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SERIES SOLUTION METHOD FOR CAUCHY PROBLEMS WITH FRACTIONAL Δ–DERIVATIVE ON TIME SCALES

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Abstract. In this paper we introduce a series solution method for Cauchy problems associated with Caputo fractional delta derivatives on time scales with delta differentiable graininess function. We also apply the method to Cauchy problems associated with dynamic equations and present some illustrative examples.

1. Introduction

Calculus on time scales and fractional calculus are two topics which gained a great interest in the past decades. Calculus on time scales unifies the discrete and continuous cases while fractional calculus deals with differentiation and integration of fractional order. These two topics have been combined recently and some studies related with fractional calculus on time scales appeared in the literature. Among the most detailed and thorough references on the subject we should mention a very recent book by Georgiev [4].

In this paper we introduce the series solution method for a Cauchy problem with Caputo fractional Δ-derivative on time scales with delta differentiable graininess function. After a brief introduction of fractional calculus on time scales we present the series solution method. We also describe the series solution method for a Cauchy problem associated with dynamic equations on time scales. As an application we consider some specific examples.

2. Basic notions of time scales and fractional calculus on time scales

In this section we briefly define some basic concepts on time scales to be used throughout the paper. For detailed information on time scale calculus we refer the reader to [1, 2, 3].

DEFINITION 1. A time scale is an arbitrary nonempty closed subset of the real numbers and is usually denoted by the symbol $\mathbb{T}$.

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DEFINITION 2. For a time scale $\mathbb{T}$,

1. the forward jump operator $\sigma : \mathbb{T} \longmapsto \mathbb{T}$ is defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

2. the backward jump operator $\rho : \mathbb{T} \longmapsto \mathbb{T}$ is defined as

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

3. the set $\mathbb{T}^\kappa$ is defined as

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if} \quad \sup\mathbb{T} < \infty \\ \mathbb{T} \quad \text{otherwise,} \end{cases}$$

4. the graininess function $\mu : \mathbb{T} \longmapsto [0,\infty)$ is defined as

$$\mu(t) = \sigma(t) - t.$$

We note that $\sigma(t) \geqslant t$ for any $t \in \mathbb{T}$ and $\rho(t) \leqslant t$ for any $t \in \mathbb{T}$. Here we set

$$\inf \emptyset = \sup \mathbb{T}, \quad \sup \emptyset = \inf \mathbb{T}.$$

DEFINITION 3. Assume that $f : \mathbb{T} \longmapsto \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. We define $f^\Delta(t)$ to be the number, provided it exists, as follows: for any $\varepsilon > 0$ there is a neighborhood U of t, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ with $\delta > 0$, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for} \quad \text{all} \quad s \in U, \quad s \neq \sigma(t).$$

We say $f^\Delta(t)$ the *delta* or *Hilger* derivative of f at t.

We say that f is *delta or Hilger differentiable*, shortly *differentiable* in $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T} \longmapsto \mathbb{R}$ is said to be the delta derivative or Hilger derivative, shortly derivative, of f in $\mathbb{T}^\kappa$.

REMARK 1. If $\mathbb{T} = \mathbb{R}$, then the delta derivative coincides with the classical derivative.

Note that the delta derivative is well-defined. For the properties of the delta derivative we refer the reader to [1], [2] and [3].

DEFINITION 4. 1. A function $f : \mathbb{T} \longmapsto \mathbb{R}$ is called regulated provided that its right-sided limits exist(finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at all left-dense points in $\mathbb{T}$.

2. A continuous function $f:\mathbb{T}\longmapsto\mathbb{R}$ is called pre-differentiable with region of differentiation D, provided that

   (a) $D\subset\mathbb{T}^{\kappa}$,

   (b) $\mathbb{T}^{\kappa}\backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$,

   (c) f is differentiable at each $t\in D$.

THEOREM 1. ([1],[2],[3]) *Let $t_0\in\mathbb{T}$, $x_0\in\mathbb{R}$, $f:\mathbb{T}^{\kappa}\longmapsto\mathbb{R}$ be given regulated map. Then there exists exactly one pre-differentiable function F satisfying*

$$F^{\Delta}(t)=f(t)\quad\text{for}\quad\text{all}\quad t\in D,\quad F(t_0)=x_0.$$

DEFINITION 5. Assume that $f:\mathbb{T}\longmapsto\mathbb{R}$ is a regulated function. Any function F by Theorem 1 is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t=F(t)+c,$$

where c is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_{\tau}^{s}f(t)\Delta t=F(s)-F(\tau)\quad\text{for}\quad\text{all}\quad\tau,s\in\mathbb{T}.$$

A function $F:\mathbb{T}\longmapsto\mathbb{R}$ is called an antiderivative of $f:\mathbb{T}\longmapsto\mathbb{R}$ provided

$$F^{\Delta}(t)=f(t)\quad\text{holds}\quad\text{for}\quad\text{all}\quad t\in\mathbb{T}^{\kappa}.$$

For properties of the delta integral we refer the reader to [1], [2] and [3].

Let $\mathbb{T}$ be a time scale with forward jump operator σ, graininess function μ and delta differential operator Δ.

DEFINITION 6. The time scale monomials $h_k(t,t_0)$, $k\in\mathbb{N}_0$, on the time scale $\mathbb{T}$ are defined as follows

$$\begin{aligned}h_0(t,t_0)&=1,\\ h_{k+1}(t,t_0)&=\int_{t_0}^{t}h_k(\tau,t_0)\Delta\tau,\end{aligned}$$

for $t,t_0\in\mathbb{T}$ and $k\in\mathbb{N}_0$.

