CONTROLLABILITY AND OBSERVABILITY OF
TIME–INVARIANT LINEAR NABLA FRACTIONAL SYSTEMS

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Abstract. In this paper, we study linear time-invariant nabla fractional discrete control systems. The nabla fractional difference operator is considered in the sense of Riemann-Liouville definition of the fractional derivative. We first give necessary and sufficient rank conditions for controllability of the discrete fractional system via Gramian and controllability matrices. We then obtain rank conditions for observability of the discrete fractional system. We illustrate main results with some numerical examples. We close the paper by stating that the rank conditions for the time-invariant linear dynamic system on time scales, fractional system in continuous time, and fractional system in discrete time coincide.

1. Introduction

Nowadays the concept of fractional order derivative and integrals have attracted increasing attention from various fields of science and engineering communities. The main reason for this is that many physical materials and processes can be properly described by using fractional order calculus. It has been proven by scientific findings that many fractional-order mathematical models are the best description for natural phenomena. Most of the research on the applications of the fractional difference/differential calculus are focused on the temporal state of physical change, image processing, viscoelastic theory, controller design, and random fractional dynamics [28, 29, 30, 31].

The study of controllability and observability plays an important role in the control theory and engineering. They have close connections to pole assignment, structural decomposition, quadratic optimal control, observer design, controller design, and so forth. In recent decades, the study of control systems have aroused great interest among applied mathematicians and engineers. For this reason, many active scholars contributed to controllability of continuous time systems [1, 19, 27] and controllability of dynamic systems on time scales [7, 8, 12, 14, 33]. Bartosiewicz and Pawluszewicz [7] proposed the controllability criteria for linear time-invariant dynamic systems on time scales, whereas Fausett and Murty [14] not only studied the controllability of dynamic systems but also obtained the observability and realizability criteria for linear time-invariant dynamic systems on time scales. Davis et al. [12] proved some basic

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results on controllability, observability, and realizability of linear time-invariant dynamic systems, and then extended their results to time-variant systems. Pawluszewicz [8] proposed a necessary and sufficient condition for positive reachability of a positive system on an arbitrary time scale considering Gramian matrix. However, when studying the controllability of dynamic systems [7, 12, 14], one must assume that the graininess function is differentiable, an assumption that is not satisfied in general for all time scales. For this reason, Wintz and Bohner [33] altered the system and obtained controllability of time-invariant linear dynamic systems without assuming differentiability of the graininess function. Due to these solid works, the controllability theory on continuous time systems, dynamic systems, and continuous fractional order systems [9, 10, 21, 25] have been well developed.

In contrast to that for the continuous-time case, the amount of literature which focus on controllability of time-invariant linear discrete systems is much less. The controllability of the linear discrete-time systems have been investigated in [13, 26, 28], and the necessary and sufficient conditions for discrete fractional order systems with the Grünwald-Letnikov operator are given in [15, 18, 32]. Kaczorek [18] introduced the notion of the positive fractional discrete-time linear system and proposed the necessary and sufficient conditions for the positivity, reachability, and controllability to zero. Guermah et al. [15] studied controllability and observability of linear discrete-time fractional-order systems that are modeled by a discrete-time linear system with delays in states. Mozyrska et al. [22] proposed the properties of the h-difference linear control systems with fractional order and developed the rank conditions for controllability and observability of fractional order systems with Caputo-Type operator. Then they extended their results to h-difference linear control systems with n different fractional orders in [23]. Mozyrska et al. [24] investigated the local controllability and observability of nonlinear discrete-time systems considering the Caputo, the Riemann-Liouville, and the Grünwald-Letnikov-type h-difference fractional operators. Atici and Nguyen [5] studied the controllability and observability of the discrete Δ-fractional time-invariant linear systems.

Fractionalizing of mathematical models in the field of applied mathematics is a method which improves the descriptive meaning of the mathematical models of the real world problems, as illustrated in many papers in the area of applied mathematics, physics, computer science, and bioengineering [6, 20, 28, 30]. So the natural question follows: Do we keep or loose the controllability of the discrete system if we fractionalize it?

Motivated by this question and the recent work in discrete time, we shall continue to develop the control theory in discrete time and search for an answer to this question in this paper. We first consider the controllability of the following time-invariant linear nabla fractional system

$$\nabla_{t_0}^{\nu} y(t) = Ay(t-1) + Bu(t-1), \quad t \in \mathbb{N}_{t_0+1}^{t_1}, \tag{1}$$

where A, B are the known constant matrices, $y(t) \in \mathbb{R}^n$ state vector, $u(t) \in \mathbb{R}^m$ is control vector, and $0 < \nu < 1$.

This paper is organized as follows: In Section 2, we recall some fundamental definitions of discrete fractional calculus and give a unique solution of the nabla fractional

order system with an initial condition. Then we state and prove the Putzer Algorithm to evaluate matrix exponential functions in discrete fractional calculus. Section 3 addresses the controllability of the time-invariant nabla fractional systems. We obtain rank conditions via the Gramian and the controllability matrices. In Section 4, we study observability of the time-invariant nabla fractional systems.

2. Preliminaries

The backward difference operator, or nabla operator (∇) for a function $f : \mathbb{N}_a \longrightarrow \mathbb{R}$ is defined by

$$(\nabla f)(t) = f(t) - f(\rho(t)),$$

where $a \in \mathbb{R}$, $\mathbb{N}_a = \{a, a+1, a+2,...\}$ and $\rho(t) = t - 1$ is known as backward jump operator on time scale calculus [11].

We define discrete interval as a set of form

$$\mathbb{N}_a^b = \{a, a+1, ..., b\}$$

where $a, b \in \mathbb{R}$ and $b - a$ is a positive integer.

Let μ be any real number. The rising factorial power $t^{\overline{\mu}}$ (read 't to the μ rising') is defined as

$$t^{\overline{\mu}} = \frac{\Gamma(t+\mu)}{\Gamma(t)},$$

where $t \in \mathbb{R} \setminus \{...,-2,-1,0\}$, $0^{\overline{\mu}} = 0$ and Γ denotes the Gamma function.

We consider the ν-th order fractional sum of f defined as

$$\nabla_a^{-\nu} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \tag{2}$$

where $a \in \mathbb{R}$, $\nu > 0$, and $t \in \mathbb{N}_a$. Further, we consider the ν-th order fractional difference (a Riemann-Liouville fractional difference) of f defined by

$$\nabla_a^{\nu} f(t) = \nabla^n (\nabla_a^{-(n-\nu)} f(t)) \tag{3}$$

where $\nu > 0$, $n - 1 < \nu < n$, n denotes a positive integer.

For further reading, we refer the reader to [2, 3, 17]

Next, we define the following function which will be used throughout the paper.

$$\widehat{y}_{\lambda,\nu}(t,a) := \sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)\nu-1}}}{\Gamma((n-a+1)\nu)},$$

where λ is any constant number, ν is positive real number and $t \in \mathbb{N}_a$.

Subsequently, we give the following theorem. The prove of this theorem can be done similarly as in the paper (Theorem 2.5, [4]).

