \quad (18)$$

with infinite point boundary condition

$$\sum_{k=1}^{\infty} \frac{1}{k^3} x\left(\frac{k^2+2k-1}{k^2+2k}\right) = x_0. \quad (19)$$

Set

$$\begin{aligned} f(t, v(t), \int_0^t g(s, I^{1-\alpha} v(s)) ds) &= t^3 e^t \\ &+ \frac{\ln\left(1 + \frac{1}{\sum_{k=1}^{\infty} \frac{1}{k^3}} [x_0 - \sum_{k=1}^{\infty} \frac{1}{k^3} \int_0^k v(s) ds] + \int_0^t v(s) ds\right)}{3+t^2} \\ &+ \int_0^t \frac{1}{9} (\cos(3s+3) + e^s I^{\frac{1}{2}} v(s)) dt. \end{aligned}$$

Then

$$\begin{aligned} |f(t, v(t), \int_0^t g(s, I^{1-\alpha} v(s)) ds)| &= |t^3 e^t| + \frac{1}{3} |v(s)| \\ &+ \int_0^t \frac{1}{3} |\cos(3s+3) + e^s I^{\frac{1}{2}} v(s)| dt, \end{aligned}$$

and also

$$|g(s, I^{1-\alpha} v(s))| \leq \frac{1}{3} |\cos(3s+3)| + \frac{1}{3} |v(s)|.$$

It is clear that the assumptions 1-4 of Theorem 1 are satisfied with $c_1(t) = t^3 e^t \in L^1[0, 1]$, $c_2(t) = \frac{1}{2} |\cos(3t+3)| \in L^1[0, 1]$, $b_1 = \frac{1}{3}$, $b_2 = \frac{1}{3}$, $\frac{2}{3} + \frac{1}{\Gamma(\frac{3}{2})} < 1$, and the series: $\sum_{k=1}^{\infty} \frac{1}{k^3}$, is convergent. Therefore, by applying to Theorem 1, the given nonlocal problem (18)-(19) has an absolutely continuous solution.

EXAMPLE 2. Consider the following nonlinear integro-differential equation

$$\begin{aligned} \frac{dx}{dt} &= t^3 + t^2 + \frac{x(t)}{\sqrt{t+9}} + \frac{1}{9} \int_0^t (\sin^2(3s+3) \\ &+ \frac{s D^{\frac{1}{2}} x(s)}{2(1+D^{\frac{1}{2}} x(s))}) dt, \quad a.e \quad t \in (0, 1), \quad (20) \end{aligned}$$

with infinite point boundary condition

$$\sum_{k=1}^{\infty} \frac{1}{k^5} x \left(\frac{k^2 + 3k - 1}{k^2 + 3k} \right) = x_0. \quad (21)$$

Set

$$\begin{aligned} f(t, v(t), \int_0^t g(s, I^{1-\alpha} v(s)) ds) &= t^3 + t^2 \\ &+ \frac{\ln(1 + \frac{1}{\sum_{k=1}^{\infty} \frac{1}{k^3}} [x_0 - \sum_{k=1}^{\infty} \frac{1}{k^3} \int_0^t v(s) ds]) + \int_0^t v(s) ds}{t + 9} \\ &+ \frac{1}{9} \int_0^t (\sin^2(3s + 3) + \frac{s I^{\frac{1}{2}} v(s)}{2(1 + I^{\frac{1}{2}} v(s))}) dt. \end{aligned}$$

Then

$$\begin{aligned} |f(t, v(t), \int_0^t g(s, I^{1-\alpha} v(s)) ds)| &\leq t^3 + t^2 + \frac{1}{3} |v| + \frac{1}{3} \int_0^t \frac{1}{3} |(\sin^2(3s + 3) \\ &+ \frac{s I^{\frac{1}{2}} v(s)}{2(1 + I^{\frac{1}{2}} v(s))})| dt, \end{aligned}$$

and also

$$|g(s, I^{\frac{1}{2}} v(s))| \leq \frac{1}{3} |(\sin^2(3s + 3) + \frac{1}{6} |v(s)|).$$

It is clear that the assumptions 1–4 of Theorem 1 are satisfied with $c_1(t) = t^3 + t^2 \in L^1[0, 1]$, $c_2(t) = \frac{1}{3} |(\sin^2(3s + 3))| \in L^1[0, 1]$, $b_1 = \frac{1}{3}$, $b_2 = \frac{1}{6}$, $\frac{2}{3} + \frac{\frac{1}{18}}{\Gamma(\frac{3}{2})} < 1$, and the series: $\sum_{k=1}^{\infty} \frac{1}{k^3}$, is convergent. Therefore, by applying to Theorem 1, the given nonlocal problem (20)-(21) has an absolutely continuous solution.

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(Received March 10, 2019)

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