Note that $h_k^{\Delta}(t,t_0)=h_{k-1}(t,t_0),\quad t,t_0\in\mathbb{T},\quad k\in\mathbb{N}.$

THEOREM 2. ([1, 3]) *(Taylor's Formula) Let $n\in\mathbb{N}$. Suppose f is n times Δ-differentiable on $\mathbb{T}^{\kappa^n}$. Let also, $t_0\in\mathbb{T}^{\kappa^{n-1}}$, $t\in\mathbb{T}$. Then*

$$f(t)=\sum_{k=0}^{n-1}h_k(t,t_0)f^{\Delta^k}(t_0)+\int_{t_0}^{\rho^{n-1}(t)}h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau.$$

THEOREM 3. ([1, 3]) *(Leibnitz Formula) Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n-k$ times Δ. If*

$$f^{\Lambda} \quad \text{exists} \quad \text{for} \quad \text{all} \quad \Lambda \in S_k^{(n)},$$

then

$$(fg)^{\Delta^n} = \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k}.$$

THEOREM 4. [3] *For every $m,n \in \mathbb{N}_0$ we have*

$$h_n(t,t_0)h_m(t,t_0) = \sum_{l=m}^{m+n} \left(\sum_{\Lambda_{l,m} \in S_m^{(l)}} h_n^{\Lambda_{l,m}}(t_0,t_0) \right) h_l(t,t_0)$$

for every $t,t_0 \in \mathbb{T}$.

We next recall some more notions and the definition of generalized exponential function on time scales.

DEFINITION 7. We say that a function $f:\mathbb{T} \to \mathbb{R}$ is regressive provided

$$1+\mu(t)f(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}$$

holds. The set of all regressive and rd-continuous functions $f:\mathbb{T} \to \mathbb{R}$ is denoted by $\mathscr{R}(\mathbb{T})$ or $\mathscr{R}$.

DEFINITION 8. 1. In $\mathscr{R}$, we define the "circle plus" addition $\oplus$ by

$$(f \oplus g)(t) = f(t)+g(t)+\mu(t)f(t)g(t).$$

2. The group $(\mathscr{R},\oplus)$ is called the regressive group.
3. For $f \in \mathscr{R}$, we define

$$(\ominus f)(t) = -\frac{f(t)}{1+\mu(t)f(t)} \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}.$$

4. In $\mathscr{R}$, we define the"circle minus" subtraction $\ominus$ by

$$(f \ominus g)(t) = (f \oplus (\ominus g))(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}.$$

Note that for $f,g \in \mathscr{R}$, we have

$$f \ominus g = \frac{f-g}{1+\mu g}.$$

Finally, we recall the definition of generalized exponential function. Let $h>0$. The Hilger complex numbers are defined by

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : \ z \neq -\frac{1}{h} \right\}.$$

Let

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : \ -\frac{\pi}{h} < \operatorname{Im}(z) \leqslant \frac{\pi}{h} \right\}.$$

For $h=0$, we set $\mathbb{Z}_0 := \mathbb{C}$ and $\mathbb{C}_0 := \mathbb{C}$. For $h>0$, we define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) := \frac{1}{h}\operatorname{Log}(1+zh),$$

where Log is the principal logarithm function. For $h=0$, we define $\xi_0(z)=z$ for all $z \in \mathbb{C}$.

DEFINITION 9. If $f \in \mathscr{R}$, then we define the generalized exponential function by

$$e_f(t,s) = e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} \quad \text{for} \quad s,t \in \mathbb{T}.$$

In fact, using the definition for the cylindrical transformation, we have

$$e_f(t,s) = e^{\int_s^t \frac{1}{\mu(\tau)}\operatorname{Log}(1+\mu(\tau)f(\tau))\Delta\tau} \quad \text{for} \quad s,t \in \mathbb{T}.$$

For more details and the properties of the generalized exponential function, we refer the reader to [1, 3]. Next we recall the Laplace transform on time scales.

DEFINITION 10. [3] Let $\mathbb{T}_0$ be a time scale such that $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = \infty$. Let $f : \mathbb{T}_0 \to \mathbb{C}$ and define the set

$$\mathscr{D}(f) = \{ z \in \mathbb{C} : \ 1+z\mu(t) \neq 0 \text{ for all } t \in \mathbb{T}_0$$

$$\text{and the improper integral } \int_0^\infty f(y)e^{\sigma}_{\ominus z}(y,0)\Delta y \text{ exists} \Big\},$$

where $e^{\sigma}_{\ominus z}(y,0) = (e_{\ominus z} \circ \sigma)(y,0) = e_{\ominus z}(\sigma(y),0)$.

The Laplace transform of the function f is defined as

$$\mathscr{L}(f)(z) = \int_0^\infty f(y)e^{\sigma}_{\ominus z}(y,0)\Delta y, \tag{1}$$

for all $z \in \mathscr{D}(f)$.

Other concepts needed in the definition of fractional Δ-derivative are the shift of a function and convolution of two functions on time scale.

DEFINITION 11. [4] For a given function $f:\mathbb{T}\to\mathbb{C}$ the shift (delay) of f is denoted by $\widehat{f}$ and defined as the solution of the shifting problem

$$\begin{aligned} u^{\Delta_t}(t,\sigma(s)) &= -u^{\Delta_s}(t,s), \quad t\in\mathbb{T}, \quad t\geqslant s\geqslant t_0, \\ u(t,t_0) &= f(t), \quad t\in\mathbb{T}, \quad t\geqslant t_0. \end{aligned} \tag{2}$$

EXAMPLE 1. [4]

1. Let $f:\mathbb{T}\to\mathbb{C}$ be any function where $\mathbb{T}$ is either $\mathbb{R}$ or $\mathbb{Z}$. Then the shift of f is

$$\widehat{f(\cdot)}(t,s)=f(t-s+t_0), \quad t\geqslant s\geqslant t_0.$$

2. The shift of $e_\lambda(t,t_0)$, where $t,t_0\in\mathbb{T}$ and $t\geqslant t_0$ is

$$\widehat{e_\lambda(\cdot,t_0)}(t,s)=e_\lambda(t,s), \quad t,s\in\mathbb{T} \text{ and are independent of } t_0.$$

3. Let $f:[t_0,\infty]\to\mathbb{C}$ be a function of the form

$$f(t)=\sum_{k=0}^{\infty}a_k h_k(t,t_0),$$

where the coefficients a_k satisfy

$$|a_k|\leqslant MR^k,$$

for some $M,R>0$ and $k\in\mathbb{N}_0$. Then the shift of f is in the form

$$\widehat{f(\cdot)}(t,s)=\sum_{k=0}^{\infty}a_k h_k(t,s), \quad t,s\in\mathbb{T}, \quad t\geqslant s\geqslant t_0.$$

In particular, we have

$$\widehat{h_k(\cdot,t_0)}(t,s)=h_k(t,s), \quad t,s\in\mathbb{T}, \quad t\geqslant s\geqslant t_0 \text{ and } k\in\mathbb{N}_0.$$

DEFINITION 12. [4] For the function $f,g:\mathbb{T}\to\mathbb{C}$, the convolution $f*g$ is defined as

$$(f*g)(t)=\int_{t_0}^{t}\widehat{f}(t,\sigma(s))g(s)\Delta s, \quad t\in\mathbb{T}, \quad t\geqslant t_0. \tag{3}$$

The convolution is associative, that is, $(f*g)*h=f*(g*h)$. For a detailed overview on Laplace transform, shifts and convolutions we refer the reader to [4].

