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FRACTIONAL INEQUALITIES FOR EXPONENTIALLY GENERALIZED (m,ω,h_1,h_2)–PREINVEX FUNCTIONS WITH APPLICATIONS

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Abstract. The aim of this paper is to introduce a new extension of preinvexity called exponentially generalized (m,ω,h_1,h_2)–preinvexity. Some new integral inequalities of Hermite–Hadamard type for exponentially generalized (m,ω,h_1,h_2)–preinvex functions via Riemann–Liouville fractional integral are established. We show that the class of exponentially generalized (m,ω,h_1,h_2)–preinvex functions includes several other classes of preinvex functions. At the end, some new error estimates for trapezoidal quadrature formula are provided as well. This results may stimulate further research in different areas of pure and applied sciences.

1. Introduction

The class of convex functions is well known in the literature and is usually defined in the following way:

DEFINITION 1. Let I be an interval in $\mathbb{R}$. A function $f: I \to \mathbb{R}$, is said to be convex on I, if the inequality

$$f(\lambda a+(1-\lambda)b) \leqslant \lambda f(a)+(1-\lambda)f(b) \tag{1}$$

holds for all $a,b \in I$ and $\lambda \in [0,1]$. Also, we say that f is concave, if the inequality in (1) holds in the reverse direction.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

THEOREM 1. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a,b \in I$ with $a<b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{2}$$

This inequality (2) is also known as trapezium inequality.

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The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (2) in the premises of newly invented definitions due to motivation of convex function. Interested readers can see the references [4]–[15],[17, 21, 22, 26, 27].

In [10], Dragomir and Agarwal proved the following results connected with the right part of (2).

LEMMA 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (interior of I), $a,b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{3}$$

THEOREM 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (interior of I), $a,b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{4}$$

Now, let us recall the following useful definitions.

DEFINITION 2. [21] A function: $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be m–MT–convex, if f is positive and for $\forall x,y \in I$, $t \in (0,1)$ and $m \in (0,1]$, satisfies the following inequality

$$f\big(tx+m(1-t)y\big) \leqslant \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{5}$$

DEFINITION 3. [3] A set $K \subset \mathbb{R}^n$ is said to be invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x,y \in K$ and $t \in [0,1]$.

DEFINITION 4. [16] Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function and $h \neq 0$. The function f on the invex set K is said to be h–preinvex with respect to η, if

$$f\big(x+t\eta(y,x)\big) \leqslant h(1-t)f(x) + h(t)f(y) \tag{6}$$

for each $x,y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

DEFINITION 5. [11] A set $K \subset \mathbb{R}^n$ is named as m–invex with respect to the bifunction $\eta : K \times K \times (0,1] \to \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x,y \in K$ and any $t \in [0,1]$.

REMARK 1. Taking $m = 1$ in Definition 5, the mapping $\eta(y,x,m)$ reduce to $\eta(y,x)$ and then we get Definition 3.

DEFINITION 6. [24] Let $K \subset \mathbb{R}$ be m–invex set respecting the bifunction $\eta : K \times K \times (0,1] \to \mathbb{R}$ and $h_1, h_2 : [0,1] \to [0,+\infty)$. A function $f : K \to \mathbb{R}$ is said to be generalized (m,h_1,h_2)–preinvex, if

$$f\big(mx + t\eta(y,x,m)\big) \leqslant mh_1(t)f(x) + h_2(t)f(y) \tag{7}$$

is valid for all $x,y \in K$ and $t \in [0,1]$, for some fixed $m \in (0,1]$.

DEFINITION 7. [14] Let $f \in L[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

An important class of convex functions, which is called exponential convex functions, was introduced and studied by Antczak in [2], Dragomir et al. in [9] and Noor et al. in [20]. Alirezai et al. in [1], have investigated their basic properties along with their potential applications in statistics and information theory. Awan et al. in [5] and Pecarić et al. in [23] defined another kind of exponential convex functions and have shown that the class of exponential convex functions unifies various unrelated concepts.

DEFINITION 8. [2, 9, 19] A function $f : K \subset \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex, if

$$e^{f((1-t)a+tb)} \leqslant (1-t)e^{f(a)} + te^{f(b)} \tag{8}$$

holds for all $a,b \in K$, $t \in [0,1]$, where f is positive.

For the applications of exponentially convex functions in different field of statistics, information theory and mathematical sciences, see [1, 2, 5, 18] and the references therein.

DEFINITION 9. [25] A function $f : K \subset \mathbb{R} \to \mathbb{R}$ is said to be exponentially m–convex, where $m \in (0,1]$, if

$$e^{f((1-t)a+mtb)} \leqslant (1-t)e^{f(a)} + mte^{f(b)} \tag{9}$$

holds for all $a,b \in K$, $t \in [0,1]$, where f is positive.