THEOREM 1. *Assume* $\lambda \in \mathbb{R}$. *The fractional difference equation of order* v *where* $v \in (0,1)$

$$\nabla_a^v y(t) = \lambda y(t-1) + f(t-1) \quad for \quad t \in \mathbb{N}_{a+1} \tag{4}$$

has the general solution

$$y(t) = \widehat{y}_{\lambda,v}(t,a)c + \sum_{s=a}^{t-1} \widehat{y}_{\lambda,v}(t+a-s-1,a)f(s), \quad t \in \mathbb{N}_a \tag{5}$$

where c is constant.

Next, we give some properties of $\widehat{y}_{\lambda,v}(t,a)$.

LEMMA 1. *The following properties hold:*

(i) $\widehat{y}_{\lambda,v}(a,a) = 1$, *where* $\lambda \in \mathbb{R}$ *and* v *is positive real number.*

(ii) $\nabla_a^v \widehat{y}_{\lambda,v}(t,a) = \lambda \widehat{y}_{\lambda,v}(t-1,a)$, *where* $0 < v < 1$ *and* $\lambda \in \mathbb{R}$.

(iii) *For* $v \geqslant 1$ *and* λ *any positive real number,* $\widehat{y}_{\lambda,v}(t,a)$ *is monotone increasing on* $\mathbb{N}_a$.

(iv) *For* $\lambda \geqslant 1$ *and* v *any positive real number,* $\widehat{y}_{\lambda,v}(t,a)$ *is increasing on* $\mathbb{N}_a$.

Proof. (i) The proof follows from the definition of $\widehat{y}_{\lambda,v}(t,a)$.

(ii) The proof is similar to the proof of Theorem 2.3 in [4].

(iii) The proof relies on the fact that if $\nabla f(t) \geqslant 0$ on $\mathbb{N}_{a+1}$, then the function f is monotone increasing on $\mathbb{N}_a$. Here we apply the nabla operator to the function $\widehat{y}_{\lambda,v}(t,a)$, we have

$$\nabla \widehat{y}_{\lambda,v}(t,a) = \nabla \sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)}$$

$$= \sum_{n=a}^{t} \nabla \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)} + \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)}\Big|_{t=t-1,n=t}$$

$$= \sum_{n=a}^{t} \frac{\lambda^{n-a}\Gamma(t-n+(n-a+1)v-1)}{\Gamma(t-n+1)\Gamma((n-a+1)v-1)},$$

where we used the identity

$$\nabla \sum_{n=0}^{t} f(t,n) = \sum_{n=0}^{t} \nabla f(t,n) + f(\rho(t),t).$$

The last quantity is positive if $v \geqslant 1$.

(iv) Let t be in $\mathbb{N}_a$. We show that $\widehat{y}_{\lambda,v}(t,a)$ is increasing if $\widehat{y}_{\lambda,v}(t+1,a) > \widehat{y}_{\lambda,v}(t,a)$.

$$\widehat{y}_{\lambda,v}(t+1,a) = \sum_{n=a}^{t+1} \frac{\lambda^{n-a}(t-n+2)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)}$$

$$= \frac{\Gamma(t-av+1)}{\Gamma(t-a+2)\Gamma(v)} + \sum_{n=a+1}^{t+1} \frac{\lambda^{n-a}(t-n+2)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)}$$

$$> \sum_{n=a}^{t} \frac{\lambda^{n+1-a}\Gamma(t-n+(n-a+2)v)}{\Gamma(t-n+1)\Gamma((n-a+2)v)}.$$

Next, we use one of the properties of the Gamma function, namely $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. We have

$$\frac{\Gamma(t-n+(n-a+2)v)}{\Gamma((n-a+2)v)}$$

$$= \frac{(t-n+(n-a+2)v-1)(t-n+(n-a+2)v-2)\cdots((n-a+2)v)\Gamma((n-a+2)v)}{\Gamma((n-a+2)v)}$$

$$\geqslant \frac{(t-n+(n-a+1)v-1)(t-n+(n-a+1)v-2)\cdots((n-a+1)v)\Gamma((n-a+1)v)}{\Gamma((n-a+1)v)}$$

$$= \frac{\Gamma(t-n+(n-a+1)v)}{\Gamma((n-a+1)v)},$$

since v is positive integer. Using this inequality, we obtain

$$\widehat{y}_{\lambda,v}(t+1,a) > \sum_{n=a}^{t} \frac{\lambda^{n+1-a}\Gamma(t-n+(n-a+1)v)}{\Gamma(t-n+1)\Gamma((n-a+1)v)}$$

$$\geqslant \sum_{n=a}^{t} \frac{\lambda^{n-a}\Gamma(t-n+(n-a+1)v)}{\Gamma(t-n+1)\Gamma((n-a+1)v)}$$

$$= \sum_{n=a}^{t} \frac{\lambda^{n-a}(t-n+1)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)}$$

$$= \widehat{y}_{\lambda,v}(t,a)$$

if $\lambda \geqslant 1$.

REMARK 1. One can easily verify that the solution of the following IVP

$$\nabla_a^v y(t) = Ay(t-1) \quad t \in \mathbb{N}_{a+1} \tag{6}$$

$$\nabla_a^{-(1-v)} y(t)|_{t=a} = y(a) = y_0, \tag{7}$$

where A is an $n \times n$ constant matrix, and y_0 and $y(.)$ are $n \times 1$ vectors, is

$$y(t) = \sum_{n=a}^{t} \frac{A^{n-a}(t-n+1)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)} y_0, \quad t \in \mathbb{N}_a. \tag{8}$$

Define

$$\widehat{y}_{A,v}(t,a) := \sum_{n=a}^{t} \frac{A^{n-a}(t-n+1)^{\overline{(n-a+1)v-1}}}{\Gamma((n-a+1)v)}, \quad t \in \mathbb{N}_a \tag{9}$$

In the following lemma we list some important properties for $\widehat{y}_{A,v}(t,a)$.

LEMMA 2. *For any given $n \times n$ matrix A, the following properties hold:*

(i) $\widehat{y}_{A,v}(a,a) = I_n$.

(ii) $\nabla_a^v \widehat{y}_{A,v}(t,a) = A\widehat{y}_{A,v}(t-1,a), \qquad t \in \mathbb{N}_{a+1}$, *where* $0 < v < 1$.

Next we want to give an algorithm to calculate $\widehat{y}_{A,v}(t,a)$ in terms of $\widehat{y}_{\lambda,v}(t,a)$ where λ is an eigenvalue of the matrix A. For this purpose, we first define the matrix exponential function in discrete fractional calculus. Then we will state and prove the Putzer algorithm for any $n \times n$ matrix.

DEFINITION 1. (Matrix exponential function) Let A be an $n \times n$ constant matrix. The unique matrix valued solution of the initial value problem(IVP)

$$\nabla_a^v Y(t) = AY(t-1) \quad \text{for} \quad t \in \mathbb{N}_{a+1} \tag{10}$$

$$\nabla_a^{-(1-v)} Y(t)|_{t=a} = Y(a) = I_n, \tag{11}$$

where I_n denotes the $n \times n$ identity matrix, is called the matrix exponential function.