In the following, we will suppose that $\mathbb{T}$ is a time scale with forward jump operator σ, graininess function μ and delta differential operator Δ and that $\mathbb{T}$ has the form

$$\mathbb{T}=\{t_n:\ n\in\mathbb{N}_0\},$$

where

$$\begin{aligned} \lim_{n\to\infty} t_n &= \infty, \\ \sigma(t_n) &= t_{n+1}, n \in \mathbb{N}_0, \\ w &= \inf_{n\in\mathbb{N}_0} \mu(t_n) > 0. \end{aligned}$$

We define next the generalized Δ-power function, the Riemann-Liouville fractional Δ-integral and Δ-derivative and the Caputo fractional Δ-derivative on the time scale $\mathbb{T}$ in the form given above. Let $\alpha \in \mathbb{R}$.

DEFINITION 13. [4] The generalized Δ-power function $h_\alpha(t,t_0)$ on $\mathbb{T}$ is defined as

$$h_\alpha(t,t_0) = \mathscr{L}^{-1}\left(\frac{1}{z^{\alpha+1}}\right)(t), \quad t \geqslant t_0,$$

for all $z \in \mathbb{C}\setminus\{0\}$ such that $\mathscr{L}^{-1}$ exists.

The fractional generalized Δ-power function $h_\alpha(t,s)$ on $\mathbb{T}$ is defined as the shift of $h_\alpha(t,t_0)$, that is,

$$h_\alpha(t,s) = \widehat{h_\alpha(\cdot,t_0)}(t,s), \quad t,s \in \mathbb{T}, \quad t \geqslant s \geqslant t_0.$$

The series solution method presented in the next section employs the following property of the generalized Δ-power functions.

THEOREM 5. [4] *Let $\alpha, \beta \in \mathbb{R}$. Then*

$$\left(h_\alpha(\cdot,t_0) * h_\beta(\cdot,t_0)\right)(t) = h_{\alpha+\beta+1}(t,t_0), \quad t \in \mathbb{T}.$$

DEFINITION 14. [4] Let $\alpha \geqslant 0$ and $\overline{[-\alpha]}$ denote the integer part of $-\alpha$. For a function $f : \mathbb{T} \to \mathbb{R}$ the Riemann-Liouville fractional Δ-integral of order α is defined as

$$\begin{aligned} (I^0_{\Delta,t_0} f)(t) &= f(t), \\ (I^\alpha_{\Delta,t_0} f)(t) &= (h_{\alpha-1}(\cdot,t_0) * f)(t) \\ &= \int_{t_0}^{t} \widehat{h_{\alpha-1}(\cdot,t_0)}(t,\sigma(u)) f(u)\Delta u \\ &= \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(u)) f(u)\Delta u, \end{aligned} \tag{4}$$

for $\alpha > 0$ and $t \geqslant t_0$.

DEFINITION 15. [4] Let $\alpha \geqslant 0$, $m = -\overline{[-\alpha]}$ and $f : \mathbb{T} \to \mathbb{R}$. For $s,t \in \mathbb{T}^{\kappa^m}$, $s < t$ the Riemann-Liouville fractional Δ-derivative of order α is defined by

$$D^\alpha_{\Delta,s} f(t) = D^m_\Delta I^{m-\alpha}_{\Delta,s} f(t), \quad t \in \mathbb{T}, \tag{5}$$

if it exists. For $\alpha < 0$, we define

$$\begin{aligned} D^\alpha_{\Delta,s} f(t) &= I^{-\alpha}_{\Delta,s} f(t), \quad t,s \in \mathbb{T}, \quad t > s. \\ I^\alpha_{\Delta,s} f(t) &= D^{-\alpha}_{\Delta,s} f(t), \quad t,s \in \mathbb{T}^{\kappa^r}, \quad t > s, \quad r = \overline{[-\alpha]} + 1. \end{aligned} \tag{6}$$

REMARK 2. If we note that the generalized monomials $h_\alpha(t,t_0)$ on the set of real numbers $\mathbb{R}$ are computed as

$$h_\alpha(t,t_0) = \mathscr{L}^{-1}\left(\frac{1}{z^{\alpha+1}}\right)(t) = \frac{(t-t_0)^\alpha}{\Gamma(\alpha)}, \quad t \geqslant t_0,$$

we observe that if $\mathbb{T} = \mathbb{R}$, that is, if Δ derivative is replaced by the classical derivative, the Riemann-Liouville fractional Δ-derivative defined in (5) becomes the usual Riemann-Liouville fractional derivative.

By using these definitions, Caputo fractional Δ-derivative is defined as follows.

DEFINITION 16. [4] For a function $f : \mathbb{T} \to \mathbb{R}$ the Caputo fractional Δ-derivative of order α is denoted by ${}^C D^\alpha_{\Delta,t_0}$ and defined via the Riemann-Liouville fractional Δ-derivative of order α as follows

$$ {}^C D^\alpha_{\Delta,t_0} = D^\alpha_{\Delta,t_0}\left(f(t) - \sum_{k=0}^{m-1} h_k(t,t_0) f^{\Delta^k}(t_0)\right), \quad t > t_0, \tag{7}$$

where $m = \overline{[\alpha]} + 1$ if $\alpha \notin \mathbb{N}$ and $m = \overline{[\alpha]}$ if $\alpha \in \mathbb{N}$.

Another representation of the Caputo fractional Δ-derivative is given in the following theorem (Theorem 7.1 in [4]).

THEOREM 6. *Let* $\alpha > 0$, $m = \overline{[\alpha]} + 1$ *if* $\alpha \notin \mathbb{N}$ *and* $m = \alpha$, *if* $\alpha \in \mathbb{N}$.

