Fractional Differential Calculus
Volume 9, Number 2 (2019), 263–278
doi:10.7153/fdc-2019-09-17

SUFFICIENT CONDITIONS FOR THE EXPONENTIAL STABILITY OF NONLINEAR FRACTIONALLY PERTURBED ODE*s* WITH MULTIPLE TIME DELAYS

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(*Communicated by N. Vasylyeva*)

Abstract. In this paper we prove some results on the exponential stability of the trivial solution of a system of fractional differential equations with multiple delays and tempered Riemann-Louville fractional integrals on its right-hand side.

1. Introduction

In this paper we consider the following class of fractional differential equations with multiple delays

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + F\Big(t, x(t)\Big) + G(t, x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_k)) \\ &+ f\Big(t, I^{(\alpha,\beta)}x(t)\Big) + g\Big(t, I^{(\alpha,\beta)}[x(t-\tau_1)], \ldots, I^{(\alpha,\beta)}[x(t-\tau_k)]\Big), \quad t \geqslant 0 \\ x(t) &= \Phi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^N$, $\quad 0 < \tau = \max_{1 \leqslant i \leqslant k} \tau_i$ and

$$I^{(\alpha,\beta)}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\beta(t-s)} x(s) ds \tag{2}$$

is so called the Riemann-Liouville tempered, or substantial fractional integral of the function $x(t)$ of order $\alpha > 0$ with a parameter $\beta > 0$ and the Caputo tempered, or substantial fractional derivative, corresponding to the tempered fractional integral, is defined as

$$\begin{aligned} D^{(\alpha,\beta)}x(t) &= \frac{e^{-\beta t}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d(e^{\beta s}x(s))}{ds} ds \\ &= \frac{e^{-\beta t}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{\beta s} \Big(\frac{d}{ds} + \beta\Big) x(s) ds, \quad 0 < \alpha < 1, \beta > 0 \end{aligned} \tag{3}$$

Mathematics subject classification (2010): 26D15, 26A51, 32F99, 41A17.
Keywords and phrases: Fractional tempered derivative, fractional tempered integral, fractional perturbation of ODE, multi-delay, exponential stability.

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(see e.g. [5], [9], [22], [31]),

$$I^{(\alpha,\beta)}[x(t-\tau_i)] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\beta(t-s)}x(s-\tau_i)ds, i=1,2,\ldots,k. \tag{4}$$

The integral $I^{\alpha}x(t) = I^{(\alpha,0)}x(t)$ is the Riemann-Liouville fractional integral of the function $x(t)$ of order α.

We establish some sufficient conditions to guarantee the trivial solution of the equation (1) is exponentially stable. We note that the trivial solution of the linear fractional differential equation

$$D^{\alpha}x(t) = Ax(t), \qquad x(t)\in\mathbb{R}^N, \quad \alpha\in(0,1), \tag{5}$$

where $D^{\alpha}x(t)$ is the Riemann-Liouville or the Caputo derivative of $x(t)$ of the order $\alpha\in(0,1)$ and A is a constant matrix, can be asymptotically stable, but not exponentially, and solutions decay towards 0 like $t^{-\alpha}$ as $t\to\infty$. The trivial solution of this equation is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ for all eigenvalues λ of the matrix A. In this case all components of $x(t)$ decay towards 0 like $t^{-\alpha}$ (see e.g. [10]).

The same problem is studied in the paper [3] for nonlinear fractional systems of equations of the following class

$$\dot{x}(t) = Ax(t) + g\big(t, x(t), {}^{RL}I^{\alpha_1}x(t), \ldots, {}^{RL}I^{\alpha_m}x(t)\big), \quad x(t)\in\mathbb{R}^N \tag{6}$$

and in the paper [4] for fractional differential equations of the type (6), where instead of the Riemann-Liouville fractional integrals there are Caputo-Fabrizio fractional integrals.

DEFINITION 1. We say that $x(t) = x_{\Phi}(t)$ is a solution of the initial value problem (1), defined on the interval $[-\tau,T)$, where $0<\tau=\max_{\{\tau_1,\tau_2,\ldots<\tau_k\}}$ if it is C^1-differentiable on the interval $(0,T)$, the fractional integrals in this equation exists, $x(t)$ fulfils the equality (1) for all $t\in(0,T)$ with $x(t)=\Phi(t)\ \forall t\in[-\tau,0]$. It is called maximal, if there is no its proper continuation, i.e. there is no $\varepsilon>0$, such that there exists a solution $y(t)$ of this problem, defined on the interval $[-\tau_k, T+\varepsilon)$ with $y(t)=x(t)$ for all $t\in[-\tau,T)$. If $T=\infty$, this solution is called global.

DEFINITION 2. The trivial solution of the equation (1) is exponentially stable with respect to the ball $\Omega(r)=\{y\in\mathbb{R}^N : \|y\|<r\}$ if there are constants $M>0, \eta>0$, such that for any solution $x(t)=x(t,\Phi)$ of the initial value problem (1) with the initial function Φ with $\|\Phi\|_{\infty}=\max_{t\in[-\tau,0)}\|\Phi(t)\|<r$, the following inequality holds

$$\|x(t,\Phi)\| \leqslant Me^{-\eta t}\|\Phi\|_{\infty}, \quad \forall t\geqslant 0. \tag{7}$$

A sufficient condition under which the zero solution of the equation

$${}^{RL}D^{\alpha}x(t) = f(t,x(t)), \quad \alpha\in(0,1), \quad x\in\mathbb{R}, \tag{8}$$

is asymptotically stable, where $f\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\phi\colon R^+ \to R^+$ are continuous functions, is proved in [7] and any solution $x(t)$ of this equation, satisfying the condition $\lim_{t\to 0} t^{1-\alpha}x(t) = x_0 \in \mathbb{R}$, is global and $|x(t)| \leqslant \frac{c}{t^{1-\alpha}}$ for some $c > 0$ and for all $t \in (0,\infty)$. It is assume there that the function $f(t,x)$ satisfies the condition

$$|f(t,x)| \leqslant t^{\mu}\phi(t)e^{-\sigma t}|u|^m \qquad \text{for all } (t,u) \in \mathbb{R}^+ \times \mathbb{R}, \tag{9}$$

where $\mu \geqslant 0$, $m > 1$, $\sigma > 0$, $\mu + (m-1)(1-\alpha) > 0, \alpha > \frac{1}{2}$ and

$$\|\phi\|_q = \int_0^{\infty} \phi(s)^q ds < L := \frac{\Gamma(\alpha)}{2^{1+m-\alpha}|b|^{m-1}}\left(\frac{2^m}{m-1}\right)^{\frac{1}{q}}\left[\frac{(\sigma p)^{\lambda_1}}{\Gamma(\lambda_1)(1+\frac{\lambda_1}{\lambda_2})}\right]^{\frac{1}{p}}, \tag{10}$$

$pq = p+q$, $q > \frac{1}{\alpha}$, $\lambda_1 = 1 + p(\mu - (1-\alpha)m]$, $\lambda_2 = 1 + p(\alpha - 1)$.