THEOREM 2. *If $\lambda_1, \lambda_2, ..., \lambda_n$ are (not necessarily distinct) eigenvalues of the $n \times n$ matrix A, with each eigenvalue repeated as many times as its multiplicity, then*

$$\widehat{y}_{A,v}(t,a) = \sum_{i=0}^{n-1} p_{i+1}(t)M_i,$$

where

$$M_0 = I_n,$$

$$M_i = (A - \lambda_i I_n)M_{i-1}, \quad (1 \leqslant i \leqslant n-1), \tag{12}$$

$$M_n = 0,$$

and vector valued function p defined by

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ \vdots \\ p_n(t) \end{bmatrix}$$

is the solution of the initial value problem

$$\nabla_a^v p(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} p(t-1) \qquad \text{for all} \qquad t \in \mathbb{N}_{a+1} \qquad (13)$$

$$\nabla_a^{-(1-v)} p(t)|_{t=a} = p(a) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad (14)$$

Proof. Let $\Phi(t) = \sum_{i=0}^{n-1} p_{i+1}(t) M_i$. We first show that Φ solves the IVP (10)-(11). First note that

$$\nabla_a^{-(1-v)} \Phi(a) = \nabla_a^{-(1-v)} p_1(a) M_0 + \nabla_a^{-(1-v)} p_2(a) M_1 + \cdots + \nabla_a^{-(1-v)} p_n(a) M_{n-1} = I_n$$

since $p(a) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T$.

$$\nabla_a^v \Phi(t) - A\Phi(t-1)$$
$$= \nabla_a^v \sum_{i=0}^{n-1} p_{i+1}(t) M_i - A \sum_{i=0}^{n-1} p_{i+1}(t-1) M_i$$
$$= \nabla_a^v p_1(t) M_0 + \nabla_a^v p_2(t) M_1 + \cdots + \nabla_a^v p_n(t) M_{n-1} - A \sum_{i=0}^{n-1} p_{i+1}(t-1) M_i,$$

since ∇_a^v is a linear operator. Next we use (13), so the last quantity equals

$$= \lambda_1 p_1(t-1) M_0 + [p_1(t-1) + \lambda_2 p_2(t-1)] M_1 + [p_2(t-1) + \lambda_3 p_3(t-1)] M_2$$
$$+ \cdots + [p_{n-1}(t-1) + \lambda_n p_n(t-1)] M_{n-1} - A \sum_{i=0}^{n-1} p_{i+1}(t-1) M_i$$
$$= [\lambda_1 M_0 + M_1 - AM_0] p_1(t-1) + [\lambda_2 M_1 + M_2 - AM_1] p_2(t-1)$$
$$+ \cdots + [\lambda_n M_{n-1} - AM_{n-1}] p_n(t-1)$$
$$= [\lambda_n I_n - A] M_{n-1} p_n(t-1),$$

since $M_i = (A - \lambda_i I_n) M_{i-1}$ for $(1 \leqslant i \leqslant n)$. The last quantity is zero matrix by the Cayley-Hamilton theorem. In fact, we have

$$(\lambda_n I_n - A) M_{n-1} p_n(t-1) = -(A - \lambda_n I_n)(A - \lambda_{n-1} I_n) \cdots (A - \lambda_1 I_n) p_n(t-1) = 0_{n \times n}.$$

Since $\widehat{y}_{A,v}(t,a)$ satisfies the IVP (10)-(11), we have

$$\Phi(t) = \widehat{y}_{A,v}(t,a)$$

by the unique solution of given initial value problem.

Next, we will give an example to illustrate the use of Putzer algorithm for 2×2 matrix.

EXAMPLE 1. Given $A = \begin{bmatrix} -0.2 & 0.5 \\ 0.6 & -0.1 \end{bmatrix}$, with eigenvalues $\lambda_1 = 0.4$, $\lambda_2 = -0.7$.

By using Theorem 1 we find that the solution of the IVP (13)-(14) is given by

$$p_1(t) = \widehat{y}_{.4,v}(t,a) \quad \text{and} \quad p_2(t) = \sum_{s=a}^{t-1} \widehat{y}_{-.7,v}(t+a-s-1,a)\widehat{y}_{.4,v}(s,a).$$

Now we compute $\widehat{y}_{A,v}(t,a)$ by using Theorem 2

$$\widehat{y}_{A,v}(t,a) = \widehat{y}_{.4,v}(t,a) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + p_2(t) \begin{bmatrix} -0.6 & 0.5 \\ 0.6 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{y}_{.4,v}(t,a) - .6p_2(t) & .5p_2(t) \\ .6p_2(t) & \widehat{y}_{.4,v}(t,a) - .5p_2(t) \end{bmatrix}.$$

We close this section by stating the variation of constant formula for the system of fractional difference equations. Since the proof of this theorem is similar to the proof of Theorem 1.1, we omit it.

THEOREM 3. *(Variation of constants.) Let $v \in \mathbb{R}$, $0 < v < 1$, A be an $n \times n$ constant matrix. Suppose $f(t)$ is an $n \times 1$ vector valued function. Then the initial value problem*

$$\nabla_a^v y(t) = Ay(t-1) + f(t-1), \qquad t \in \mathbb{N}_{a+1}$$

$$\nabla_a^{-(1-v)} y(t)|_{t=a} = y(a) = y_0,$$

has a unique solution. Moreover, the solution is given by

$$y(t) = \widehat{y}_{A,v}(t,a)y_0 + \sum_{s=a}^{t-1} \widehat{y}_{A,v}(t+a-s-1,a)f(s), \quad t \in \mathbb{N}_a. \tag{15}$$

3. Controllability

In this section, we establish the criterion for controllability of the linear discrete-fractional time-invariant system

$$\nabla_{t_0}^v y(t) = Ay(t-1) + Bu(t-1), \quad t \in \mathbb{N}_{t_0+1}^{t_1}, \tag{16}$$

where $y(t_0) = y_0$ is initial state, A is an $n \times n$ constant matrix, $y(t)$ is an $n \times 1$ state vector, B is an $n \times m$ constant matrix and $u(t)$ is an $m \times 1$ control vector, $m \leqslant n$ and $0 < v < 1$. Because the output does not play any role in controllability, the output equation is disregarded in this study. By Theorem 3 the corresponding solution of the system (16) is

$$y(t) = \widehat{y}_{A,v}(t,t_0)y_0 + \sum_{s=t_0}^{t-1} \widehat{y}_{A,v}(t + t_0 - s - 1, t_0)Bu(s). \tag{17}$$

In the following definitions we assume that $t_0, t_1 \in \mathbb{R}^+$ and $t_1 - t_0 \in \mathbb{Z}^+$.

DEFINITION 2. A system modeled by (16) or pair $\{A, B\}$ is said to be completely controllable, if it is possible to construct a control signal $u(t)$ that will transfer any initial state $y(t_0)$ to any final state $y(t_1)$ in finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1 - 1}$. Otherwise the system (16) or $\{A, B\}$ is said to be uncontrollable.

DEFINITION 3. If every non-zero initial state $y(t_0)$ can be transfered to final state $y(t_1) = 0_{n \times 1}$, by control signal $u(t)$ in finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1 - 1}$, then the system (16) is said to be controllable to the origin.