1. *If* $\alpha \notin \mathbb{N}$ *then*

$$ {}^C D^\alpha_{\Delta,t_0} f(t) = I^{m-\alpha}_{\Delta,t_0} D^m_{\Delta,t_0} f(t), \quad t \in \mathbb{T}, t > t_0.$$

2. *If* $\alpha \in \mathbb{N}$ *then*

$$ {}^C D^\alpha_{\Delta,t_0} f(t) = f^{\Delta^m}(t), \quad t \in \mathbb{T}, t > t_0.$$

REMARK 3. Regarding the result of the Theorem 6, if $\mathbb{T} = \mathbb{R}$, the Caputo fractional Δ-derivative defined in (7) becomes the usual Caputo fractional derivative.

Now we continue with some more preliminary definitions and notations. Let $\mathbb{T}$ be a time scale with a differentiable graininess function. Consider an infinite series of the form

$$\sum_{i=0}^{\infty} A_i h_i(t,t_0), \quad t, t_0 \in \mathbb{T}, \quad t > t_0. \tag{8}$$

Define the constants $C_{r,k,l}$ as

$$C_{r,k,l} = \sum_{\Lambda_{r,k-l} \in S^{(r)}_{k-l}} h_l^{\Lambda_{r,k-l}}(t_0,t_0), \tag{9}$$

where $r \in \{k-l,\ldots,k\}$, $l \in \{0,\ldots,k\}$ and $k \geqslant r$. Also, define the constants $A_{n,r}$ as

$$\begin{aligned} A_{1,r} &= A_r, \\ A_{n,r} &= \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} A_{n-1,l} A_{1,k-l} C_{r,k,l}, \end{aligned} \tag{10}$$

where $r,k \in \mathbb{N}_0$, $k \geqslant r$ and $n \in \mathbb{N}, n > 1$. Using these notation and the result in Theorem 4 we compute the following.

$$\begin{aligned} &\left(\sum_{i=0}^{\infty} A_i h_i(t,t_0)\right)^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_i A_j h_i(t,t_0) h_j(t,t_0) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} A_l A_{k-l} h_l(t,t_0) h_{k-l}(t,t_0) \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} A_l A_{k-l} \sum_{r=k-l}^{k}\left(\sum_{\Lambda_{r,k-l}\in S_{k-l}^{(r)}} h_l^{\Lambda_{r,k-l}}(t_0,t_0)\right) h_r(t,t_0)\right). \end{aligned}$$

Now, we employ the constants $C_{r,k,l}$ defined in (9) and we get

$$\begin{aligned} &\left(\sum_{i=0}^{\infty} A_i h_i(t,t_0)\right)^2 = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} A_l A_{k-l} \sum_{r=k-l}^{k} C_{r,k,l} h_r(t,t_0)\right) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{r=k-l}^{k} A_l A_{k-l} C_{r,k,l} h_r(t,t_0) = \sum_{r=0}^{\infty}\sum_{k=r}^{\infty}\sum_{l=k-r}^{k} A_l A_{k-l} C_{r,k,l} h_r(t,t_0) \\ &= \sum_{r=0}^{\infty}\sum_{k=r}^{\infty}\sum_{l=k-r}^{k} A_{1,l} A_{1,k-l} C_{r,k,l} h_r(t,t_0) = \sum_{r=0}^{\infty} A_{2,r} h_r(t,t_0). \end{aligned}$$

In a similar way, we compute

$$\begin{aligned} &\left(\sum_{i=0}^{\infty} A_i h_i(t,t_0)\right)^3 = \left(\sum_{i=0}^{\infty} A_i h_i(t,t_0)\right)^2 \left(\sum_{j=0}^{\infty} A_j h_j(t,t_0)\right) \\ &= \left(\sum_{i=0}^{\infty} A_{2,i} h_i(t,t_0)\right)\left(\sum_{j=0}^{\infty} A_{1,j} h_j(t,t_0)\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{r=k-l}^{k} A_{2,l} A_{1,k-l} C_{r,k,l} h_r(t,t_0) \\ &= \sum_{r=0}^{\infty}\sum_{k=r}^{\infty}\left(\sum_{l=k-r}^{k} A_{2,l} A_{1,k-l} C_{r,k,l}\right) h_r(t,t_0) = \sum_{r=0}^{\infty} A_{3,r} h_r(t,t_0). \end{aligned}$$

Generalizing this representation we end up with

$$\left(\sum_{i=0}^{\infty} A_i h_i(t,t_0)\right)^n = \sum_{r=0}^{\infty} A_{n,r} h_r(t,t_0), \tag{11}$$

for $n \in \mathbb{N}$.

3. The Series solution method for a Cauchy problem associate with Caputo fractional dynamic equation

In this section we develop the series solution method for a Cauchy problem associated with Caputo fractional dynamic equations.

Suppose that $\mathbb{T}$ is a time scales with forward jump operator σ, graininess function μ and delta differential operator Δ, and that $\mathbb{T}$ has the form

$$\mathbb{T} = \{t_n : n \in \mathbb{N}_0\},$$

where

$$\begin{aligned} &\lim_{n\to\infty} t_n = \infty, \\ &\sigma(t_n) = t_{n+1}, n \in \mathbb{N}_0, \\ &w = \inf_{n\in\mathbb{N}_0} \mu(t_n) > 0. \end{aligned}$$

Assume that the graininess function μ is delta differentiable. Let ${}^{C}D^{\alpha}_{\Delta,t_0}$ denote the Caputo fractional Δ-derivative. Suppose that $\alpha > 0$ and that $m = -\overline{[-\alpha]}$. We will consider the Cauchy problem associated with Caputo fractional Δ-derivative given as

$$\begin{cases} {}^{C}D^{\alpha}_{\Delta,t_0} y(t) = f(t, y(t)), \quad t > t_0, \\ {}^{C}D^{k}_{\Delta,t_0} y(t) = b_k, \quad k \in \{0, \dots, m-1\}, \end{cases} \tag{12}$$

where $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a given function, $b_k \in \mathbb{R}$ for $k \in \{0, \dots, m-1\}$ are given constants. The existence and uniqueness of solution of the problem (12) and its dependance on the initial data has been studied in [4]. The cases of homogeneous and nonhomogeneous constant coefficient dynamic equations with Caputo fractional delta derivative are also discussed in the same reference [4]. In general, finding explicit solutions for nonlinear fractional dynamic equations is very complicated and it depends on the nature of the right-hand side of the considered equation. Obviously, in the classical case many researchers use different numerical methods for the nonlinear case. To our knowledge, there are not methods for finding the exact or approximate solution for the considered Cauchy problem of the nonlinear Caputo fractional dynamic equations on time scales. On the other hand, the usual fractional differential equations with Caputo derivative are treated by numerical methods in several studies [6, 8].