Our conditions (H3)–(H6), concerning the nonlinearities of the system (1), also contain the exponential functions of the form $e^{-\omega t}, \omega > 0$, however with $\omega > \rho$, where $\rho > 0$ is the number from the condition (H2). It seems that such conditions are necessary for the exponential stability of the trivial solution of the system (1).

It is proven in the paper [23] that solutions of the equation

$$u''(t) + b^C D^{\alpha}u(t) + cu(t) = 0, \quad \alpha \in (0,1), a > 0, b > 0 \tag{11}$$

have similar asymptotic properties as the equation (8). An equation of this type is the well-known Bagley-Torvik equation

$$u''(t) + A^C D^{3/2}u(t) = au(t) + \phi(t), \tag{12}$$

modeling the motion of a rigid plate immersing in a viscous liquid with the fractional damping term $A^C D^{3/2}u(t)$ (see [30]). A two-point boundary value problems for the following generalized Bagley-Torvik equation

$$u''(t) + A^C D^{\alpha}u(t) = f(t, u(t), {}^C D^{\beta}u(t), u'(t)) \tag{13}$$

is studied e. g. in [29]. We were motivated by the paper [25], where an equation of the form

$$Bu''(t) + \sum_{k=1}^{N} B_k{}^C D^{\alpha_k}u(t)) = g(t,u) \tag{14}$$

is studied. In particular, this equation represent a nonlinear damped pendulum with N fractional dampers. Systems of single-mass oscillators with different fractional damping terms are also studied in the paper [26].

In the book [27] a distributed-order fractional mass-spring viscoelastic damper system with mass m, spring constant K and assembly of viscoelastic dampers of damping coefficients $c_i\,(1 \leqslant i \leqslant n)$, subjected to the spring force $-kx(t)$ and damping force $-\sum_{i=1}^{n} c_i^C D^{\alpha_i}$ is studied.

We study the equation (1) with the Riemann-Liouville tempered integrals because they have better asymptotic properties than the Riemann-Liouville ones. In the proofs of the main result we need to estimate the integral $\int_0^t (t-s)^{\alpha-1}e^{-\eta(t-s)}g(s)ds$ for a nonnegative nondecreasing function $g(t)$, where $\alpha \in (0,1), \eta > 0$. Obviously,

$$\int_0^t (t-s)^{\alpha-1}e^{-\eta(t-s)}g(s)ds \leqslant g(t)\int_0^t \sigma^{\alpha-1}e^{-\eta\sigma}d\sigma \leqslant g(t)\frac{\Gamma(\alpha)}{\eta^\alpha}. \tag{15}$$

In the paper [3] the integral $\int_0^t (t-s)^{\alpha-1}\|x(s)\|ds$, where $x(t)$ is a solution of the equation (6), is estimated as follows:

$$\begin{aligned}\int_0^t (t-s)^{\alpha-1}\|x(s)\|ds &\leqslant \left(\int_0^t (t-s)^{p(\alpha-1)}e^{ps}\right)^{\frac{1}{p}}\left(\int_0^t e^{-qs}\|x(s)\|^q ds\right)^{\frac{1}{q}}\\ &\leqslant K_p\left(\int_0^t e^{-qs}\|x(s)\|^q ds\right)^{\frac{1}{q}},\end{aligned} \tag{16}$$

where $q = \frac{p}{p-1}, p(\alpha-1)+1 > 0$ and

$$K_p = \left[\frac{\Gamma(p(\alpha-1)+1)}{p^{p(\alpha-1)+1)}}\right]^{\frac{1}{p}}. \tag{17}$$

For the final estimate of $\|x(t)\|$ some further procedures are applied in the proof of [3, Theorem 3.2].

It seems that the Riemann-Liouville tempered fractional integral is more convenient for applications to many real models and their qualitative and numerical analysis (see e.g. [1], [5], [9], [20]). This is also the reason why we study the equation (1) with this type of fractional integrals.

The Medina's method [12] applied to the delay systems of differential equations is very powerful technique because the characteristic equation of the linear parts of these equations is not necessary to analyze. In the papers [16], [17], [18], [19] the problem of the exponential stability of the zero solution of systems of nonlinear differential equations with multiple time delays whose linear parts are defined by pairwise permutable matrices is studied. In this case it is not necessary to know roots of the characteristic equations of the linear parts. Sufficient conditions for the exponential stability of the trivial solution for a class of fractionally perturbed ordinary differential equations without delays with right-hand sides involving the Riemann-Liouville tempered fractional integrals of different orders are proved in [14] (see also [15]). We extend results on exponential stability of the trivial solution of a non-fractional system proved in the papers [12], [13] to a class of fractional systems of differential equations, represented by the system (1).

A system of the form (1) can be obtained from the following linear multi-order fractional pendulum equation

$$\begin{aligned}&u''(t) + a(t)^C D^{(\alpha,\beta)}u(t)\\ &+\lambda_1(t)^C D^{(\alpha,\beta)}[u(t-\tau_1)]\ldots+\lambda_m(t)^C D^{(\alpha,\beta)}[u(t-\tau_m)]+\lambda u'(t)+\omega^2 u(t)=0,\end{aligned} \tag{18}$$

which can be written as a system of the form (1) with

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\lambda \end{pmatrix},$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

$$f(t, I^{(\gamma,\beta)}x(t)) = \begin{pmatrix} 0 \\ -a(t)[\beta I^{(\gamma,\beta)}x_1(t) + I^{(\gamma,\beta)}x_2(t)] \end{pmatrix},$$

$$g(t, I^{(\gamma,\beta)}[x(t-\tau_1)], \dots, I^{(\gamma,\beta)}[x(t-\tau_m)])$$
$$= \begin{pmatrix} 0 \\ -\sum_{i=1}^{m} \lambda_i(t)\left[\beta I^{(\gamma,\beta)}[x_1(t-\tau_i)] + I^{(\gamma,\beta)}[x_2(t-\tau_i)]\right] \end{pmatrix},$$

where $x_1(t) = u(t)$, $x_2(t) = u'(t)$, $\alpha \in (0,1), \beta > 0, \gamma = 1-\alpha$, $a(t), \lambda_i(t)$ $i = 1,2,\dots,m$ are continuous functions on $[0,\infty)$ and $\lambda > 0$, $\omega > 0$ are constants.