Motivated by the above literatures, the main objective of this article is to establish in Section 2 fractional integral inequalities using a new class of preinvex functions called exponentially generalized (m,ω,h_1,h_2)–preinvex. Also, using a new identity pertaining differentiable functions defined on m-invex set as auxiliary result, some new

Hermite–Hadamard inequalities for exponentially generalized (m,ω,h_1,h_2)–preinvex functions via Riemann–Liouville fractional integral will obtain. Various special cases will be discussed. In Section 3, some new error estimates for trapezoidal quadrature formula will be given. In Section 4, a briefly conclusion is provided as well. This results may stimulate further research in different areas of pure and applied sciences.

2. Main results

Now, we are in position to introduce the new class of functions called exponentially generalized (m,ω,h_1,h_2)–preinvex as follows:

DEFINITION 10. Let $K \subset \mathbb{R}$ be m–invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$, for some fixed $m \in (0,1]$ and $h_1,h_2 : [0,1] \to [0,+\infty)$. A function $f : K \to (0,+\infty)$ is called exponentially generalized (m,ω,h_1,h_2)-preinvex, if

$$e^{f(mx+t\eta(y,mx))} \leqslant mh_1(t)e^{\omega f(x)} + h_2(t)e^{\omega f(y)} \tag{10}$$

holds for all $x,y \in K$, $t \in [0,1]$ and $\omega \in \mathbb{R}$.

REMARK 2. In Definition 10, if we choose $\omega = 1$, $h_1(t) = 1-t$, $h_2(t) = t$ and $\eta(y,mx) = y - mx$, we obtain Definition 9.

REMARK 3. Under the conditions of Remark 2, taking $m = 1$, we get Definition 8.

REMARK 4. Let us discuss some special cases in Definition 10 as follows:

(I) Taking $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, then we get exponentially generalized (m,ω,h)–preinvex functions.

(II) Taking $h_1(t) = h_2(t) = t(1-t)$, then we get exponentially generalized (m,ω,tgs)–preinvex functions.

(III) Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get exponentially generalized (m,ω)–MT–preinvex functions.

In this section, we obtain Hermite–Hadamard type inequalities for exponentially generalized (m,ω,h_1,h_2)–preinvex function via Riemann–Liouville fractional integral.

For brevity, we denote

$$K = [ma, ma + \eta(b,ma)], \quad \text{where} \quad \eta(b,ma) > 0.$$

THEOREM 3. *Let $K \subset \mathbb{R}$ be m–invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$ for some fixed $m \in (0,1]$, where $a < b$. Suppose $h_1, h_2 : [0,1] \to [0,+\infty)$ be continuous functions. Let $f, g : K \to (0,+\infty)$ be exponentially generalized (m,ω,h_1,h_2)–preinvex functions. If $f, g \in L(K)$, then for $\omega \in \mathbb{R}$ and $\alpha > 0$, the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,ma)}\left\{J^{\alpha}_{(ma+\eta(b,ma))^-}e^{f(ma)} + J^{\alpha}_{(ma+\eta(b,ma))^-}e^{g(ma)}\right\}\\ &\leqslant m\left(e^{\omega f(a)} + e^{\omega g(a)}\right)H_1(\alpha) + \left(e^{\omega f(b)} + e^{\omega g(b)}\right)H_2(\alpha),\end{aligned} \tag{11}$$

where

$$H_i(\alpha) = \int_0^1 t^{\alpha-1}h_i(t)dt, \quad \forall i = 1,2.$$

Proof. From exponentially generalized (m,ω,h_1,h_2)–preinvexity of f and g for all $t \in [0,1]$, we have

$$e^{f(ma+t\eta(b,ma))} \leqslant mh_1(t)e^{\omega f(a)} + h_2(t)e^{\omega f(b)}$$

and

$$e^{g(ma+t\eta(b,ma))} \leqslant mh_1(t)e^{\omega g(a)} + h_2(t)e^{\omega g(b)}.$$

Adding both sides of the above inequalities, we get

$$e^{f(ma+t\eta(b,ma))} + e^{g(ma+t\eta(b,ma))} \leqslant m\left(e^{\omega f(a)} + e^{\omega g(a)}\right)h_1(t) + \left(e^{\omega f(b)} + e^{\omega g(b)}\right)h_2(t). \tag{12}$$

Multiplying both sides of inequality (12) with $t^{\alpha-1}$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}\left[e^{f(ma+t\eta(b,ma))} + e^{g(ma+t\eta(b,ma))}\right]dt\\ &\leqslant m\left(e^{\omega f(a)} + e^{\omega g(a)}\right)\int_0^1 t^{\alpha-1}h_1(t)dt + \left(e^{\omega f(b)} + e^{\omega g(b)}\right)\int_0^1 t^{\alpha-1}h_2(t)dt.\end{aligned}$$

Using Definition 7, we get the required result (11).