To give necessary and sufficient conditions for controllability of the linear system (16) we will define controllability matrix and controllability Gramian matrix.

The *controllability matrix* $\widehat{W}$ of the system (16) is defined as an $n \times (nm)$ matrix

$$\widehat{W} := \begin{bmatrix} B\ AB\ A^2B\ \cdots\ A^{n-1}B \end{bmatrix},$$

and we define *controllability Gramian matrix* $\mathscr{P}$ of the system (16) as an $n \times n$ matrix

$$\mathscr{P}(t,t_0) := \sum_{s=t_0}^{t-1} \widehat{y}_{A,v}(s,t_0)BB^T [\widehat{y}_{A,v}(s,t_0)]^T.$$

THEOREM 4. *The following statements are equivalent:*

(i) *The system $\nabla_{t_0}^v y(t) = Ay(t-1) + Bu(t-1)$ is completely controllable on discrete time interval $\mathbb{N}_{t_0+1}^{t_1}$.*

(ii) *The $n \times n$ controllability Gramian matrix $\mathscr{P}(t_1, t_0)$ has rank n.*

(iii) *The controllability matrix $\widehat{W}$ has rank n.*

Proof. $(i) \Leftrightarrow (ii)$

First we show that if a given system is completely controllable then controllability Gramian matrix $\mathscr{P}(t_1, t_0)$ of the given system has rank n. Let us prove this by

contradiction. Suppose that $\text{rank}(\mathscr{P}(t_1,t_0)) < n$. And then there exists nonzero vector $\eta \in \mathbb{R}^n$ such that $\eta^T \mathscr{P}(t_1,t_0) = 0_{1 \times n}$. Then it follows that

$$0 = \eta^T \mathscr{P}(t_1,t_0)\eta = \sum_{s=t_0}^{t_1-1} \eta^T \widehat{y}_{A,v}(s,t_0)BB^T [\widehat{y}_{A,v}(s,t_0)]^T \eta = \sum_{s=t_0}^{t_1-1} ||\eta^T \widehat{y}_{A,v}(s,t_0)B||_2^2,$$

where $||\cdot||_2$ defines the Euclidean norm. Hence

$$\eta^T \widehat{y}_{A,v}(t,t_0)B = 0_{1 \times m}, \qquad t \in \mathbb{N}_{t_0}^{t_1-1}. \tag{18}$$

From the controllable assumption there exists control signal $u(t)$ that will transfer initial state $y(t_0) = y_0$ to final state $y(t_1) = y_f = \widehat{y}_{A,v}(t_1,t_0)y_0 + \eta$. By substitution initial and final state to (17) the solution of the given system becomes

$$\widehat{y}_{A,v}(t_1,t_0)y_0 + \eta = \widehat{y}_{A,v}(t_1,t_0)y_0 + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s)$$

$$\eta = \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s).$$

Multiplying though by η^T and using (18) yields

$$\eta^T \eta = \sum_{s=t_0}^{t_1-1} \eta^T \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s) = 0,$$

which contradicts the assumption that η is a nonzero vector in $\mathbb{R}^n$ Thus, the controllability Gramian matrix $\mathscr{P}(t_1,t_0)$ has rank n.

Conversely, suppose $\mathscr{P}(t_1,t_0)$ has rank n. Then it follows that $\mathscr{P}(t_1,t_0)$ is invertible. Therefore, for the given any initial state $y(t_0) = y_0$ and final state $y(t_1) = y_f$ we can choose the control signal $u(t)$ as

$$u(t) = B^T [\widehat{y}_{A,v}(t_1 + t_0 - t - 1, t_0)]^T [\mathscr{P}(t_1,t_0)]^{-1}[y_f - \widehat{y}_{A,v}(t_1,t_0)y_0].$$

The corresponding solution of the system at $t = t_1$ can be written as

$$y(t_1) = \widehat{y}_{A,v}(t_1,t_0)y_0 + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s)$$

$$= \widehat{y}_{A,v}(t_1,t_0)y_0 + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)BB^T [\widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)]^T$$

$$\times [\mathscr{P}(t_1,t_0)]^{-1}[y_f - \widehat{y}_{A,v}(t_1,t_0)y_0].$$

By performing the above last summation we obtain

$$\sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)BB^T [\widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)]^T$$

$$= \widehat{y}_{A,v}(t_1-1,t_0)BB^T[\widehat{y}_{A,v}(t_1-1,t_0)]^T + \widehat{y}_{A,v}(t_1-2,t_0)BB^T[\widehat{y}_{A,v}(t_1-2,t_0)]^T$$
$$+ \cdots + \widehat{y}_{A,v}(t_0,t_0)BB^T[\widehat{y}_{A,v}(t_0,t_0)]^T$$
$$= \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(s,t_0)BB^T[\widehat{y}_{A,v}(s,t_0)]^T = \mathscr{P}(t_1,t_0).$$

Hence we have

$$y(t_1) = \widehat{y}_{A,v}(t_1,t_0)y_0 + \mathscr{P}(t_1,t_0)[\mathscr{P}(t_1,t_0)]^{-1}[y_f - \widehat{y}_{A,v}(t_1,t_0)y_0] = y_f.$$

This shows that if the controllability Gramian matrix $\mathscr{P}(t_1,t_0)$ has rank n, then a given system is completely controllable on given discrete time interval.

$(i) \Leftrightarrow (iii)$ First we note that for all $N \geqslant n$ the rank of matrix $\widehat{W}(N) = [B \; AB \; A^2B \; \cdots \; A^{N-1}B]$ is equal to the rank of the controllability matrix $\widehat{W}$. By the Cayley-Hamilton theorem

$$A^n = \sum_{s=0}^{n-1} p_s A^s,$$

where $-p_s$ are coefficients of characteristic polynomial of A. Multiplying the above last expression by the matrix B we obtain

$$A^n B = \sum_{s=0}^{n-1} p_s A^s B.$$

Thus columns of $A^n B$ are linearly dependent as the columns of $\widehat{W}$ and rank($\widehat{W}(n+1)$) = rank($\widehat{W}$). Multiplying last equation by the matrix A we obtain

$$A^{n+1}B = \sum_{s=0}^{n-1} p_s A^{s+1} B.$$

Consequently, rank($\widehat{W}(n+2)$) = rank($\widehat{W}(n+1)$) = rank($\widehat{W}$). Proceeding forward, we can concludes that rank($\widehat{W}(N)$) = rank($\widehat{W}$) for all $N \geqslant n$. Here we assume that $t_1 - t_0 = n$.