Let us note that up to now little attention has been paid to time-delayed fractionally damped oscillators (see e. g. [21], [24]).

2. Main results

In the paper [3] a local existence result for the system (6) (Theorem 2.1) is proved by using the Picard's method of successive approximations. This method is not quite standard because the mapping g on the right-hand side of this equation contains the fractional integrals. It was applied in recently published paper [19] in the proof of a local existence theorem (Theorem 3.1) which can be formulated also for our system (1) as follows:

THEOREM 1. *Let $I = [0,T) \subset \mathbb{R}$ for some $T > 0, D \subset \mathbb{R}^N$ be a region, $H_1 \subset \mathbb{R}^N$ be a region containing $0 \in \mathbb{R}^N, H_2 \subset \mathbb{R}^{kN}$ be a region containing $0 \in \mathbb{R}^{kN}, F \in C(I \times D, \mathbb{R}^N), G \in C(I \times D^k, \mathbb{R}^N) f \in C(I \times H_1, \mathbb{R}^N), g \in C(I \times H_2, \mathbb{R}^N)$ be continuous locally Lipschitz mappings. Then for any $\Phi \in C([-\tau, 0], D)$ there exists $\varepsilon > 0$ such that the initial value problem (1) has a unique solution $x(t)$ on the interval $[-\tau, \varepsilon)$.*

In the paper [19] the stability problem for an equation of the type (1) is solved, however under the assumption that its linear part is defined by pairwise permutable matrices. In this case some results from [19] (see also [18],[16], [17]) on the representation of solutions of this equation are used. We are avoiding this assumption to obtain more readable results and more convenient for applications.

Except this theorem we shall also need the following integral inequality, proved in [11] for integer powers. However it is clear that this inequality holds also for arbitrary real powers greater or qual 1. We formulate its generalization without its proof as follows

LEMMA 1. *Let $c \geqslant 0$ be a constant, $f_i(t), i = 1,2,\dots,n$ be nonnegative continuous functions on the interval $I = [a,b), 1 = k_1 < k_2 \leqslant k_3 \leqslant \dots \leqslant k_n$ be real numbers* *and $v(t)$ be a nonnegative continuously differentiable real-valued function satisfying the inequality:*

$$v(t) \leqslant c + \int_a^t \sum_{i=1}^n f_i(s) v^{k_i}(s) ds, \quad t \in I = [a,b). \tag{19}$$

Then

$$v(t) \leqslant \frac{c \exp\left(\int_a^t f_1(s) ds \right)}{\left[1 - (k_n - 1) \sum_{i=2}^n c^{k_i - 1} \int_a^t f_i(s) \exp\left((k_n - 1) \int_a^s f_1(\sigma) d\sigma \right) ds \right]^{\frac{1}{k_n - 1}}}, \tag{20}$$

under the assumption

$$\sum_{i=2}^n c^{k_i - 1} \int_a^t f_i(s) \exp\left((k_n - 1) \int_a^s f_1(\sigma) d\sigma \right) ds < \frac{1}{k_n - 1}, \quad \forall t \in I. \tag{21}$$

To establish the main results we make the following assumptions:

(H1) There are positive numbers q, Θ, δ and $\tau \geqslant 0$ such that

$$\|A(t) - A(\tau)\| \leqslant q e^{-\delta|t-\tau|} |t - \tau|^\Theta \quad \forall t \geqslant 0, \tag{22}$$

where $\|.\|$ denotes a norm in $\mathbb{R}^N$;

(H2)

$$\|e^{A(\tau)t}\| \leqslant K e^{-\rho t} \quad \forall t \geqslant 0, \tag{23}$$

where $K > 0, \rho > 0$ are constants.

We assume that there exist constants $0 < r \leqslant \infty, 1 < M_1 < \ldots < M_m$ such that the following conditions hold:

(H3) There exist constants $\gamma_0, \delta_0 > \rho, \gamma_j > 0, j = 1, 2, \ldots, m$ such that

$$\|F(t,u)\| \leqslant \gamma_0 \|u\| e^{-\delta_0 t} + \sum_{j=1}^m \gamma_j \|u\|^{M_j}, \ \forall t \geqslant 0 \ \forall u \in \Omega(r); \tag{24}$$

(H4) There exist constants $\omega_{i0} > \rho, \eta_{i0} > 0, \eta_{ij} > 0, i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, m$ such that

$$\begin{aligned} \|G(t, u_1, u_2, \ldots, u_k)\| &\leqslant \sum_{i=1}^k \eta_{i0} e^{-\omega_{i0} t} \|u_i\| \\ &+ \sum_{i=1}^k \sum_{j=1}^m \eta_{ij} \|u_i\|^{M_j} \ \forall t \geqslant 0, \forall u_i \in \Omega(r); \end{aligned} \tag{25}$$

(H5) There exist constants $\xi_0 > 0, \xi_j, \mu_j > \rho, j = 1,2,\dots,m$ such that

$$\|f(t,u)\| \leqslant \xi_0 e^{-\mu_0 t}\|u\| + \sum_{j=1}^{m} \xi_j e^{-\mu_j t}\|u\|^{M_j}, \ \forall t \geqslant 0 \, \forall u \in \Omega(r); \tag{26}$$

(H6) There exist constants $\Delta_{i0} > 0, \nu_{i0} > \rho, \Delta_{ij} > 0, \nu_{ij} > \rho, i = 1,2,\dots,k;$ $j = 1,2,\dots,m$ such that

$$\begin{aligned} \|g(t,u_1,u_2,\dots,u_k)\| &\leqslant \sum_{i=1}^{k} \Delta_{i0} e^{-\nu_{i0} t}\|u_i\| \\ &+ \sum_{i=1}^{k}\sum_{j=1}^{m} \Delta_{ij} e^{-\nu_{ij} t}\|u_i\|^{M_j} \, \forall t \geqslant 0, \forall u_i \in \Omega(r), \end{aligned} \tag{27}$$

where $\Omega(r) = \{h \in \mathbb{R}^N : \|h\| < r\}, r \leqslant \infty$.