We point out some special cases of Theorem 3.

COROLLARY 1. *In Theorem 3, if we choose $m = 1$ and $\eta(b,ma) = b - ma$, we have*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left\{J^{\alpha}_{b^-}e^{f(a)} + J^{\alpha}_{b^-}e^{g(a)}\right\} \leqslant \left(e^{\omega f(a)} + e^{\omega g(a)}\right)H_1(\alpha) + \left(e^{\omega f(b)} + e^{\omega g(b)}\right)H_2(\alpha).$$

COROLLARY 2. *In Theorem 3, if we choose* $\alpha = 1$, *we get*

$$\frac{1}{\eta(b,ma)}\int_{ma}^{ma+\eta(b,ma)}\left[e^{f(t)}+e^{g(t)}\right]dt \leqslant m\left(e^{\omega f(a)}+e^{\omega g(a)}\right)H_1+\left(e^{\omega f(b)}+e^{\omega g(b)}\right)H_2,$$

where

$$H_i = \int_0^1 h_i(t)dt, \quad \forall i = 1,2.$$

THEOREM 4. *Let* $K \subset \mathbb{R}$ *be* m*–invex set with respect to the bifunction* $\eta : K \times K \to \mathbb{R}$ *for some fixed* $m \in (0,1]$, *where* $a < b$. *Suppose* $h_1, h_2 : [0,1] \to [0,+\infty)$ *be continuous functions. Let* $f, g : K \to (0,+\infty)$ *be exponentially generalized* (m,ω,h_1,h_2)*–preinvex functions. If* $f, g \in L(K)$, *then for* $\omega \in \mathbb{R}$ *and* $\alpha > 0$, *the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,ma)}\left\{J^{\alpha}_{(ma)^+}e^{f(ma+\eta(b,ma))}+J^{\alpha}_{(ma+\eta(b,ma))^-}e^{g(ma)}\right\}\\ \leqslant & m\left(e^{\omega f(a)}C_1(\alpha)+e^{\omega g(a)}H_1(\alpha)\right)+e^{\omega f(b)}C_2(\alpha)+e^{\omega g(b)}H_2(\alpha),\end{aligned} \tag{13}$$

where

$$C_i(\alpha) = \int_0^1 (1-t)^{\alpha-1}h_i(t)dt, \quad \forall i = 1,2$$

and $H_1(\alpha), H_2(\alpha)$ *are defined in Theorem 3.*

Proof. From exponentially generalized (m,ω,h_1,h_2)–preinvexity of f and g for all $t \in [0,1]$, we have

$$e^{f(ma+t\eta(b,ma))} \leqslant mh_1(t)e^{\omega f(a)}+h_2(t)e^{\omega f(b)}$$

and

$$e^{g(ma+t\eta(b,ma))} \leqslant mh_1(t)e^{\omega g(a)}+h_2(t)e^{\omega g(b)}.$$

Multiplying first above inequality with $(1-t)^{\alpha-1}$, the second inequality with $t^{\alpha-1}$ and adding both sides, we get

$$\begin{aligned}&(1-t)^{\alpha-1}e^{f(ma+t\eta(b,ma))}+t^{\alpha-1}e^{g(ma+t\eta(b,ma))}\\ \leqslant&(1-t)^{\alpha-1}\left[mh_1(t)e^{\omega f(a)}+h_2(t)e^{\omega f(b)}\right]+t^{\alpha-1}\left[mh_1(t)e^{\omega g(a)}+h_2(t)e^{\omega g(b)}\right].\end{aligned} \tag{14}$$

Integrating over $[0,1]$ both sides of inequality (14) and using Definition 7, we get the required result (13).

We point out some special cases of Theorem 4.

COROLLARY 3. *In Theorem 4, if we choose* $m = 1$ *and* $\eta(b,ma) = b - ma$, *we obtain*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left\{J^{\alpha}_{a^+}e^{f(b)}+J^{\alpha}_{b^-}e^{g(a)}\right\}\\ \leqslant&e^{\omega f(a)}C_1(\alpha)+e^{\omega g(a)}H_1(\alpha)+e^{\omega f(b)}C_2(\alpha)+e^{\omega g(b)}H_2(\alpha).\end{aligned}$$

COROLLARY 4. *In Theorem 4, if we choose* $\alpha = 1$, *we get Corollary 2.*

REMARK 5. Under the conditions of Theorems 3 and 4, using Remark 4, we can get several new integral inequalities. We omit their proofs and the details are left to the interested readers.