First we show that if the given system is completely controllable, then the controllability matrix has full rank n. Since given system completely controllable, there exists $u(t)$ control signal that will transfer any given initial state $y(t_0) = y_0 \in \mathbb{R}^n$ to any final state $y(n+t_0) = y_f \in \mathbb{R}^n$. Plugging $t_1 = n+t_0$ into the solution (17) yields

$$y_f = \widehat{y}_{A,v}(n+t_0,t_0)y_0 + \sum_{s=t_0}^{n+t_0-1} \widehat{y}_{A,v}(n+2t_0-s-1,t_0)Bu(s).$$

By performing the sum we obtain

$$y(n+t_0) - \widehat{y}_{A,v}(n+t_0,t_0)y_0$$
$$= \sum_{s=t_0}^{n+t_0-1} \widehat{y}_{A,v}(s,t_0)Bu(n+2t_0-s-1)$$

$$= \sum_{s=t_0}^{n+t_0-1} \sum_{\tau=t_0}^{s} \frac{A^{\tau-t_0}(s-\tau+1)^{\overline{(\tau-t_0+1)v-1}}}{\Gamma((\tau-t_0+1)v)} Bu(n+2t_0-s-1)$$

$$= \sum_{\tau=t_0}^{n+t_0-1} A^{\tau-t_0}B \sum_{s=\tau}^{n+t_0-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)v-1}}}{\Gamma((\tau-t_0+1)v)} u(n+2t_0-s-1),$$

where we interchanged the order of the summations. Next, we define $F(\tau)$ for $t_0 \leqslant \tau \leqslant n+t_0-1$ by

$$F(\tau) = \sum_{s=\tau}^{n+t_0-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)v-1}}}{\Gamma((\tau-t_0+1)v)} u(n+2t_0-s-1).$$

Substituting $F(\tau)$ into last equation, we have

$$y_f - \widehat{y}_{A,v}(n+t_0,t_0)y_0 = \sum_{\tau=t_0}^{n+t_0-1} A^{\tau-t_0}BF(\tau)$$

$$y_f - \widehat{y}_{A,v}(n+t_0,t_0)y_0 = \begin{bmatrix} B & AB & A^2B \cdots A^{n-1}B \end{bmatrix} \begin{bmatrix} F(t_0) \\ F(t_0+1) \\ F(t_0+2) \\ \vdots \\ F(t_0+n-1) \end{bmatrix} = \widehat{W}F_1(n). \quad (19)$$

Suppose the controllability matrix $\widehat{W}$ has rank less than n, then this implies that there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\eta^T\widehat{W} = 0_{1\times(mn)}$. Hence, multiplying both sides of (19) by η^T yields $\eta^T(y_f - \widehat{y}_{A,v}(n+t_0,t_0)y_0) = 0_{1\times n}$ regardless of control signal $u(t)$. Since the given system is completely controllable, we choose $y_f = \widehat{y}_{A,v}(n+t_0,t_0)y_0 + \eta$. Then $\eta^T\eta = 0$ which contradicts the assumption that η is a nonzero vector. Therefore, $\text{rank}(\widehat{W}) = n$.

For the converse, suppose $\text{rank}(\widehat{W}) = n$, but for the sake of a contradiction, we assume that the given system is uncontrollable. Since the system is uncontrollable, then the controllability Gramian matrix $\mathscr{P}(t_0+n,t_0)$ has rank less than n. Hence there exists $\eta \in \mathbb{R}^n$ such that $\eta^T\mathscr{P}(t_0+n,t_0) = 0_{1\times n}$. Then we have

$$0 = \eta^T\mathscr{P}(t_0+n,t_0)\eta = \sum_{s=t_0}^{n+t_0-1} \eta^T\widehat{y}_{A,v}(s,t_0)BB^T[\widehat{y}_{A,v}(s,t_0)]^T\eta$$

$$= \sum_{s=t_0}^{n+t_0-1} ||\eta^T\widehat{y}_{A,v}(s,t_0)B||_2^2,$$

which implies that

$$\eta^T\widehat{y}_{A,v}(t,t_0)B = 0_{1\times m} \qquad \text{for all} \quad t \in \mathbb{N}_{t_0}^{t_0+n-1}. \quad (20)$$

Setting $t = t_0$ and using Lemma 2 (i) we have

$$\eta^TB = 0_{1\times m}.$$

Applying v-th order fractional difference operator to the each side of the last equality and using Lemma 2 we have $\eta^T A \widehat{y}_{A,v}(t-1,t_0)B = 0_{1 \times m}$ for all $t \in \mathbb{N}_{t_0+1}^{t_0+n-1}$. Hence we have

$$\eta^T A \widehat{y}_{A,v}(t,t_0)B = 0_{1 \times m} \quad \text{for all} \quad t \in \mathbb{N}_{t_0}^{t_0+n-2}.$$

Setting $t = t_0$ and using Lemma 2 we have

$$\eta^T A B = 0_{1 \times m}.$$

Repeating the same step up to $n-1$ times, we have

$$\eta^T A^k B = 0_{1 \times m} \quad \text{for} \quad k = 0, 1, ..., n-1.$$

Then we have

$$\eta^T \left[B \ AB \ A^2 B \ \cdots \ A^{n-1} \right] = \eta^T \widehat{W} = 0_{1 \times (mn)}.$$

This contradicts the assumption that $\mathrm{rank}(\widehat{W}) = n$. Thus the controllability Gramian matrix has rank n implies that the given system is completely controllable.

REMARK 2. Note that, for every $\eta \in \mathbb{R}^n$

$$\eta^T \mathscr{P}(t_1, t_0) \eta = \sum_{s=t_0}^{t_1-1} ||\eta^T \widehat{y}_{A,v}(s,t_0)B||^2.$$

Hence the Controllability Gramian matrix $\mathscr{P}(t_1, t_0)$ is a non-negative symmetric matrix.

Next, we provide an example to illustrate the applicability of the Theorem 4.

EXAMPLE 2. Consider the following system

$$\nabla_{t_0}^v y(t) = \begin{bmatrix} -1 & 1 & 1 \\ 5 & -9 & 1 \\ 6 & -3 & -1 \end{bmatrix} y(t-1) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t-1),$$

where $0 < v < 1$.

Then the controllability matrix of the system is

$$\widehat{W} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & -4 & 1 & 39 & -5 \\ 0 & 1 & 3 & -1 & 9 & 4. \end{bmatrix}.$$

It can be easily verified that the rank of $\widehat{W}$ is 3. Thus, by Theorem 4 the given linear fractional order system is completely controllable.

Next we give an extra assumption on $\widehat{y}$ to prove that completely controllability and controllability to the origin are equivalent concepts for the given system (16).

THEOREM 5. *If $\widehat{y}_{A,v}(\cdot, t_0)$ in (16) is non-singular on discrete time interval $t \in \mathbb{N}_{t_0+1}^{t_1}$, then the given system is completely controllable if and only if the system is controllable to the origin.*

Proof. Suppose that the system (16) is completely controllable. Choose final state as $y(t_1) = 0_{n \times 1}$. Then by the Definition 3 the given system is controllable to the origin.

Assume that $\widehat{y}_{A,v}(\cdot, t_0)$ in (16) is non-singular on discrete time interval $\mathbb{N}_{t_0+1}^{t_1}$ and the system (16) is controllable to the origin. For given any initial state $y(t_0)$ and any final state $y(t_1)$, define

$$x(t_0) := y(t_0) - [\widehat{y}_{A,v}(t_1, t_0)]^{-1} y(t_1) \qquad x(t_1) := 0_{n \times 1}.$$

Then we obtain a system with initial state $x(t_0)$ and final state $x(t_1)$, by assumption there exists $u(t)$ in finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1-1}$, such that $x(t_0)$ can be transferred to $x(t_1)$. By Theorem 3 we have,

$$x(t_1) = \widehat{y}_{A,v}(t_1, t_0)x(t_0) + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s),$$

$$0_{n \times 1} = \widehat{y}_{A,v}(t_1, t_0)[y(t_0) - [\widehat{y}_{A,v}(t_1, t_0)]^{-1} y(t_1)] + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s),$$

$$0_{n \times 1} = \widehat{y}_{A,v}(t_1, t_0)y(t_0) - y(t_1) + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s),$$

$$y(t_1) = \widehat{y}_{A,v}(t_1, t_0)y(t_0) + \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(t_1 + t_0 - s - 1, t_0)Bu(s),$$

which for any given initial state $y(t_0)$ and any final state $y(t_1)$ there exists control vector $u(t)$. This means that the given system is completely controllable.