THEOREM 2. *Suppose that the conditions $(H1)-(H6)$ are satisfied. In addition, let*

$$D(\Phi) := (k_m - 1)e^{(k_m-1)R_0} \sum_{j=1}^{m} C(\Phi)^{k_j - 1} R_j < 1 \tag{28}$$

for any Φ with $\|\Phi\|_\infty = \max_{t\in[-\tau,0]} \|\Phi(t)\| < r$, where

$$\begin{aligned} R_0 &= 2Kq\frac{\Gamma(\Theta+1)}{\delta^{\Theta+1}} + \frac{K\gamma_0}{\delta_0} + K\sum_{i=1}^{k} \eta_{i0}\frac{e^{\rho\tau_i}}{\omega_{i0}} \\ &+ \frac{K\xi_0}{\beta^\alpha(\mu_0-\rho)} + K\sum_{i=1}^{k}\Delta_{i0}\frac{e^{\rho\tau_i}}{\nu_{i0}-\rho} < \infty, \\ R_j &= \frac{K}{[M_j-1]\rho}\left(\gamma_j + k\eta_j\right) + \frac{K\xi_j}{\beta^{M_j\alpha}} + \frac{Kk\Delta_j}{\nu_j-\rho}, \ j = 1,2,\dots,m, \end{aligned} \tag{29}$$

where $\eta_j = e^{M_m\rho\tau}\max_{1\leqslant i\leqslant k}\eta_{ij}, \Delta_j = e^{M_m\rho\tau}\max_{1\leqslant i\leqslant k}\Delta_{ij}, \nu_j = \min_{1\leqslant i\leqslant k}\nu_{ij}$ and

$$C(\Phi) = \|\Phi\|_\infty K\left[1 + \sum_{i=1}^{k}\left(\frac{1}{\omega_{i0}-\rho} + \frac{1}{\nu_{i0}-\rho}\right) + \sum_{i=1}^{k}\sum_{j=1}^{m}\frac{\|\Phi\|_\infty^{M_j-1}}{\beta^{\alpha M_j}(\nu_{ij}-\rho)}\right]. \tag{30}$$

Then there is a constant $M > 0$ such that any solution $x(t) = x(t,\Phi)$ of the initial value problem (1) with $\|\Phi\|_\infty < r$, satisfies the inequality

$$\|x(t,\Phi)\| \leqslant Me^{-\rho t}\|\Phi\|_\infty \, \forall t \geqslant 0. \tag{31}$$

Proof. Let $x(t)$ be the maximal solution of the system (1) on an interval $[0,T)$ with the initial value $x(0) \in \Omega(r), 0 < T \leqslant \infty$. From Theorem 1 it follows that this solution exists. Let us rewrite this system in the form

$$
\begin{aligned}
\dot{x}(t) &= A(\tau)x(t) + [A(t) - A(\tau)]x(t) \\
&+ F\left(t, x(t)\right) + G(t, x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_k)) \\
&+ f\left(t, I^{(\alpha,\beta} x(t)\right) + g\left(t, I^{(\alpha,\beta)}[x(t-\tau_1)], \ldots, I^{(\alpha,\beta)}[x(t-\tau_k)]\right).
\end{aligned}
\tag{32}
$$

For now, let us assume that $r = \infty$. The case $r < \infty$ is postponed to the end of the proof. Then for $t \in [0,T)$ we have

$$
\begin{aligned}
x(t) &= e^{A(\tau)t}x(0) + \int_0^t e^{A(\tau)(t-s)}[A(s) - A(\tau)]x(s)ds + \int_0^t e^{A(\tau)(t-s)}F(s,x(s))ds \\
&+ \int_0^t e^{A(\tau)(t-s)}G(s, x(s-\tau_1), x(s-\tau_1), \ldots, x(s-\tau_k))ds \\
&+ \int_0^t e^{A(\tau)(t-s)} f\left(s, I^{(\alpha,\beta)}x(s)\right)ds \\
&+ \int_0^t e^{A(\tau)(t-s)} g\left(s, I^{(\alpha,\beta)}[x(s-\tau_1)], \ldots, I^{(\alpha,\beta)}[x(s-\tau_k)]\right)ds.
\end{aligned}
\tag{33}
$$

The conditions (H1)-(H6) yield

$$
\begin{aligned}
\|x(t)\| &\leqslant Ke^{-\rho t}\|\Phi\|_\infty + Kqe^{-\rho t}\int_0^t e^{\rho s}e^{-\delta|s-\tau|}|s-\tau|^\Theta \|x(s)\|ds \\
&+ K\gamma_0 e^{-\rho t}\int_0^t e^{-[\delta_0-\rho]s}\|x(s)\|ds + Ke^{-\rho t}\sum_{j=1}^m \gamma_j \int_0^t e^{\rho s}\|x(s)\|^{M_j}ds \\
&+ Ke^{-\rho t}\sum_{i=1}^k \eta_{i0}\int_0^t e^{-[\omega_{i0}-\rho]s}\|x(s-\tau_i)\|ds + Ke^{-\rho t}\sum_{i=1}^k\sum_{j=1}^m \eta_{ij}\int_0^t e^{\rho s}\|x(s-\tau_i)\|^{M_j}ds \\
&+ K\xi_0 e^{-\rho t}\int_0^t e^{-[\mu_0-\rho]s}\|I^{(\alpha,\beta)}x(s)\|ds + Ke^{-\rho t}\sum_{j=1}^m \xi_j\int_0^t e^{\rho s}\|I^{(\alpha,\beta)}x(s)\|^{M_j}ds \\
&+ Ke^{-\rho t}\sum_{i=1}^k \Delta_{i0}\int_0^t e^{-[\nu_{i0}-\rho]s}\|I^{(\alpha,\beta)}[x(s-\tau_i)]\|ds \\
&+ Ke^{-\rho t}\sum_{i=1}^k\sum_{j=1}^m \Delta_{ij}\int_0^t e^{-[\nu_{ij}-\rho]s}\|I^{(\alpha,\beta)}[x(s-\tau_i)]\|^{M_j}ds.
\end{aligned}
\tag{34}
$$