We suppose that the nonlinear function f is in the form

$$f(t, y(t)) = \left(h_{-\alpha-1}(\cdot, t_0) * \left(\sum_{p=1}^{n} a_p(\cdot)(y(\cdot))^p + a_0(\cdot) \right) \right)(t),$$

where

$$a_p(t) = \sum_{i=0}^{\infty} A_{i,p} h_i(t, t_0), \quad p \in \{0, \dots, n\}, \tag{13}$$

and the coefficients $A_{i,p}$ are given real constants for $i \in \mathbb{N}_0$, $p \in \{0, \dots, n\}$.

It is shown in [4] that the Cauchy problem (12) is equivalent to an integral equation of the form

$$\begin{aligned} y(t) &= \sum_{j=0}^{m-1} h_j(t,t_0)b_j \\ &+ \left(h_{\alpha-1}(\cdot,t_0) * \left(h_{-\alpha-1}(\cdot,t_0) * \left(\sum_{p=1}^{n} a_p(\cdot)(y(\cdot))^p + a_0(\cdot)\right)\right)\right)(t) \\ &= \sum_{j=0}^{m-1} h_j(t,t_0)b_j + \left(h_{-1}(\cdot,t_0) * \left(\sum_{p=1}^{n} a_p(\cdot)(y(\cdot))^p + a_0(\cdot)\right)\right)(t). \end{aligned} \tag{14}$$

We will search a solution of the equation (14) of the form

$$y(t) = \sum_{i=0}^{\infty} B_i h_i(t,t_0), \tag{15}$$

where B_i, $i \in \mathbb{N}_0$, are constants which will be determined below. Let $B_{1,r} = B_r$ and

$$B_{s,r} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} B_{s-1,l} B_{1,k-l} C_{r,k,l}, \tag{16}$$

for $r,k,s \in \mathbb{N}_0$, $k \geqslant r$, $s \geqslant 2$. Then, using (11), we obtain

$$(y(t))^p = \sum_{r=0}^{\infty} B_{p,r} h_r(t,t_0), \quad p \in \{1,\dots,n\}. \tag{17}$$

Consequently,

$$\begin{aligned} a_p(y)(y(t))^p &= \sum_{i=0}^{\infty} A_{i,p} h_i(t,t_0) \sum_{j=0}^{\infty} B_{p,j} h_j(t,t_0) \\ = \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{r=k-l}^{k} A_{l,p} B_{p,k-l} C_{r,k,l} h_r(t,t_0) &= \sum_{r=0}^{\infty}\left(\sum_{k=r}^{\infty}\sum_{l=k-r}^{k} A_{l,p} B_{p,k-l} C_{r,k,l}\right) h_r(t,t_0), \end{aligned} \tag{18}$$

where $p \in \{1,\dots,n\}$. Let

$$D_{r,p} = \sum_{k=r}^{\infty}\sum_{l=k-r}^{k} A_{l,p} B_{p,k-l} C_{r,k,l}. \tag{19}$$

Then

$$a_p(y)(y(t))^p = \sum_{r=0}^{\infty} D_{r,p} h_r(t,t_0), \tag{20}$$

where $p \in \{1,\dots,n\}$. Hence, we obtain

$$\begin{aligned} \sum_{i=0}^{\infty} B_i h_i(t,t_0) &= \sum_{j=0}^{\infty} b_j h_j(t,t_0) \\ &+ \left(h_{-1}(\cdot,t_0) * \left(\sum_{p=1}^{n}\left(\sum_{r=0}^{\infty} D_{r,p} h_r(\cdot,t_0)\right) + \sum_{r=0}^{\infty} A_{0,r} h_r(\cdot,t_0)\right)\right)(t) \\ &= \sum_{j=0}^{\infty} b_j h_j(t,t_0) + \sum_{p=1}^{n}\sum_{r=0}^{\infty} D_{r,p} h_r(t,t_0) + \sum_{r=0}^{\infty} A_{0,r} h_r(t,t_0), \end{aligned} \tag{21}$$

and then

$$
\begin{aligned}
B_i &= b_i + \sum_{p=1}^{n} D_{i,p} + A_{0,i}, \quad i \in \{0,\dots,m-1\},\\
B_i &= \sum_{p=1}^{n} D_{i,p} + A_{0,i}, \quad i \in \{m,\dots,\}.
\end{aligned} \tag{22}
$$

Next we consider some particular examples of Cauchy problems associated with Caputo fractional dynamic equations on time scales.

EXAMPLE 2. Consider the problem

$$
\begin{cases}
{}^{C}D_{\Delta,0}^{\frac{11}{4}} y(t) = \left(h_{-\frac{15}{4}}(\cdot,0) * \left(\left(\sum_{i=0}^{\infty} \frac{1}{i^2+3} h_i(\cdot,0)\right) y(\cdot) \right.\right.\\
\qquad\qquad + \left.\left.\left(\sum_{i=0}^{\infty} \frac{i}{i^2+i+1} h_i(\cdot,0)\right)(y(\cdot))^2\right)\right)(t), \quad t > 0,\\
\\
y(0) = 1, \quad y^{\Delta}(0) = -1, \quad y^{\Delta^2}(0) = 2.
\end{cases} \tag{23}
$$