REMARK 3.   (*i*) In [5], a criterion for controllability of the following discrete fractional system has been obtained.

$$\Delta_{v-1}^v y(t) = Ay(t + v - 1) + Bu(t + v - 1), \tag{21}$$

where $y(v - 1) = y_0$ and A, B, and $u(t + v - 1)$ are $n \times n$, $n \times m$ and $m \times 1$ matrices respectively.

The controllability matrix of this system was given as:

$$\hat{\mathbf{W}} = [E_A(v - 1)B \quad E_A(v)B \quad E_A(v + 1)B \quad \dots \quad E_A(n + v - 2)B],$$

where

$$E_A(t) = \sum_{i=0}^{\infty} \frac{A^i}{\Gamma((i+1)v)}(t + i(v - 1))^{((i+1)v-1)}.$$

The controllability criterion: The system (21) is completely controllable $\Leftrightarrow \text{rank}(\hat{\mathbf{W}}) = n$.

Using some basics of linear algebra, one can show that

$$\text{rank}(\hat{\mathbf{W}}) = \text{rank}(\begin{bmatrix} B \ AB \ A^2B \ \cdots \ A^{n-1}B \end{bmatrix}).$$

(*ii*) Using Theorem (4.3) in a recent paper by Goodrich and Lizama [16], we observe that the systems (16) and (21) coincide. In other word, one system can be obtained from other using the operator of translation (see [16] for its definition).

4. Observability

In this section we discuss the observability of the following linear discrete fractional system

$$\begin{cases} \nabla_{t_0}^{\nu} y(t) = Ay(t-1) + Bu(t-1), & t \in \mathbb{N}_{t_0+1}^{t_1} \\ z(t) = Cy(t) + Du(t) \end{cases} \tag{22}$$

where $z(t)$ is an $r \times 1$ the output vector, C is an $r \times n$ constant matrix, D is an $r \times m$ constant matrix $A, B, y(\cdot), u(\cdot)$ are defined as in (16).

Suppose we are given $z(t)$ and $u(t)$ for $t \in \mathbb{N}_{t_0}^{t_1}$. We substitute the solution of the state system (20) into the output measurement and we obtain

$$z(t) = Cy(t) + Du(t)$$
$$= C[\hat{y}_{A,\nu}(t_1,t_0)y_0 + \sum_{s=t_0}^{t_1-1} \hat{y}_{A,\nu}(t_1+t_0-s-1,t_0)Bu(s)] + Du(t).$$

Hence we have

$$C\hat{y}_{A,\nu}(t_1,t_0)y_0 = z(t) - C\sum_{s=t_0}^{t_1-1} \hat{y}_{A,\nu}(t_1+t_0-s-1,t_0)Bu(s) - Du(t).$$

Since A, B, C, D matrices and control vector $u(t)$ are given, the last two terms on the right-hand side of this equation are known quantities. Thus, we can subtract known terms from observed value of output vector $z(t)$ and we define right-hand side by $z_1(t)$. Then response of the system (22) can be written as

$$C\hat{y}_{A,\nu}(t_1,t_0)y_0 = z_1(t). \tag{23}$$

DEFINITION 4. The system (22) is said to be completely observable, if every state $y(t_0)$ can be uniquely determined from the observation of $z(t)$ over a finite discrete time interval $t \in \mathbb{N}_{t_0}^{t_1}$. Otherwise the system (22) or $\{A,C\}$ is said to be unobservable.

We define the *observability matrix* $\hat{O}$ of this system as $(nr) \times n$ matrix

$$\hat{O} := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Next, we define *observability Gramian matrix* $\mathscr{R}(t,t_0)$ of the system (22) as an $n \times n$ matrix

$$\mathscr{R}(t,t_0) := \sum_{s=t_0}^{t-1} \widehat{y}_{A,v}(s,t_0)^T C^T C \widehat{y}_{A,v}(s,t_0).$$

THEOREM 6. *The following statements are equivalent:*

(i) *The system* (22) *is completely observable on* $\mathbb{N}_{t_0}^{t_1}$.

(ii) *The observability Gramian matrix* $\mathscr{R}(t_1,t_0)$ *has rank n.*

(iii) *The observability matrix* $\widehat{O}$ *has rank n.*

Proof. $(i) \Longleftrightarrow (ii)$.

First we show that if the given system is completely observable, then rank$(\mathscr{R}(t_1,t_0))$ $= n$. We will prove the contrapositive, suppose rank$(\mathscr{R}(t_1,t_0)) < n$, then there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\mathscr{R}(t_1,t_0)\eta = 0_{n \times 1}$. Then we have

$$0 = \eta^T \mathscr{R}(t_1,t_0)\eta = \sum_{s=t_0}^{t_1-1} \eta^T \widehat{y}_{A,v}(s,t_0)^T C^T C \widehat{y}_{A,v}(s,t_0)\eta = \sum_{s=t_0}^{t_1-1} ||C\widehat{y}_{A,v}(s,t_0)\eta||_2^2,$$

which implies $C\widehat{y}_{A,v}(t,t_0)\eta = 0_{r \times 1}$ for all $t \in \mathbb{N}_{t_0}^{t_1-1}$. Thus $y(t_0) = y_0 + \eta$ yields same response for the system as $y(t_0) = y_0$ and contradicts the assumption that the given system is completely observable. Therefore rank$(\mathscr{R}(t_1,t_0)) = n$.

On the other hand, suppose the matrix $\mathscr{R}(t_1,t_0)$ has rank n. Multiplying both sides of (23) by $\widehat{y}_{A,v}(t,t_0)^T C^T$ and taking summation over the discrete interval $t \in \mathbb{N}_{t_0}^{t_1-1}$, we obtain

$$\sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(s,t_0)^T C^T C \widehat{y}_{A,v}(s,t_0)y_0 = \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(s,t_0)^T C^T z_1(s),$$

$$\mathscr{R}(t_1,t_0)y_0 = \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(s,t_0)^T C^T z_1(s).$$

Since rank$(\mathscr{R}(t_1,t_0)) = n$, the matrix is invertible and

$$y_0 = \mathscr{R}(t_1,t_0)^{-1} \sum_{s=t_0}^{t_1-1} \widehat{y}_{A,v}(s,t_0)^T C^T z_1(s).$$

Hence, the given system is completely observable.

$(i) \Leftrightarrow (iii)$.