If $u(t) = e^{\rho t}\|x(t)\|$, then

$$
\begin{aligned}
&\|x(t)\| = e^{-\rho t}u(t), \|x(t)\|^{M_j} = e^{-M_j\rho t}u(t)^{M_j}, \\
&\|x(t-\tau_i)\| = e^{\rho\tau_i}e^{-\rho t}u(t-\tau_i), \|x(t-\tau_i)\|^{M_j} = e^{M_j\rho\tau_i}e^{-M_j\rho t}u(t-\tau_i)^{M_j}
\end{aligned}
\tag{35}
$$

and we can write the above inequality as the following inequality for $u(t)$:

$$\begin{aligned}
u(t) \leqslant\; & Ke^{-\rho t}\|\Phi\|_\infty + Kq\int_0^t e^{-\delta|s-\tau|}|s-\tau|^{\Theta}u(s)ds + K\gamma_0\int_0^t e^{-\delta_0 s}u(s)ds \\
& + K\sum_{j=1}^{m}\gamma_j\int_0^t e^{-[M_j-1]\rho s}u(s)^{M_j}ds + K\sum_{i=1}^{k}\eta_{i0}e^{\rho\tau_i}\int_0^t e^{-\omega_{i0}s}u(s-\tau_i)ds \\
& + K\sum_{i=1}^{k}\sum_{j=1}^{m}\eta_{ij}e^{M_j\rho\tau_i}\int_0^t e^{-[M_j-1]\rho s}u(s-\tau_i)^{M_j}ds \\
& + K\xi_0\int_0^t e^{-[\mu_0-\rho]s}\|I^{(\alpha,\beta)}x(s)\|ds + K\sum_{j=1}^{m}\xi_j\int_0^t e^{\rho s}\|I^{(\alpha,\beta)}x(s)\|^{M_j}ds \\
& + K\sum_{i=1}^{k}\Delta_{i0}\int_0^t e^{-[\nu_{i0}-\rho]s}\|I^{(\alpha,\beta)}[x(s-\tau_i)]\|ds \\
& + K\sum_{i=1}^{k}\sum_{j=1}^{m}\Delta_{ij}\int_0^t e^{-[\nu_{ij}-\rho]s}\|I^{(\alpha,\beta)}[x(s-\tau_i)]\|^{M_j}ds.
\end{aligned} \tag{36}$$

Now let us estimate the integrals with delays on the intervals $[-\tau_i, 0]$.

$$\begin{aligned}
& \int_0^{\tau_i} e^{-[\omega_{i0}-\rho]s}\|x(s-\tau_i)\|ds = \int_0^{\tau_i} e^{-[\omega_{i0}-\rho]s}\|\Phi(s-\tau_i)\|ds \leqslant \frac{\|\Phi\|_\infty}{\omega_{i0}-\rho}; \\
& \int_0^{\tau_i} e^{-[\nu_{i0}-\rho]s}\|I^{(\alpha,\beta)}[x(s-\tau_i)]\|ds \\
& \leqslant \frac{1}{\Gamma(\alpha)}\int_0^{\tau_i} e^{-[\nu_{i0}-\rho]s}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}\|x(\sigma-\tau_i)\|d\sigma\right)ds \\
& \leqslant \frac{\|\Phi\|_\infty}{\Gamma(\alpha)}\left(\int_0^\infty e^{-[\nu_{i0}-\rho]s}ds\right)\int_0^\infty \eta^{\alpha-1}e^{-\beta\eta}d\eta \leqslant \frac{\|\Phi\|_\infty}{\beta^\alpha(\nu_{i0}-\rho)}; \\
& \int_0^{\tau_i} e^{-[\nu_{ij}-\rho]s}\|I^{(\alpha,\beta)}[x(s-\tau_i)]\|^{M_j}ds \leqslant \frac{\|\Phi\|_\infty^{M_j}}{\beta^{\alpha M_j}(\nu_{ij}-\rho)}.
\end{aligned} \tag{37}$$

Let $C(\Phi)$ be given by the formula (30). Then the inequalities (36), (37) yield

$$\begin{aligned}
u(t) \leqslant\; & C(\Phi) + Kq\int_0^t e^{-\delta|s-\tau|}|s-\tau|^{\Theta}u(s)ds + K\gamma_0\int_0^t e^{-\delta_0 s}u(s)ds \\
& + K\sum_{i=1}^{k}\eta_{i0}e^{\rho\tau_i}\int_{\tau_i}^t e^{-\omega_{i0}s}u(s-\tau_i)ds \\
& + K\sum_{j=1}^{m}\gamma_j\int_0^t e^{-[M_j-1]\rho s}u(s)^{M_j}ds \\
& + K\sum_{i=1}^{k}\sum_{j=1}^{m}\eta_{ij}e^{M_j\rho\tau_i}\int_{\tau_i}^t e^{-[M_j-1]\rho s}u(s-\tau_i)^{M_j}ds
\end{aligned} \tag{38}$$

$$+K\frac{\xi_0}{\Gamma(\alpha)}\int_0^t e^{-[\mu_0-\rho]s}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}u(\sigma)d\sigma\right)ds$$
$$+K\sum_{j=1}^m \xi_j\int_0^t e^{-[\mu_j-\rho]s}\frac{1}{\Gamma(\alpha)^{M_j}}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}u(\sigma)d\sigma\right)^{M_j}ds$$
$$+K\sum_{i=1}^k \Delta_{i0}\int_{\tau_i}^t e^{-[\nu_{i0}-\rho]s}\frac{1}{\Gamma(\alpha)}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}u(\sigma)d\sigma\right)ds$$
$$+K\sum_{i=1}^k\sum_{j=1}^m \Delta_{ij}\int_{\tau_i}^t e^{-[\nu_{ij}-\rho]s}\frac{1}{\Gamma(\alpha)^{M_j}}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}u(\sigma-\tau_i)d\sigma\right)^{M_j}ds.$$