For establishing some new results regarding generalizations of Hermite–Hadamard type integral inequalities associated with exponentially generalized (m,ω,h_1,h_2)–preinvexity via Riemann–Liouville fractional integrals, we need the following lemma.

LEMMA 2. *Let* $K \subset \mathbb{R}$ *be* m*-invex set with respect to the bifunction* $\eta : K \times K \to \mathbb{R}$ *for some fixed* $m \in (0,1]$, *where* $a < b$. *Let* $f : K \to \mathbb{R}$ *be differentiable function on* K° *(the interior of* K*) such that* $f' \in L(K)$. *Then for* $\alpha > 0$, *the following equality holds:*

$$\frac{e^{f(ma)} + e^{f(ma+\eta(b,ma))}}{2} - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,ma)}\left\{J^\alpha_{(ma)^+}e^{f(ma+\eta(b,ma))} + J^\alpha_{(ma+\eta(b,ma))^-}e^{f(ma)}\right\}$$
$$= \frac{\eta(b,ma)}{2}\int_0^1 \left[t^\alpha - (1-t)^\alpha\right] e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt. \tag{15}$$

We denote

$$\Xi^\alpha_f(\eta;m,a,b) = \frac{\eta(b,ma)}{2} \times \int_0^1 \left[t^\alpha - (1-t)^\alpha\right] e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt. \tag{16}$$

Proof. From (16), we have

$$\begin{aligned}\Xi^\alpha_f(\eta;m,a,b) =& \frac{\eta(b,ma)}{2} \times \left[\int_0^1 t^\alpha e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt\right.\\ &\left.- \int_0^1 (1-t)^\alpha e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt\right]\\ =& \frac{\eta(b,ma)}{2}\left[\Xi^\alpha_{f,1}(\eta;m,a,b) - \Xi^\alpha_{f,2}(\eta;m,a,b)\right],\end{aligned} \tag{17}$$

where

$$\Xi^\alpha_{f,1}(\eta;m,a,b) = \int_0^1 t^\alpha e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt \tag{18}$$

and

$$\Xi^\alpha_{f,2}(\eta;m,a,b) = \int_0^1 (1-t)^\alpha e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))dt. \tag{19}$$

Now, integrating by parts (18), changing the variable of integration and using Definition 7, we get

$$\Xi^\alpha_{f,1}(\eta;m,a,b) = \frac{t^\alpha e^{f(ma+t\eta(b,ma))}}{\eta(b,ma)}\bigg|_0^1 - \frac{\alpha}{\eta(b,ma)}\int_0^1 t^{\alpha-1} e^{f(ma+t\eta(b,ma))}dt$$

$$= \frac{e^{f(ma+\eta(b,ma))}}{\eta(b,ma)} - \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,ma)} J^{\alpha}_{(ma+\eta(b,ma))^-} e^{f(ma)}. \tag{20}$$

Similarly, using (19), we obtain

$$\Xi^{\alpha}_{f,2}(\eta;m,a,b) = -\frac{e^{f(ma)}}{\eta(b,ma)} + \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,ma)} J^{\alpha}_{(ma)^+} e^{f(ma+\eta(b,ma))}. \tag{21}$$

Substituting (20) and (21) in (17), we get the required result (15).

Using Lemma 2, we now state the following theorems.

THEOREM 5. *Let $K \subset \mathbb{R}$ be m–invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$ for some fixed $m \in (0,1]$, where $a < b$. Suppose $h_1, h_2 : [0,1] \to [0,+\infty)$ be continuous functions. Let $f : K \to (0,+\infty)$ be a differentiable exponentially generalized (m,ω,h_1,h_2)–preinvex function on K° (the interior of K) such that $f' \in L(K)$ and $\omega \in \mathbb{R}$. If $|f'|^q$ is generalized (m,h_1,h_2)–preinvex function, then for $\alpha > 0$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds:*

$$\left|\Xi^{\alpha}_{f}(\eta;m,a,b)\right| \leqslant \frac{\eta(b,ma)}{2} \sqrt[p]{B(p,\alpha)} \tag{22}$$
$$\times \sqrt[q]{m^2 e^{q\omega f(a)} \left|f'(a)\right|^q G_1 + m\Delta(q;\omega,a,b)F + e^{q\omega f(b)} \left|f'(b)\right|^q G_2},$$

where

$$B(p,\alpha) = \int_0^1 \left|t^{\alpha} - (1-t)^{\alpha}\right|^p dt,$$
$$F = \int_0^1 h_1(t)h_2(t)dt, \quad G_i = \int_0^1 [h_i(t)]^2 dt, \ \forall i = 1,2$$

and

$$\Delta(q;\omega,a,b) = e^{q\omega f(a)} \left|f'(b)\right|^q + e^{q\omega f(b)} \left|f'(a)\right|^q.$$