Here we have $\alpha = \frac{11}{4}$ and $m = -\left[-\frac{11}{4}\right] = 3$. Employing the integral equation form given in (14), we can rewrite the problem (23) as

$$
\begin{aligned}
y(t) &= h_0(t,0) - h_1(t,0) + 2h_2(t,0) + \Big(h_{\frac{7}{4}}(\cdot,0) * h_{-\frac{15}{4}}(\cdot,0)\\
&* \left(\sum_{i=0}^{\infty} \frac{1}{i^2+3} h_i(\cdot,0)\right) y(\cdot) + \left(\sum_{i=0}^{\infty} \frac{i}{i^2+i+1} h_i(\cdot,0)\right)(y(\cdot))^2\Big)(t)\\
&= h_0(t,0) - h_1(t,0) + 2h_2(t,0) + \Big(h_{-1}(\cdot,0) * \Big(\left(\sum_{i=0}^{\infty} \frac{1}{i^2+3} h_i(\cdot,0)\right) y(\cdot)\\
&+ \left(\sum_{i=0}^{\infty} \frac{i}{i^2+i+1} h_i(\cdot,0)\right)(y(\cdot))^2\Big)\Big)(t)\\
&= h_0(t,0) - h_1(t,0) + 2h_2(t,0) + \Big(\left(\sum_{i=0}^{\infty} \frac{1}{i^2+3} h_{-1}(\cdot,0) * h_i(\cdot,0)\right) y(\cdot)\\
&+ \left(\sum_{i=0}^{\infty} \frac{i}{i^2+i+1} h_{-1}(\cdot,0) * h_i(\cdot,0)\right)(y(\cdot))^2\Big)(t)\\
&= h_0(t,0) - h_1(t,0) + 2h_2(t,0) + \left(\sum_{i=0}^{\infty} \frac{1}{i^2+3} h_i(t,0)\right) y(t)\\
&+ \left(\sum_{i=0}^{\infty} \frac{i}{i^2+i+1} h_i(t,0)\right)(y(t))^2.
\end{aligned} \tag{24}
$$

Assume that

$$
y(t) = \sum_{r=0}^{\infty} B_r h_r(t,0), \tag{25}
$$

where the coefficients B_r are going to be obtained. Then, by (16) we have

$$(y(t))^2 = \sum_{r=0}^{\infty} B_{2,r} h_r(t,0), \tag{26}$$

where

$$B_{1,r} = B_r, \quad B_{2,r} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} B_{1,l} B_{1,k-l} C_{r,k,l},$$

and $r \in \mathbb{N}_0$. On the other hand,

$$D_{r,1} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} \frac{1}{l^2+3} B_{1,k-l} C_{r,k,l}, \quad D_{r,2} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} \frac{l}{l^2+l+1} B_{2,k-l} C_{r,k,l}. \tag{27}$$

Then we get

$$\left(\sum_{i=0}^{\infty} \frac{1}{i^2+3} h_i(t,0)\right) y(t) = \sum_{r=0}^{\infty} D_{r,1} h_r(t,0),$$
$$\left(\sum_{i=0}^{\infty} \frac{i}{i^2+i+1} h_i(t,0)\right) (y(t))^2 = \sum_{r=0}^{\infty} D_{r,2} h_r(t,0).$$

We substitute these expressions and (25) into the equation (24) and we obtain

$$\sum_{r=0}^{\infty} B_r h_r(t,0) = h_0(t,0) - h_1(t,0) + 2h_2(t,0) + \sum_{r=0}^{\infty} (D_{r,1} + D_{r,2}) h_r(t,0). \tag{28}$$

From this equation we conclude

$$\begin{aligned} B_0 &= 1 + D_{0,1} + D_{0,2}, \quad B_1 = -1 + D_{1,1} + D_{1,2}, \quad B_2 = 2 + D_{2,1} + D_{2,2}, \\ B_r &= D_{r,1} + D_{r,2}, \quad r \in \{3,4,\ldots\}. \end{aligned} \tag{29}$$

EXAMPLE 3. Consider the problem

$$\begin{cases} {}^{C}D_{\Delta,1}^{\frac{9}{5}} y(t) = \Bigg(h_{-\frac{14}{5}}(\cdot,1) * \Bigg(\sum_{i=0}^{\infty} \frac{1}{2+i} h_i(\cdot,1) \\ \qquad + \left(\sum_{i=0}^{\infty} \frac{i-1}{i+4} h_i(\cdot,1) \right) y(\cdot) \\ \qquad + \left(\sum_{i=0}^{\infty} \frac{i+1}{2i+3} h_i(\cdot,1) \right) (y(\cdot))^3 \Bigg) \Bigg)(t), \quad t > 1, \\ y(1) = 0, \quad y^{\Delta}(1) = 1. \end{cases} \tag{30}$$

Here we have $\alpha = \dfrac{9}{5}$ and $m = -\overline{\left[-\dfrac{9}{5}\right]} = 2$. Employing the integral equation form

(14), we can rewrite the problem (30) as

$$
\begin{aligned}
y(t) &= h_1(t,1) + \left(h_{\frac{4}{5}}(\cdot,1) * h_{-\frac{14}{5}}(\cdot,1) * \left(\sum_{i=0}^{\infty} \frac{1}{2+i} h_i(\cdot,1) \right.\right.\\
&\quad \left.\left. + \left(\sum_{i=0}^{\infty} \frac{i-1}{i+4} h_i(\cdot,1) \right) y(\cdot) + \left(\sum_{i=0}^{\infty} \frac{i+1}{2i+3} h_i(\cdot,1) \right) (y(\cdot))^3 \right) \right)(t) \\
&= h_1(t,1) + \left(h_{-1}(\cdot,1) * \left(\sum_{i=0}^{\infty} \frac{1}{2+i} h_i(\cdot,1) \right.\right.\\
&\quad \left.\left. + \left(\sum_{i=0}^{\infty} \frac{i-1}{i+4} h_i(\cdot,1) \right) y(\cdot) + \left(\sum_{i=0}^{\infty} \frac{i+1}{2i+3} h_i(\cdot,1) \right) (y(\cdot))^3 \right) \right)(t).
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
y(t) &= h_1(t,1) + \left(\left(\sum_{i=0}^{\infty} \frac{1}{2+i} h_{-1}(\cdot,1) * h_i(\cdot,1) \right.\right.\\
&\quad + \left(\sum_{i=0}^{\infty} \frac{i-1}{i+4} h_{-1}(\cdot,1) * h_i(\cdot,1) \right) y(\cdot) \\
&\quad \left.\left. + \left(\sum_{i=0}^{\infty} \frac{i+1}{2i+3} h_{-1}(\cdot,1) * h_i(\cdot,1) \right) (y(\cdot))^3 \right) \right)(t) \\
&= h_1(t,1) + \left(\left(\sum_{i=0}^{\infty} \frac{1}{2+i} h_i(t,1) \right.\right.\\
&\quad \left.\left. + \left(\sum_{i=0}^{\infty} \frac{i-1}{i+4} h_i(t,1) \right) y(t) + \left(\sum_{i=0}^{\infty} \frac{i+1}{2i+3} h_i(t,1) \right) (y(t))^3 \right) \right).
\end{aligned}
\tag{31}
$$