Denote by $z(t)$ the right-hand side of this inequality. This function is nondecreasing and hence $u(s-\tau_i)\leqslant g(s-\tau_i)\leqslant g(s)$ for all $s\geqslant\tau_i$ and hence we have

$$u(t)\leqslant z(t)\leqslant C(\Phi)+Kq\int_0^t e^{-\delta|s-\tau|}|s-\tau|^{\Theta}z(s)ds+K\gamma_0\int_0^t e^{-\delta_0 s}z(s)ds$$
$$+K\sum_{i=1}^k \eta_{i0}e^{\rho\tau_i}\int_{\tau_i}^t e^{-\omega_{i0}s}z(s)ds+K\sum_{j=1}^m \gamma_j\int_0^t e^{-[M_j-1]\rho s}z(s)^{M_j}ds$$
$$+K\sum_{i=1}^k\sum_{j=1}^m \eta_{ij}e^{M_j\rho\tau_i}\int_{\tau_i}^t e^{-[M_j-1]\rho s}z(s)^{M_j}ds$$
$$+K\frac{\xi_0}{\Gamma(\alpha)}\int_0^t e^{-[\mu_0-\rho]s}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}z(\sigma)d\sigma\right)ds$$
$$+K\sum_{j=1}^m \xi_j\int_0^t e^{-[\mu_j-\rho]s}\frac{1}{\Gamma(\alpha)^{M_j}}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}z(\sigma)d\sigma\right)^{M_j}ds$$
$$+K\sum_{i=1}^k \Delta_{i0}\int_{\tau_i}^t e^{-[\nu_{i0}-\rho]s}\frac{1}{\Gamma(\alpha)}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}z(\sigma)d\sigma\right)ds$$
$$+K\sum_{i=1}^k\sum_{j=1}^m \Delta_{ij}\int_{\tau_i}^t e^{-[\nu_{ij}-\rho]s}\frac{1}{\Gamma(\alpha)^{M_j}}\left(\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}z(\sigma)d\sigma\right)^{M_j}ds. \tag{39}$$

Obviously

$$\int_0^s (s-\sigma)^{\alpha-1}e^{-\beta(s-\sigma)}d\sigma=\int_0^s \eta^{\alpha-1}e^{-\beta\eta}d\eta\leqslant\frac{\Gamma(\alpha)}{\beta^{\alpha}} \tag{40}$$

and since $z(t)$ is nondecreasing, $z(\sigma)\leqslant z(s)$ for all $s\geqslant\sigma$ this yields the following inequality:

$$u(t)\leqslant z(t)\leqslant C(\Phi)+Kq\int_0^t e^{-\delta|s-\tau|}|s-\tau|^{\Theta}z(s)ds+K\gamma_0\int_0^t e^{-\delta_0 s}z(s)ds$$
$$+K\sum_{j=1}^m \gamma_j\int_0^t e^{-[M_j-1]\rho s}z(s)^{M_j}ds+K\sum_{i=1}^k \eta_{i0}e^{\rho\tau_i}\int_0^t e^{-\omega_{i0}s}z(s)ds$$
$$+K\sum_{i=1}^k\sum_{j=1}^m \eta_{ij}e^{M_j\rho\tau_i}\int_0^t e^{-[M_j-1]\rho s}z(s)^{M_j}+\frac{K\xi_0}{\beta^{\alpha}}\int_0^t e^{-[\mu_0-\rho]s}z(s)ds \tag{41}$$

$$+K\sum_{j=1}^{m}\frac{\xi_j}{\beta^{M_j\alpha}}\int_0^t e^{-[\nu_j-\rho]s}z(s)^{M_j}ds+K\sum_{i=1}^{k}\Delta_{i0}e^{\rho\tau_i}\int_0^t e^{-[\nu_{i0}-\rho]s}z(s)ds$$
$$+K\sum_{i=1}^{k}\sum_{j=1}^{m}\Delta_{ij}e^{M_j\rho\tau_i}\int_0^t e^{-[\nu_{ij}-\rho]s}z(s)^{M_j}ds.$$

Let

$$\begin{aligned}
&\eta_j=\max_{1\leqslant i\leqslant k}\eta_{ij}e^{M_j\rho\tau_i},\Delta_j=\max_{1\leqslant i\leqslant k}\Delta_{ij}e^{M_j\rho\tau_i},\nu_j=\min_{1\leqslant i\leqslant k}\nu_{ij},\\
&F_0(t)=Kqe^{-\delta|t-\tau|}|t-\tau|^{\Theta}+K\gamma_0e^{-\delta_0t}\\
&+K\sum_{i=1}^{k}\eta_{i0}e^{\rho\tau_i}e^{-\omega_{i0}t}+\frac{K\xi_0}{\beta^{\alpha}}e^{-[\mu_0-\rho]t}+K\sum_{i=1}^{k}\Delta_{i0}e^{\rho\tau_i}e^{-[\nu_{i0}-\rho]t},\\
&F_j(t)=K\gamma_je^{-[M_j-1]\rho t}+Kk\eta_je^{M_j\rho\tau_i}\int_0^t e^{-[M_j-1]\rho t}\\
&+K\frac{\xi_j}{\beta^{M_j\alpha}}e^{-[\nu_j-\rho]t}+Kk\Delta_je^{-[\nu_j-\rho]t}.
\end{aligned}\tag{42}$$

Then we obtain the inequality

$$z(t)\leqslant C(\Phi)+\int_0^t F_0(s)z(s)ds+\sum_{j=1}^{m}\int_0^t F_j(s)z(s)^{M_j}ds,\ t\in[0,T).\tag{43}$$

Obviously,

$$\int_0^t F_0(s)ds<\int_0^\infty F_0(s)ds\leqslant R_0,\tag{44}$$

where

$$R_0=2Kq\frac{\Gamma(\Theta+1)}{\delta^{\Theta+1}}+\frac{K\gamma_0}{\delta_0}+K\sum_{i=1}^{k}\eta_{i0}\frac{e^{\rho\tau_i}}{\omega_{i0}}+\frac{K\xi_0}{\beta^{\alpha}(\mu_0-\rho)}+K\sum_{i=1}^{k}\Delta_{i0}\frac{e^{\rho\tau_i}}{\nu_{i0}-\rho}<\infty,\tag{45}$$

$$\int_0^t F_j(s)ds<\int_0^\infty F_j(s)ds<R_j,\ j=1,2,\dots,m,\tag{46}$$

where

$$R_j=\frac{K\gamma_j}{[M_j-1]\rho}+\frac{Kk\eta_j}{[M_j-1]\rho}+\frac{K\xi_j}{\beta^{M_j\alpha}}+\frac{Kk\Delta_j}{\nu_j-\rho}\tag{47}$$