Firstly, for all $N \geqslant n$ the rank of matrix

$$\begin{bmatrix} C & CA & CA^2 & \cdots & CA^{N-1} \end{bmatrix}^T$$

is equal to the rank of observability matrix $\widehat{O}$. The proof follows from the Cayley-Hamilton theorem and similar to controllability case. Here we assume $t_1 - t_0 = n$.

Assume that the system (22) is completely observable. Multiplying both sides of state response (23) by $\widehat{y}_{A,v}(t, t_0)^T C^T$ and taking summation over the discrete interval $t \in \mathbb{N}_{t_0}^{t_0+n-1}$, we obtain

$$
\begin{aligned}
\mathscr{R}(t_0+n, t_0)y_0 &= \sum_{s=t_0}^{t_0+n-1} \widehat{y}_{A,v}(s, t_0)^T C^T z_1(s) \\
&= \sum_{s=t_0}^{t_0+n-1} \sum_{\tau=t_0}^{s} \frac{[A^{\tau-t_0}]^T (s-\tau+1)^{\overline{(\tau-t_0+1)v-1}}}{\Gamma((\tau-t_0+1)v)} C^T z_1(s) \\
&= \sum_{\tau=t_0}^{t_0+n-1} [A^{\tau-t_0}]^T C^T \sum_{s=\tau}^{t_0+n-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)v-1}}}{\Gamma((\tau-t_0+1)v)} z_1(s),
\end{aligned}
$$

where we interchanged order of the summation. Next, we define $G(\tau)$ for all $\tau \in \mathbb{N}_{t_0}^{t_0+n-1}$ by

$$
G(\tau) = \sum_{s=\tau}^{t_0+n-1} \frac{(s-\tau+1)^{\overline{(\tau-t_0+1)v-1}}}{\Gamma((\tau-t_0+1)v)} z_1(s).
$$

Substituting $G(\tau)$ into the last equality, we obtain

$$
\mathscr{R}(t_0+n, t_0)y_0 = \sum_{s=t_0}^{t_0+n-1} [A^{\tau-t_0}]^T C^T G(\tau)
$$

$$
\mathscr{R}(t_0+n, t_0)y_0 = \begin{bmatrix} C^T & A^T C^T & (A^2)^T C^T & \cdots & (A^{n-1})^T C^T \end{bmatrix} \begin{bmatrix} G(0) \\ G(1) \\ G(2) \\ \vdots \\ G(n-1) \end{bmatrix} = \widehat{O}^T G_1(n). \quad (24)
$$

Since the system is completely observable and $(i) \Leftrightarrow (ii)$, then $\mathscr{R}(t_0+n, t_0)$ has full rank n, thus $\mathscr{R}(t_0+n, t_0)y_0 \in \mathbb{R}^n$. Since $\mathrm{rank}(\widehat{O}^T G_1(n)) \leqslant \mathrm{rank}(\widehat{O}^T)$ we have $\mathbb{R}^n \subseteq Im(\widehat{O}^T) \subseteq \mathbb{R}^n$. Therefore, $\mathrm{rank}(\widehat{O}^T) = n = \mathrm{rank}(\widehat{O})$.

Conversely, we show that if $\mathrm{rank}(\widehat{O}) = n$, then the given system is completely observable. We assume to the contrary that the given system is unobservable. Since the given system is unobservable, by $(i) \Leftrightarrow (ii)$ the observability Gramian matrix has rank less than n, and there exists a nonzero vector $\eta \in \mathbb{R}^n$ such that $\eta^T \mathscr{R}(t_0+n, t_0) = 0_{1 \times n}$. Then we have

$$
0 = \eta^T \mathscr{R}(t_0+n, t_0)\eta = \sum_{s=t_0}^{t_0+n-1} \eta^T \widehat{y}_{A,v}(s, t_0)^T C^T C\widehat{y}_{A,v}(s, t_0)\eta = \sum_{s=t_0}^{t_0+n-1} ||C\widehat{y}_{A,v}(s, t_0)\eta||_2^2
$$

which implies that

$$
C\widehat{y}_{A,v}(t, t_0)\eta = 0_{r \times 1} \qquad \text{for all} \quad t \in \mathbb{N}_{t_0}^{t_0+n-1}.
$$

Now setting $t = t_0$ and using Lemma (2) (i) yields

$$C\eta = 0_{r \times 1}.$$

Applying v-th order fractional difference operator to both sides of the last equality and using Lemma 2 yield $CA\widehat{y}_{A,v}(t-1,t_0)\eta = 0_{r \times 1}$ for all $t \in \mathbb{N}_{t_0+1}^{t_0+n-1}$ and shifting each side one unit left we obtain

$$CA\widehat{y}_{A,v}(t,t_0)\eta = 0_{r \times 1} \quad \text{for all} \quad t \in \mathbb{N}_{t_0}^{t_0+n-2}.$$

Setting $t = t_0$ and using Lemma 2 one has

$$CA\eta = 0_{r \times 1}.$$

Repeating the same step up to $n-1$ times, we have

$$CA^k\eta = 0 \qquad \text{for} \qquad k = 0, 1, 2, ..., n-1.$$

Then

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \eta = \widehat{O}\eta = 0_{(rn) \times 1}.$$

This contradicts the assumption $\text{rank}(\widehat{O}) = n$. Therefore the observability Gramian matrix has full rank implies that the given system is completely observable.

The following example illustrates the applicability of Theorem 6.

EXAMPLE 3. Consider the following system

$$\begin{cases} \nabla_{t_0}^v y(t) = \begin{bmatrix} -1 & 1 & 1 \\ 5 & -9 & 1 \\ 6 & -3 & -1 \end{bmatrix} y(t-1) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t-1) \\ \\ z(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} y(t) \end{cases},$$

where $0 < v < 1$.

Using Theorem 6, we get observability matrix of the system

$$\widehat{O} = \begin{bmatrix} 1 & 0 & 1 \\ 5 & -2 & 0 \\ -15 & 23 & 3 \end{bmatrix}$$

whose rank is 3. Thus, by Theorem 6 the given linear fractional order system is completely observable.

REMARK 4. In [5], a criterion for observability of the following discrete fractional system has been obtained

$$\begin{cases} \Delta^v_{v-1}y(t) = Ay(t+v-1) + Bu(t+v-1), & t = 0,1,2,\dots \\ z(t) = Cy(t), & t = v-1,v,\dots \end{cases} \tag{25}$$

where $z(t) = Cy(t)$ is the output measurement, $z(t)$ is an $r \times 1$ matrix, and C is an $r \times n$ matrix.

The observability matrix of this system was given as:

$$\hat{\mathbf{O}} = \begin{bmatrix} CE_A(v-1) \\ CE_A(v) \\ CE_A(v+1) \\ \dots \\ CE_A(n+v-2) \end{bmatrix}.$$

The observability criterion: The system (25) is observable if and only if $\operatorname{rank}(\hat{\mathbf{O}}) = n$.

Using some basics of linear algebra, one can show that

$$\operatorname{rank}(\hat{\mathbf{O}}) = \operatorname{rank}(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}).$$

REMARK 5. Let $\mathbb{T}$ be a time scale and v be a real number such that $0 < v < 1$. The following control systems have same controllability and observability matrices.