Assume that

$$
y(t) = \sum_{r=0}^{\infty} B_r h_r(t,1), \tag{32}
$$

where the coefficients B_r are going to be obtained. Then, by (16) we have

$$
(y(t))^2 = \sum_{r=0}^{\infty} B_{2,r} h_r(t,1), \quad (y(t))^3 = \sum_{r=0}^{\infty} B_{3,r} h_r(t,1), \tag{33}
$$

where $B_{1,r} = B_r$,

$$
B_{2,r} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} B_{1,l} B_{1,k-l} C_{r,k,l}, \quad B_{3,r} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} B_{2,l} B_{1,k-l} C_{r,k,l},
$$

and $r \in \mathbb{N}_0$. Set

$$
D_{r,1} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} \frac{l-1}{l+4} B_{1,k-l} C_{r,k,l}, \quad D_{r,3} = \sum_{k=r}^{\infty} \sum_{l=k-r}^{k} \frac{l+1}{2l+3} B_{3,k-l} C_{r,k,l}. \tag{34}
$$

Then we get

$$\left(\sum_{i=0}^{\infty}\frac{i-1}{i+4}h_i(t,1)\right)y(t) = \sum_{r=0}^{\infty}D_{r,1}h_r(t,1),$$
$$\left(\sum_{i=0}^{\infty}\frac{i+1}{2i+3}h_i(t,1)\right)(y(t))^3 = \sum_{r=0}^{\infty}D_{r,3}h_r(t,1).$$

We substitute these expressions and (32) into the equation (31) which gives

$$\sum_{r=0}^{\infty}B_rh_r(t,1) = h_1(t,1)+\sum_{r=0}^{\infty}\left(\frac{1}{2+r}+D_{r,1}+D_{r,3}\right)h_r(t,1). \tag{35}$$

From this equation we conclude

$$\begin{aligned} B_0 &= \frac{1}{2}+D_{0,1}+D_{0,3}, \quad B_1 = 1+\frac{1}{3}+D_{1,1}+D_{1,3},\\ B_r &= \frac{1}{2+r}+D_{r,1}+D_{r,3}, \quad r\in\{2,3,\dots\}. \end{aligned} \tag{36}$$

4. Series solution method for Cauchy problems associated with dynamic equations

In this section we suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differential operator σ and Δ, respectively, and that the graininess function μ is Δ-differentiable.

Consider the Cauchy problem

$$\begin{cases} y^{\Delta}(t) = f(t,y(t)), \quad t>t_0,\\ y(t_0) = y_0, \end{cases} \tag{37}$$

where $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a given function, $y_0\in\mathbb{R}$ is a given constant. Suppose that the nonlinear function f is in the form

$$f(t,y(t)) = \sum_{p=1}^{n}a_p(t)(y(t))^p+a_0(t),$$

where

$$a_p(t) = \sum_{i=0}^{\infty}A_{i,p}h_i(t,t_0), \quad p\in\{0,\dots,n\}, \tag{38}$$

and the coefficients $A_{i,p}$ are given real constants for $i\in\mathbb{N}_0$, $p\in\{0,\dots,n\}$. The problem (37) is equivalent to the integral equation

$$y(t) = y_0+\int_{t_0}^{t}f(u,y(u))\Delta u,$$

or,

$$y(t) = y_0+\int_{t_0}^{t}\left(\sum_{p=1}^{n}a_p(u)(y(u))^p+a_0(u)\right)\Delta u.$$

We assume that $y(t)$ has the form (15), that is,

$$y(t) = \sum_{i=0}^{\infty} B_i h_i(t,t_0).$$

Now, using (19) and (20), we get

$$\begin{aligned}\sum_{i=0}^{\infty} B_i h_i(t,t_0) &= y_0 + \int_{t_0}^{t} \left(\sum_{p=1}^{n} \left(\sum_{r=0}^{\infty} D_{r,p} h_r(u,t_0) \right) + \sum_{r=0}^{\infty} A_{0,r} h_r(u,t_0) \right) \Delta u \\ &= y_0 + \sum_{p=1}^{n} \sum_{r=0}^{\infty} D_{r,p} h_{r+1}(t,t_0) + \sum_{r=0}^{\infty} A_{0,r} h_{r+1}(t,t_0),\end{aligned} \tag{39}$$

where $B_0 = y_0$,

$$B_i = \sum_{p=1}^{n} D_{i-1,p} + A_{0,i-1}, \quad i \in \mathbb{N}. \tag{40}$$

EXAMPLE 4. As a particular example we will apply the series solution method to a population growth model known as the Logistic model. The Logistic model has also been generalized in the case of fractional derivative [7, 9]. The Logistic model on an arbitrary time scales is described by the Cauchy problem

$$\begin{cases} N^{\Delta}(t) = \dfrac{\alpha N(t)}{\mu(t)} \left(1 - \dfrac{N(t)}{K} \right), & t \geqslant t_0, \\ N(t_0) = N_0. \end{cases} \tag{41}$$

Here $N(t)$ is the size of the population of a certain species at time t and $N(t_0) = N_0$ is the initial size of the population. The constant α represents the proportionality constant which is large for quickly growing species like bacteria and small for slowly growing populations like elephants. The constant K stands for the carrying capacity of the system, that is, the size of the population that the environment can long term sustain.

This model has been discussed and treated via the series solution method in [5] on the time scales $\mathbb{T} = \mathbb{Z}$. We will consider this model on the more general case when $\mathbb{T} = a\mathbb{Z}$ for some positive constant a.