and from Lemma 1 it follows

$$\begin{aligned}
&u(t)\leqslant z(t)\\
&\leqslant\frac{C(\Phi)\exp\int_0^t F_0(s)ds}{\left[1-(k_m-1)\sum_{j=1}^{m}C(\Phi)^{k_j-1}\int_0^t F_j(s)\exp\left((k_m-1)\int_0^s F_0(\tau)d\tau\right)ds\right]^{\frac{1}{k_m-1}}}\\
&\leqslant M\|\Phi\|_\infty\,\forall t\in[0,T),
\end{aligned}\tag{48}$$

where

$$M = M(\Phi) = \frac{R_0K[1+\sum_{i=1}^{k}\left(\frac{1}{\omega_{i0}-\rho}+\frac{1}{\nu_{i0}-\rho}\right)+\sum_{i=1}^{k}\sum_{j=1}^{m}\frac{\|\Phi\|_\infty^{M_j-1}}{\beta^{\alpha M_j}(\nu_{ij}-\rho)}]}{\left[1-(k_m-1)\exp(k_m-1)R_0\sum_{j=1}^{m}C(\Phi)^{k_j-1}R_j\right]^{\frac{1}{k_m-1}}} < \infty, \quad (49)$$

under the assumption

$$D(\Phi) := (k_m-1)\exp[(k_m-1)R_0]\sum_{j=1}^{m}C(\Phi)^{k_j-1}R_j < 1, \quad (50)$$

i. e. $u(t) \leqslant M\|\Phi\|_\infty$. This yields the inequality

$$\|x(t)\| \leqslant Me^{-\rho t}\|\Phi\|_\infty \; \forall t \in [0,T). \quad (51)$$

From this inequality it follows that $\lim_{t\to T^-}\|x(t)\| = d < \infty$ and by Theorem 1 there exists an $\varepsilon > 0$ such that the solution $x(t)$ can be extended to the interval $[0,T+\varepsilon)$ and this is a contradiction with the maximality of the solution $x(t), t \in [0,T)$. This means that the solution $x(t)$ exists on $[0,\infty)$ and since the right-hand side of the inequality (51) is independent of t, the inequality (51) holds for all $t \in [0,\infty)$. Finally, if $r < \infty$, then using the Urysohn's lemma [2, Lemma 10.2], the nonlinearities F,G,f,g can be modified by functions $\tilde{F},\tilde{G},\tilde{f},\tilde{g}$, equal to F,G,f,g, respectively, equal to F,G,f,g on the ball $\Omega(r)$ and equal to zero outside a ball $\Omega(\tilde{r})$ with $r < \tilde{r} < \infty$ and so the assertion of theorem follows from the previous case.

Now, we extend the results proved in [12], [13] and [14] (see also [15]). Using the Medina's method and the method of integral inequalities applied in the proof of Theorem (2), we will prove the following theorem.

THEOREM 3. *Suppose that the conditions $(H1)-(H6)$ are satisfied. Let $R_0, R_j, j = 1,2,\ldots,m, C(\Phi)$ be as in Theorem 2 and assume that the following conditions are satisfied:*

(C1)

$$R_0 < 1, \quad (52)$$

(C2)

$$H(\Phi) := (k_m-1)\sum_{j=1}^{m}[VC(\Phi)]^{k_j-1}(\tilde{R}_j) < 1, \quad (53)$$

where $V = (1-R_0)^{-1}, \tilde{R}_j = VR_j, j = 1,2,\ldots,m$, for any Φ with $\|\Phi\|_\infty < r$.

Then there is a constant $M > 0$ such that any solution $x(t) = x(t,\Phi)$ of the initial value problem (1) with $\|\Phi\|_\infty < r$, satisfies the inequality

$$\|x(t,\Phi)\| \leqslant Me^{-\rho t}\|\Phi\|_\infty \; \forall t \geqslant 0. \quad (54)$$

Proof. In the proof of Theorem 2 we have proved the inequality (43) for $z(t)$. Since this function is nondecreasing, this inequality yields

$$z(t) \leqslant C(\Phi) + R_0 z(t) + \sum_{j=1}^{m} \int_0^t F_j(s) z(s)^{k_j} ds, \tag{55}$$

where by the assumption (C1) $R_0 = \int_0^\infty F_0(s) ds < 1$ and so we have the inequality

$$z(t) \leqslant VC(\Phi) + \sum_{j=1}^{m} \int_0^t VF_j(s) z(s)^{k_j} ds, \tag{56}$$

where $V = (1 - R_0)^{-1}$. Hence, we have obtained the integral inequality for $z(t)$ without linear terms in its right-hand side, where we have $\tilde{C}(\Phi) = VC(\Phi)$ instead of $C(\Phi)$ and $\tilde{F}_j = VF_j$ instead of F_j. Therefore the assertion of the theorem can be proved by using Lemma 1 in an analogous way as in the proof of Theorem 2.

3. Illustrative example

From the practical point of view it is useful to work with the logarithmic norm $\mu(B)$, of a square $N \times N$ matrix $B = (b_{ij})$. We will use this norm in an illustrative example given below. This norm is defined by

$$\mu(B) = \lim_{\varepsilon \to 0^+} \frac{\|I + \varepsilon B\| - 1}{\varepsilon}, \tag{57}$$

where I is the unit matrix and $\|.\|$ is a norm on $\mathbb{R}^N$. For example,

$$\mu(B) = \mu_1(B) = \max \left\{ b_{jj} + \sum_{i \neq j}^{n} |b_{ij}| \right\}, \tag{58}$$

with respect to the $1-$norm $\|x\| := \|x\|_1 = \sum_{i=1}^N |x_i|, x = (x_1, x_2, \ldots, x_N)$ (see [12, Lemma 5]). We will apply the following Coppel's inequality:

$$\|e^{Bt}\| \leqslant e^{\mu(B)t}, \ \forall t \geqslant 0. \tag{59}$$

Consider the system

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) \\ &+ F\Big(x(t)\Big) + G(t, x(t-\tau)) + f\big(t, I^{(\alpha,\beta)} x(t)\big) + g\big(t, I^{(\alpha,\beta)}[x(t-\tau)]\big), \ t \geqslant 0, \\ &x(t) = \Phi(t) = (\Phi_1(t), \Phi_2(t)), \ t \in [-\tau, 0], \end{aligned} \tag{60}$$

where $x(t) = (x_1(t), x_2(t))^T$,

$$A(t) = \begin{bmatrix} -[a_1 + d_1(t)] & d_2(t) \\ d_1(t) & -[a_2 + d_2(t)] \end{bmatrix}, \tag{61}$$

a_1, a_2 are positive constants and $d_1(t), d_2(t)$ are continuous nonnegative and bounded functions, satisfying the condition

$$|d_i(t) - d_i(\tau)| \leqslant q_i e^{-\delta|t-\tau|t}|t-\tau|^{\Theta}, i = 1,2 \quad \forall t,s \geqslant 0, \tag{62}$$

where $q_1 > 0, q_2 > 0, \Theta > 0, \tau \geqslant 0$ are constants. This system is a fractional version of the system from [12, Example 9, pp. 4].