(i) The linear dynamic time-invariant system on $\mathbb{T}$

$$\begin{cases} y^\Delta(t) = Ay(t) + Bu(t), & t \in [t_0,t_1] \cap \mathbb{T}, \\ z(t) = Cy(t) + Du(t), \end{cases} \tag{26}$$

where $t_0, t_1 \in \mathbb{T}$.

(ii) The linear ∇-discrete fractional time-invariant system

$$\begin{cases} \nabla^v_{t_0}y(t) = Ay(t-1) + Bu(t-1), & t \in \mathbb{N}^{t_1}_{t_0+1}, \\ z(t) = Cy(t) + Du(t). \end{cases} \tag{27}$$

(iii) The linear Δ-discrete fractional time invariant system

$$\begin{cases} \Delta^v_{v-1}y(t) = Ay(t+v-1) + Bu(t+v-1), & t = 0,1,2,\dots. \\ z(t) = Cy(t) + Du(t), & t = v-1,v,\dots \end{cases} \tag{28}$$

(iv) The continuous fractional time-invariant system

$$\begin{cases} D^\nu y(t) = Ay(t) + Bu(t) & t \in [t_0, t_1], \\ z(t) = Cy(t) + Du(t). \end{cases} \qquad (29)$$

The system (26) in [12], the system (27) in this paper, the system (28) in [5], and the last system (29) in [21] have the same controllability matrix

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

Additionally, observability also studied in the mentioned papers and all have the same observability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

REFERENCES

[1] P. J. ANTSAKLIS AND A. N. MICHEL, *Linear Systems*, Birkhäuser Boston, Inc., 2006.

[2] F. M. ATICI AND P. W. ELOE, *Discrete fractional calculus with the nabla operator*, Electronic Journal of Qualitative Theory of Differential Equations, Spec. Ed. I, 2009, no. 3, 1–12.

[3] F. M. ATICI AND N. ACAR, *Exponential functions of discrete fractional calculus*, Applicable Analysis and Discrete Mathematics, 7 (2013), no. 2, 343–353.

[4] F. M. ATICI, M. ATICI, N. NGUYEN, T. ZHOROEV AND G. KOCH, *A study on discrete and discrete fractional pharmacokinetics-pharmacodynamics models for tumor growth and anti-cancer effects*, Computational Mathematical Biophysics, 7(2019), 10–24.

[5] F. M. ATICI AND D. M. NGUYEN, *Rank conditions for controllability of discrete fractional time-invariant linear systems*, Journal of Difference Equations and Applications, 25(2019), Issue 6, Special Issue: Fractional Calculus, Guest Edited by Allan Peterson, 869–881. doi:10.1080/10236198.2019.1596265.

[6] D. BALEANU, K. DIETHELM, E. SCALAS AND J. J. TRUJILLO, *Fractional Calculus. Models and Numerical Methods*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[7] Z. BARTOSIEWICZ AND E. PAWLUSZEWICZ, *Linear control systems on time scales: unification of continuous and discrete*, Proc. of 10th IEEE Int. Conference MMAR, 2004, 263–266.

[8] Z. BARTOSIEWICZ, *Linear positive control systems on time scales; controllability*, Mathematics of Control, Signals, and Systems, 25 (2013), no. 3, 327–343.

[9] M. BETTAYEB AND S. DJENNOUNCE, *A note on the controllability and observability of fractional dynamic systems*, Proceeding of the 2nd IFAC workshop on fractional differntiation and its applications, 39 (2013), no. 11, 493–498.

[10] M. BETTAYEB AND S. DJENNOUNE, *New results on the controllability and observability of fractional dynamical systems*, Journal of Vibrating and Control, 14 (2008), no. 9–10, 1531–1541.

[11] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkhäuser Boston, Inc., 2001.

[12] J. M. DAVIS, I. A. GRAVAGNE, B. J. JACKSON AND R. J. MARKS II, *Controllability, observability, realizability, and stability of dynamic linear systems*, Electron. J. Diff. Equ., 2009, no. 37, 1–32.

[13] S. N. ELAYDI, *An introduction to Difference Equations*, Springer-Verlag, New York, 1999.

[14] L. V. FAUSETT AND K. N. MURTY, *Controllability, observability and realizability criteria on time scale dynamical systems*, Nonlinear Stud, 11 (2004), no. 14, 627–638.

[15] S. Guermah, S. Djennoune and M. Bettayeb, *Controllability and observability of linear discrete–time fractional order systems*, International Journal of Applied Mathematics and Computer Science, 18 (2008), no. 2, 213–222.

[16] C. S. Goodrich and C. Lizama, *A transference principle for nonlocal operators using a convolutional approac: Fractional monotonicity and convexity*, Israel Journal of Mathematics, to appear.

[17] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer International Publishing, 2015.

[18] T. Kaczorek, *Reachability and controllability to zero of positive fractional discrete–time systems*, European Control Conference, 2007, 1708–1712.

[19] R. E. Kalman, *On the general theory of control systems*, Proc. First IFAC Congress on Automatic Control, Moscow, 1960, 481–492.

[20] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House, 2016.

[21] D. Matignon and B. d'Andrea Novel, *Some results on controllability and observability of finite–dimensional fractional differential systems*, Proceedings of the Computational Engineering in Systems and Application Multiconference, 1996, no. 2, 952–956.

[22] D. Mozyrska and E. Pawłuszewicz, *Controllability of h-difference linear control systems with two fractional orders*, International Journal of Systems Science, 46 (2015), no. 4, 662–669.

[23] D. Mozyrska, E. Pawłuszewicz and M. Wyrwas, *The h-difference approach to controllability and observability of fractional linear systems with Caputo–type operator*, Asian Journal of Control, 17 (2015), no. 4, 1163–1173.

[24] D. Mozyrska, E. Pawłuszewicz and M. Wyrwas, *Local observability and controllability of nonlinear discrete-time fractional order systems based on their linearisation*, International Journal of Systems Science, 48 (2017), no. 4, 788–794.

[25] D. Mozyrska and D. F. M. Torres, *Minimal modified energy control for fractional linear control systems with the Caputo derivative*, Carpathian J. Math., 26 (2010), no. 2, 210–221.

[26] K. Ogata, *Discrete-Time Control Systems*, Prentice Hall, 1987.

[27] K. Ogata, *Modern Control Engineering*, Prentice Hall, 2010.

[28] P. Ostalczyk, *Discrete Fractional Calculus. Applications in Control and Image Processing*, World Scientific, 2015.

[29] I. Podlubny, *Fractional-Order Systems and $PI^\lambda D^\nu$-Controllers*, IEEE Transactions on Automatic Control, 44 (1999), no. 1, 208–214.

[30] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[31] Y. Pu, J. Zhou and X. Yuan, *Fractional differential mask. A fractional differential-based approach for multiscale texture enhancement*, IEEE Trans. on Image Processing, 19 (2010), no. 2, 491–511.

[32] D. Serociuk and D. Dzielinski, *Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation*, Int. J. Appl. Math. Comp. Sci., 16 (2006), no. 1, 129–140.

[33] N. Wintz and M. Bohner, *Controllability and observability of time-invariant linear dynamic systems*, Mathematica Bohemica, 137 (2012), no. 2, 149–163.

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