Proof. From Lemma 2, exponentially generalized (m,ω,h_1,h_2)–preinvexity of f, generalized (m,h_1,h_2)–preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|\Xi^{\alpha}_{f}(\eta;m,a,b)\right| \leqslant \frac{\eta(b,ma)}{2} \times \int_0^1 \left|t^{\alpha} - (1-t)^{\alpha}\right| \left|e^{f(ma+t\eta(b,ma))} f'(ma+t\eta(b,ma))\right| dt$$
$$\leqslant \frac{\eta(b,ma)}{2} \sqrt[p]{B(p,\alpha)} \left(\int_0^1 e^{qf(ma+t\eta(b,ma))} \left|f'(ma+t\eta(b,ma))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leqslant \frac{\eta(b,ma)}{2} \sqrt[p]{B(p,\alpha)} \times \left(\int_0^1 \left[mh_1(t)e^{q\omega f(a)} + h_2(t)e^{q\omega f(b)}\right]\right.$$
$$\left.\left[mh_1(t)\left|f'(a)\right|^q + h_2(t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&= \frac{\eta(b,ma)}{2}\sqrt[p]{B(p,\alpha)} \\
&\times \sqrt[q]{m^2 e^{q\omega f(a)}\left|f'(a)\right|^q G_1 + m\Delta(q;\omega,a,b)F + e^{q\omega f(b)}\left|f'(b)\right|^q G_2}.
\end{aligned}
$$

The proof of Theorem 5 is completed.

We point out some special cases of Theorem 5.

COROLLARY 5. *In Theorem 5, if we choose $m = 1$ and $\eta(b,ma) = b - ma$, we get*

$$
\begin{aligned}
&\left|\frac{e^{f(a)} + e^{f(b)}}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left\{J_{a^+}^\alpha e^{f(b)} + J_{b^-}^\alpha e^{f(a)}\right\}\right| \\
&\leqslant \frac{(b-a)}{2}\sqrt[p]{B(p,\alpha)} \times \sqrt[q]{e^{q\omega f(a)}\left|f'(a)\right|^q G_1 + \Delta(q;\omega,a,b)F + e^{q\omega f(b)}\left|f'(b)\right|^q G_2}.
\end{aligned}
$$

COROLLARY 6. *In Theorem 5, if we choose $\alpha = 1$, we obtain*

$$
\begin{aligned}
&\left|\frac{e^{f(ma)} + e^{f(ma+\eta(b,ma))}}{2} - \frac{1}{\eta(b,ma)}\int_{ma}^{ma+\eta(b,ma)} e^{f(t)}dt\right| \\
&\leqslant \frac{\eta(b,ma)}{2\sqrt[p]{p+1}} \times \sqrt[q]{m^2 e^{q\omega f(a)}\left|f'(a)\right|^q G_1 + m\Delta(q;\omega,a,b)F + e^{q\omega f(b)}\left|f'(b)\right|^q G_2}.
\end{aligned}
$$

THEOREM 6. *Let $K \subset \mathbb{R}$ be m–invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$ for some fixed $m \in (0,1]$, where $a < b$. Suppose $h_1, h_2 : [0,1] \to [0,+\infty)$ be continuous functions. Let $f : K \to (0,+\infty)$ be a differentiable exponentially generalized (m,ω,h_1,h_2)–preinvex function on K° (the interior of K) such that $f' \in L(K)$ and $\omega \in \mathbb{R}$. If $|f'|^q$ is generalized (m,h_1,h_2)–preinvex function, then for $\alpha > 0$ and $q \geqslant 1$, the following inequality holds:*

$$
\begin{aligned}
&\left|\Xi_f^\alpha(\eta;m,a,b)\right| \\
&\leqslant \frac{\eta(b,ma)}{2}\left[B(1,\alpha)\right]^{1-\frac{1}{q}} \\
&\times \sqrt[q]{m^2 e^{q\omega f(a)}\left|f'(a)\right|^q P_1(\alpha) + m\Delta(q;\omega,a,b)S(\alpha) + e^{q\omega f(b)}\left|f'(b)\right|^q P_2(\alpha)},
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
S(\alpha) &= \int_0^1 \left|t^\alpha - (1-t)^\alpha\right| h_1(t)h_2(t)dt, \\
P_i(\alpha) &= \int_0^1 \left|t^\alpha - (1-t)^\alpha\right| [h_i(t)]^2 dt, \ \forall i = 1,2
\end{aligned}
$$

and $B(1,\alpha)$ (for value $p = 1$) and $\Delta(q;\omega,a,b)$ are defined in Theorem 5.