Then

$$\|A(t) - A(\tau)\| \leqslant q e^{-\delta|t-\tau|}|t-\tau|^{\Theta} \quad \forall t,s \geqslant 0, \tag{63}$$

where $q = \max\{2q_1, 2q_2\}$.

By [12, (44)]

$$\mu(A(0)) = -\min\{a_1, a_2\}, t \geqslant 0. \tag{64}$$

From the inequality (59) with $B = A(0)$ it follows that the inequality (23) in the condition $(H2)$ is satisfied with $K = 1$ and $\rho := \min\{a_1, a_2\}$.

Let us define the nonlinearities of the system (60) as follows:

$$F(t,u) = \big(F_1(t,u), F_2(t,u)\big) = \gamma_0 u + \gamma_1(u_{11}^2, u_{12}^2), \ \forall t \geqslant 0, x \in \mathbb{R}^2, \tag{65}$$

where $\gamma_1, \gamma_2, \delta_1$ are positive constants, $\delta_0 > \rho, u = (u_{11}, u_{12})$,

$$G(t,u_1) = \eta_{10} e^{-\omega_{10}t} u_1 + \eta_{11}(u_{11}^2, u_{12}^2), \forall t \geqslant 0, u_1, u_2 \in \mathbb{R}^2, \tag{66}$$

where $u_1 = (u_{11}, u_{12}), \eta_{11}$ is a positive constant, $\omega_{10} > \rho$,

$$f(t,u) = \xi_0 e^{-\mu_0 t} u + \xi_1 e^{-\mu_1 t}(u_{11}^2, u_{12}^2), \ \forall t \geqslant 0, u \in \mathbb{R}^2 \tag{67}$$

ξ_0, ξ_1, μ_1 are positive constants, $\mu_0 > \rho, \mu_1 > \rho$ and $u = (u_{11}, u_{12})$,

$$g(t,u_1) = \Delta_{10} e^{-\nu_{10}t} u_1 + \Delta_{11} e^{-\nu_{11}t}(u_{11}^2, u_{12}^2), \tag{68}$$

where $\Delta_{10}, \Delta_{11}, \nu_{10} > \rho$ are positive constants, $\nu_{11} > \rho, u_1 = (u_{11}, u_{12})$. Obviously,

$$\|F(t,u)\| \leqslant \gamma_0\|u\| + \gamma_1 e^{-\delta_1 t}\|u\|^2, \forall t \geqslant 0, x \in \mathbb{R}^2, \tag{69}$$

$$\|G(t,u_1,u_2)\| \leqslant \eta_{10} e^{-\omega_{10}t}\|u_1\| + \eta_{11}\|u_2\|^2 \ \forall t \geqslant 0, u_1, u_2 \in \mathbb{R}^2, \tag{70}$$

$$\|f(t,u)\| \leqslant \xi_0 e^{-\mu_0 t}\|u\| + \xi_1 e^{-\mu_1 t}\|u\|^2 \quad \forall u \in \mathbb{R}^2, \tag{71}$$

$$\|g(t,u_1)\| = \Delta_{10} e^{-\nu_{10}t}\|u_1\| + \Delta_{11} e^{-\nu_{11}t} u_1^2 \, \forall u_1, u_2 \in \mathbb{R}^2. \tag{72}$$

We note that in all these nonlinearities $m = 1, 1 < M_1 = 2$. Since in the equation (60) there is only one delay we have $k = 1$. Now let us express the formulas for $R_0, R_1, C(\Phi)$ and $D(\Phi)$ as follows:

$$R_0 = 2q\frac{\Gamma(\Theta+1)}{\delta^{\Theta+1}} + \frac{\gamma_0}{\delta_0} + \eta_{10}\frac{e^{\rho\tau}}{\omega_{10}} + \frac{\xi_0}{\beta^{\alpha}(\mu_0 - \rho)} + \Delta_{10}\frac{e^{\rho\tau}}{\nu_{10} - \rho}, \tag{73}$$

$$R_1 = \frac{\gamma_1}{\rho} + \frac{\eta_{11}}{\rho} + \frac{\xi_1}{\beta^{2\alpha}} + \Delta_{11}\frac{e^{\rho\tau}}{\nu_{10} - \rho}, \tag{74}$$

$$C(\phi) = \|\Phi\|_\infty \left[1 + \frac{1}{\omega_{10} - \rho} + \frac{1}{\nu_{10} - \rho} + \frac{\|\Phi\|_\infty}{\beta^{2\alpha}(\nu_{11} - \rho)}\right], \tag{75}$$

$$D(\Phi) = (M_1 - 1)e^{(M_1-1)R_0}C(\Phi)R_1 = e^{R_0}C(\Phi)R_1. \tag{76}$$

If $D(\Phi) = e^{R_0}C(\Phi)R_1 < 1$, then by Theorem 2 $\|x(t)\| \leqslant Me^{-\rho t}\|\Phi\|_\infty$, where

$$M = R_0 \frac{1 + \frac{1}{\omega_{10}-\rho} + \frac{1}{\nu_0} + \frac{\|\Phi\|_\infty}{\beta^\alpha(\nu_{11}-\rho)}}{1 - e^{R_0}C(\Phi)R_1}. \tag{77}$$

Acknowledgements. This work was supported by the Slovak Research and Development Agency under the contract APVV -14-0378 and by the Slovak Grant Agency VEGA-MŠ, project No. 1/0078/17.

The authors thank the anonymous reviewers for careful reading of the manuscript and also for valuable comments which significantly improved the original manuscript.

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(Received April 1, 2019)

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