As noted above, the Cauchy problem (41) can be written as an integral equation of the form

$$N(t) = N_0 + \int_{t_0}^{t} \frac{\alpha N(u)}{a} \left(1 - \frac{N(u)}{K} \right) \Delta u, \quad t_0 \in a\mathbb{Z}, \tag{42}$$

which is a nonlinear Volterra integral equation of the second kind. We will take the initial time as $t_0 = 0$ and the initial population as N_0 and apply the series solution method to solve this integral equation. Let

$$N(t) = \sum_{i=0}^{\infty} B_i h_i(t,0) = \sum_{i=0}^{\infty} B_{1,i} h_i(t,0), \quad t \in a\mathbb{Z}.$$

Then,

$$(N(t))^2 = \left(\sum_{i=0}^{\infty} B_{1,i}h_i(t,0)\right)\left(\sum_{j=0}^{\infty} B_{1,j}h_j(t,0)\right) = \sum_{r=0}^{\infty} B_{2,r}h_r(t,0), \quad t \in a\mathbb{Z},$$

where

$$B_{2,r} = \sum_{k=r}^{\infty}\sum_{l=k-r}^{k} B_{1,l}B_{1,k-l}C_{r,k,l},$$

for $r \in \mathbb{N}_0$. We insert these series into the equation (42) and we get

$$\begin{aligned}\sum_{r=0}^{\infty} B_r h_r(t,0) &= N_0 + \int_0^t \frac{\alpha}{a}\left(\sum_{r=0}^{\infty} B_r h_r(u,0) - \frac{1}{K}\sum_{r=0}^{\infty} B_{2,r}h_r(u,0)\right)\Delta u \\ &= N_0 + \frac{\alpha}{a}\left(\sum_{r=0}^{\infty} B_r h_{r+1}(t,0) - \frac{1}{K}\sum_{r=0}^{\infty} B_{2,r}h_{r+1}(t,0)\right) \\ &= N_0 + \sum_{r=0}^{\infty}(\frac{\alpha}{a}B_r - \frac{\alpha}{aK}B_{2,r})h_{r+1}(t,0), \quad t \in a\mathbb{Z}.\end{aligned}$$

Therefore, we have $B_0 = N_0$ and

$$B_{r+1} = \frac{\alpha}{a}B_r - \frac{\alpha}{aK}B_{2,r}, \text{ for all } r \in \mathbb{N}_0. \tag{43}$$

We recall that on $\mathbb{T} = a\mathbb{Z}$ the forward jump operator is $\sigma(t) = t + a$. The first 5 monomials $h_n(t,0)$, $n = 0,1,2,3,4$, are in the form

$$h_0(t,0) = 1, \quad h_1(t,0) = t, \quad h_2(t,0) = \int_0^t x\Delta x = \frac{t(t-a)}{2},$$

$$h_3(t,0) = \int_0^t \frac{x(x-a)}{2}\Delta x = \frac{t(t-a)(t-2a)}{6},$$

$$h_4(t,0) = \int_0^t \frac{x(x-a)(x-2a)}{6}\Delta x = \frac{t(t-a)(t-2a)(t-3a)}{24}, \quad t \in a\mathbb{Z}.$$

To compute the first few coefficients $B_{2,r}$, we consider the series expansion of N^2

$$\begin{aligned}(N(t))^2 &= (B_0h_0(t,0) + B_1h_1(t,0) + B_2h_2(t,0) + B_3h_3(t,0) + \cdots)^2 \\ &= B_0B_0 + (B_0B_1 + B_1B_0)t + B_1B_1t^2 + (B_0B_2 + B_2B_0)\frac{t(t-a)}{2} \\ &\quad + (B_1B_2 + B_2B_1)\frac{t^2(t-a)}{2} + B_2B_2\frac{t^2(t-a)^2}{4} + \cdots, \quad t \in a\mathbb{Z}.\end{aligned}$$

Note that we have

$$\begin{aligned} t^2 &= ah_1(t,0) + 2h_2(t,0), \quad \frac{t^2(t-a)}{2} = 2ah_2(t,0) + 3h_3(t,0), \\ \frac{t^2(t-a)^2}{4} &= a^2h_2(t,0) + 6ah_3(t,0) + 6h_4(t,0), \quad t \in a\mathbb{Z}.\end{aligned}$$

As a result, we obtain

$$(N(t))^2 = \sum_{n=0}^{\infty} B_{2,n}h_n(t,0) = B_0B_0h_0(t,0) + (B_0B_1 + B_1B_0 + aB_1B_1)h_1(t,0)$$
$$+ (B_0B_2 + B_2B_0 + 2B_1B_1 + 2aB_1B_2 + 2aB_2B_1 + a^2B_2B_2)h_2(t,0) + \cdots, \quad t \in a\mathbb{Z}.$$

Then, the recurrence relation (43) yields $B_0 = N_0$ and

$$\begin{aligned} B_1 &= \frac{\alpha}{a}B_0 - \frac{\alpha}{aK}B_{2,0} = \frac{\alpha}{a}B_0 - \frac{\alpha}{aK}B_0B_0, \\ B_2 &= \frac{\alpha}{a}B_1 - \frac{\alpha}{aK}B_{2,1} = \frac{\alpha}{a}B_1 - \frac{\alpha}{aK}(B_0B_1 + B_1B_0 + aB_1B_1), \\ B_3 &= \frac{\alpha}{a}B_2 - \frac{\alpha}{aK}B_{2,2} \\ &= \frac{\alpha}{a}B_2 - \frac{\alpha}{aK}(B_0B_2 + B_2B_0 + 2B_1B_1 + 2aB_1B_2 + 2aB_2B_1 + a^2B_2B_2). \end{aligned}$$

Then, the solution $N(t)$ has the form

$$N(t) = B_0 + B_1t + B_2\frac{t(t-a)}{2} + B_3\frac{t(t-a)(t-2a)}{6} + \cdots, \quad t \in a\mathbb{Z}.$$

5. Conclusion

The series solution method introduced in this paper is the generalization of the series solution method of the Cauchy problems associated with ordinary differential equations. It can be easily seen that when $\mathbb{T} = \mathbb{R}$ and the order α of the fractional derivative is 1, we get the well known first order differential equation. The most difficult part is the presence of the constants $C_{r,k,l}$ which do not have a general representation and their computation depends solely on the time scale under consideration.

An important feature of the method is that it can be extended to higher order dynamic equations and fractional dynamic equations of more general form.

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