Proof. From Lemma 2, exponentially generalized (m,ω,h_1,h_2)–preinvexity of f, generalized (m,h_1,h_2)–preinvexity of $|f'|^q$, the well–known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|\Xi_f^{\alpha}(\eta;m,a,b)\right|\\
\leqslant&\frac{\eta(b,ma)}{2}\times\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|e^{f(ma+t\eta(b,ma))}f'(ma+t\eta(b,ma))\right|dt\\
\leqslant&\frac{\eta(b,ma)}{2}\left[B(1,\alpha)\right]^{1-\frac{1}{q}}\times\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|e^{qf(ma+t\eta(b,ma))}\left|f'(ma+t\eta(b,ma))\right|^q dt\right)^{\frac{1}{q}}\\
\leqslant&\frac{\eta(b,ma)}{2}\left[B(1,\alpha)\right]^{1-\frac{1}{q}}\\
&\times\left[\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left[mh_1(t)e^{q\omega f(a)}+h_2(t)e^{q\omega f(b)}\right]\left[mh_1(t)\left|f'(a)\right|^q+h_2(t)\left|f'(b)\right|^q\right]dt\right]^{\frac{1}{q}}\\
=&\frac{\eta(b,ma)}{2}\left[B(1,\alpha)\right]^{1-\frac{1}{q}}\\
&\times\sqrt[q]{m^2e^{q\omega f(a)}\left|f'(a)\right|^qP_1(\alpha)+m\Delta(q;\omega,a,b)S(\alpha)+e^{q\omega f(b)}\left|f'(b)\right|^qP_2(\alpha)}.
\end{aligned}$$

The proof of Theorem 6 is completed.

We point out some special cases of Theorem 6.

COROLLARY 7. *In Theorem 6, if we choose* $m=1$ *and* $\eta(b,ma)=b-ma$, *we get*

$$\begin{aligned}
&\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J_{a^+}^{\alpha}e^{f(b)}+J_{b^-}^{\alpha}e^{f(a)}\right\}\right|\\
\leqslant&\frac{(b-a)}{2}\left[B(1,\alpha)\right]^{1-\frac{1}{q}}\\
&\times\sqrt[q]{e^{q\omega f(a)}\left|f'(a)\right|^qP_1(\alpha)+\Delta(q;\omega,a,b)S(\alpha)+e^{q\omega f(b)}\left|f'(b)\right|^qP_2(\alpha)}.
\end{aligned}$$

COROLLARY 8. *In Theorem 6, if we choose* $\alpha=1$, *we obtain*

$$\begin{aligned}
&\left|\frac{e^{f(ma)}+e^{f(ma+\eta(b,ma))}}{2}-\frac{1}{\eta(b,ma)}\int_{ma}^{ma+\eta(b,ma)}e^{f(t)}dt\right|\\
\leqslant&2^{\frac{1-2q}{q}}\eta(b,ma)\times\sqrt[q]{m^2e^{q\omega f(a)}\left|f'(a)\right|^q\Omega_1+m\Delta(q;\omega,a,b)\Theta+e^{q\omega f(b)}\left|f'(b)\right|^q\Omega_2},
\end{aligned}$$

where

$$\Theta=\int_0^1|2t-1|h_1(t)h_2(t)dt,\quad \Omega_i=\int_0^1|2t-1|[h_i(t)]^2dt,\ \forall i=1,2.$$

REMARK 6. Under the conditions of Theorems 5 and 6, using Remark 4, we can get several new integral inequalities. We omit their proofs and the details are left to the interested readers.

3. Applications

In this section, we provide some new error estimates for trapezoidal quadrature formula. Let P be the partition of the points $a = x_0 < x_1 < \ldots < x_k = b$ of the interval $[a,b]$. Let consider the following quadrature formula:

$$\int_a^b e^{f(x)}dx = T(f,P) + E(f,P),$$

where

$$T(f,P) = \sum_{i=0}^{k-1}\left[\frac{e^{f(x_i)} + e^{f(x_{i+1})}}{2}\right](x_{i+1} - x_i)$$

is the trapezoidal version and $E(f,P)$ is denote the associated approximation error.

PROPOSITION 1. *Let $f : [a,b] \to (0,+\infty)$ be a differentiable exponentially generalized (ω,h_1,h_2)–convex function on (a,b), where $a < b$ and $\omega \in \mathbb{R}$. Suppose $h_1,h_2 : [0,1] \to [0,+\infty)$ be continuous functions. If $|f'|^q$ is generalized (m,h_1,h_2)–convex on $[a,b]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|E(f,P)\right| \leqslant \frac{1}{2\sqrt[p]{p+1}}\sum_{i=0}^{k-1}(x_{i+1} - x_i)^2 \times \sqrt[q]{e^{q\omega f(x_i)}\left|f'(x_i)\right|^q G_1 + \Delta(q;\omega,x_i,x_{i+1})F + e^{q\omega f(x_{i+1})}\left|f'(x_{i+1})\right|^q G_2}, \tag{24}$$

where

$$\Delta(q;\omega,x_i,x_{i+1}) = e^{q\omega f(x_i)}\left|f'(x_{i+1})\right|^q + e^{q\omega f(x_{i+1})}\left|f'(x_i)\right|^q$$

and F, G_1, G_2 are defined in Theorem 5.

Proof. Applying Theorem 5 for $\alpha = m = 1$ and $\eta(b,ma) = b - ma$ on the subintervals $[x_i,x_{i+1}]$ $(i = 0,\ldots,k-1)$ of the partition P, we have

$$\left|\frac{e^{f(x_i)} + e^{f(x_{i+1})}}{2} - \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} e^{f(x)}dx\right| \leqslant \frac{(x_{i+1} - x_i)}{2\sqrt[p]{p+1}} \times \sqrt[q]{e^{q\omega f(x_i)}\left|f'(x_i)\right|^q G_1 + \Delta(q;\omega,x_i,x_{i+1})F + e^{q\omega f(x_{i+1})}\left|f'(x_{i+1})\right|^q G_2}. \tag{25}$$

Hence from (25), we get

$$\left|E(f,P)\right| = \left|\int_a^b e^{f(x)}dx - T(f,P)\right|$$

$$\leqslant \left| \sum_{i=0}^{k-1} \left\{ \int_{x_i}^{x_{i+1}} e^{f(x)} dx - \left[\frac{e^{f(x_i)} + e^{f(x_{i+1})}}{2} \right] (x_{i+1} - x_i) \right\} \right|$$
$$\leqslant \sum_{i=0}^{k-1} \left| \left\{ \int_{x_i}^{x_{i+1}} e^{f(x)} dx - \left[\frac{e^{f(x_i)} + e^{f(x_{i+1})}}{2} \right] (x_{i+1} - x_i) \right\} \right|$$
$$\leqslant \frac{1}{2\sqrt[p]{p+1}} \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2$$
$$\times \sqrt[q]{e^{q\omega f(x_i)} \left| f'(x_i) \right|^q G_1 + \Delta(q;\omega,x_i,x_{i+1}) F + e^{q\omega f(x_{i+1})} \left| f'(x_{i+1}) \right|^q G_2}.$$

The proof of Proposition 1 is completed.

PROPOSITION 2. *Let $f : [a,b] \to (0,+\infty)$ be a differentiable exponentially generalized (ω,h_1,h_2)–convex function on (a,b), where $a < b$ and $\omega \in \mathbb{R}$. Suppose $h_1,h_2 : [0,1] \to [0,+\infty)$ be continuous functions. If $|f'|^q$ is generalized (m,h_1,h_2)–convex on $[a,b]$ for $q \geqslant 1$, then the following inequality holds:*

$$\left| E(f,P) \right| \leqslant 2^{\frac{1-2q}{q}} \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \tag{26}$$
$$\times \sqrt[q]{e^{q\omega f(x_i)} \left| f'(x_i) \right|^q \Omega_1 + \Delta(q;\omega,x_i,x_{i+1}) \Theta + e^{q\omega f(x_{i+1})} \left| f'(x_{i+1}) \right|^q \Omega_2},$$

where $\Delta(q;\omega,x_i,x_{i+1})$ is defined from Proposition 1 and Θ, Ω_1, Ω_2 are defined from Corollary 8.

Proof. The proof is analogous as to that of Proposition 1, taking $\alpha = m = 1$ and $\eta(b,ma) = b - ma$ in Theorem 6.

REMARK 7. Under the conditions of Theorems 5 and 6, using Remark 4, we can deduce several new bounds for the trapezoidal quadrature formula using above ideas and techniques. We omit their proofs and the details are left to the interested readers.

4. Conclusion

This new class of functions called exponentially generalized (m,ω,h_1,h_2)–preinvexity can be applied to obtain several new results in convex analysis, related optimization theory, etc. The authors hope that these work may stimulate further research in different areas of pure and applied sciences.

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REFERENCES

[1] G. ALIREZAEI AND R. MATHAR, *On exponentially concave functions and their impact in information theory*, Information Theory and Applications Workshop, San Diego, California, USA, (2018).

[2] T. ANTCZAK, *(p,r)–invex sets and functions*, J. Math. Anal. Appl., **263** (2001), 355–379.

[3] T. ANTCZAK, *Mean value in invexity analysis*, Nonlinear Anal., **60** (2005), 1473–1484.

[4] S. M. ASLANI, M. R. DELAVAR AND S. M. VAEZPOUR, *Inequalities of Fejér type related to generalized convex functions with applications*, Int. J. Anal. Appl., **16** (1) (2018), 38–49.

[5] M. U. AWAN, M. A. NOOR AND K. I. NOOR, *Hermite–Hadamard inequalities for exponentially convex functions*, Appl. Math. Inform. Sci., **2** (12) (2018), 405–409.

[6] F. X. CHEN AND S. H. WU, *Several complementary inequalities to inequalities of Hermite–Hadamard type for s–convex functions*, J. Nonlinear Sci. Appl., **9** (2) (2016), 705–716.

[7] Y.-M. CHU, M. ADIL KHAN, T. U. KHAN AND T. ALI, *Generalizations of Hermite–Hadamard type inequalities for MT–convex functions*, J. Nonlinear Sci. Appl., **9** (5) (2016), 4305–4316.

[8] M. R. DELAVAR AND M. DE LA SEN, *Some generalizations of Hermite–Hadamard type inequalities*, SpringerPlus, **5** (1661) (2016).

[9] S. S. DRAGOMIR, *On some new inequalities of Hermite–Hadamard type for m-convex functions*, Tamkang J. Math., **33** (1) (2002), 45–55.

[10] S. S. DRAGOMIR AND R. P. AGARWAL, *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., **11** (5) (1998), 91–95.

[11] T.-S. DU, J. G. LIAO AND Y. J. LI, *Properties and integral inequalities of Hadamard–Simpson type for the generalized (s,m)–preinvex functions*, J. Nonlinear Sci. Appl., **9** (2016), 3112–3126.

[12] M. JLELI AND B. SAMET, *On Hermite–Hadamard type inequalities via fractional integral of a function with respect to another function*, J. Nonlinear Sci. Appl., **9** (2016), 1252–1260.

[13] A. KASHURI AND R. LIKO, *Some new Hermite–Hadamard type inequalities and their applications*, Stud. Sci. Math. Hung., **56** (1) (2019), 103–142.

[14] W. LIU, W. WEN AND J. PARK, *Hermite–Hadamard type inequalities for MT–convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9** (2016), 766–777.

[15] C. LUO, T.-S. DU, M. ADIL KHAN, A. KASHURI AND Y. SHEN, *Some k–fractional integrals inequalities through generalized $\lambda_{\phi m}$–MT–preinvexity*, J. Comput. Anal. Appl., **27** (4) (2019), 690–705.

[16] M. MATLOKA, *Inequalities for h–preinvex functions*, Appl. Math. Comput., **234** (2014), 52–57.

[17] M. V. MIHAI, *Some Hermite–Hadamard type inequalities via Riemann–Liouville fractional calculus*, Tamkang J. Math., **44** (4) (2013), 411–416.

[18] M. A. NOOR AND K. I. NOOR, *Exponential convex functions*, Preprint.

[19] M. A. NOOR, K. I. NOOR AND S. RASHID, *Fractal exponential convex functions and inequalities*, Preprint.

[20] M. A. NOOR, K. I. NOOR AND S. RASHID, *Exponential r–convex functions and inequalities*, Preprint.

[21] O. OMOTOYINBO AND A. MOGBODEMU, *Some new Hermite–Hadamard integral inequalities for convex functions*, Int. J. Sci. Innovation Tech., **1** (1) (2014), 1–12.

[22] M. E. ÖZDEMIR, S. S. DRAGOMIR AND C. YILDIZ, *The Hadamard's inequality for convex function via fractional integrals*, Acta Math. Sci., Ser. B, Engl. Ed., **33** (5) (2013), 153–164.

[23] J. PEČARIĆ AND J. JAKŠETIĆ, *Exponential convexity, Euler–Radau expansions and Stolarsky means*, Rad Hrvat. Akad. Znan. Umjet., Mat. Znan., **515** (2013), 81–94.

[24] C. PENG, C. ZHOU AND T.-S. DU, *Riemann–Liouville fractional Simpson's inequalities through generalized (m,h_1,h_2)–preinvexity*, Ital. J. Pure Appl. Math., **38** (2017), 345–367.

[25] S. RASHID, M. A. NOOR AND K. I. NOOR, *Fractional exponentially m–convex functions and inequalities*, Int. J. Anal. Appl., **17** (3) (2019), 464–478.

[26] H., WANG, T.-S. DU AND Y. ZHANG, *k–fractional integral trapezium–like inequalities through (h,m)–convex and (α,m)–convex mappings*, J. Inequal. Appl., **2017**(311) (2017), 20 pp.

[27] Y. ZHANG, T.-S. DU, H. WANG, Y. J. SHEN AND A. KASHURI, *Extensions of different type parameterized inequalities for generalized (m,h)–preinvex mappings via k–fractional integrals*, J. Inequal. Appl., **2018** (49) (2018), 30 pp.

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