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μ–PSEUDO ALMOST AUTOMORPHIC MILD SOLUTIONS FOR TWO TERMS ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the existence and uniqueness of $\mu-$pseudo almost automorphic mild solutions for two term fractional order differential equations in a Banach space with $\mu-$pseudo almost automorphic forcing terms. The fractional derivative is understood in the sense of Weyl. We use classical tools to obtain our results.

1. Introduction

The concept of almost automorphy was introduced in the literature by S. Bochner in [7, 8, 9]. It turns out to be a generalization of almost periodicity in the sense of Bohr. Veech [21] and Zaki [23] studied almost automorphic functions respectively on groups and the real number set. In his paper [17], N'Guérékata introduced the concept of asymptotically almost automorphic functions. For more information on the concept of almost automorphy and its application to evolution equations, cf [19, 20]. Many authors have produced extensive literature on the theory of almost automorphy with usefull generalizations. In [22], Xiao et al. introduced the notion of pseudo almost automorphy as suggested by N'Guérékata in [19]. Later on, the notion of weighted pseudo almost automorphy was introduced by Blot et al. in [6]. Recently, Blot et al. in [4] introduced the concept of $\mu-$pseudo almost automorphy which is more general than the class of weighted pseudo almost automorphic functions. The existence of pseudo almost automorphic, weighted pseudo almost automorphic and $\mu-$pseudo almost automorphic solutions of fractional differential equations has become an interesting field due to a lot of applications. Therefore, many authors have made important contributions on these topics [1, 2, 4, 6, 10, 11, 12, 14, 22].

Motivated by the above papers, we would like to study the existence and uniqueness of $\mu-$pseudo almost automorphic mild solutions for the following classes of two-term time fractional differential equations with iterated deviating arguments and integral forcing terms.

$$D_t^{\alpha+1} y(t) + \nu D_t^{\beta} y(t) = Ay(t) + D_t^{\alpha} F(t, y(t)) \tag{1}$$

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and

$$D_t^{\alpha+1}y(t)+\nu D_t^{\beta}y(t)=Ay(t)+D_t^{\alpha}G\left(t,\int_{-\infty}^{t}\mathscr{K}(t-s)h(s,y(s))ds\right), \tag{2}$$

where $0<\alpha\leqslant\beta\leqslant 1,\ \nu\geqslant 0;\ t\geqslant 0$ and $A:\ D(A)\subset\mathbb{X}\to\mathbb{X}$ is a sectorial operator of angle $\beta\pi/2$. The forcing terms F, G and h are suitable functions satisfying some appropriate conditions in their arguments mentioned later in assumptions. Here, D_t^{η} represents the Weyl fractional derivative of order η and $\mathscr{K}\in L^1(\mathbb{R})$ with $|\mathscr{K}(t)|\leqslant C_{\mathscr{K}}e^{-bt}$, where $b,\ C_{\mathscr{K}}>0$.

The paper is organized as follows. In Section 2, we recall some basic definitions and fundamental results about the notion of $\mu-$pseudo almost automorphic functions. The Section 3 is devoted to the main results. In this section, we first study the linearized cases of problems (1) and (2). Secondly, we use the Banach contraction principle to show that problem (1) admits a unique $\mu-$pseudo almost automorphic mild solution. Finally, we prove the existence and uniqueness of the solution of problem (2) using again the Banach contraction principle.

2. Preliminaries

In this section, we recall some basic definitions and preliminary results on $\mu-$pseudo almost automorphic functions.

In this paper, $(\mathbb{X},\|\cdot\|)$ will stand for a Banach space. We denote by $\mathscr{B}(\mathbb{R},\mathbb{X})$, $\mathscr{C}(\mathbb{R},\mathbb{X})$ and $\mathscr{BC}(\mathbb{R},\mathbb{X})$ the collections of bounded functions, continuous functions and continuous and bounded functions from $\mathbb{R}$ to $\mathbb{X}$, respectively. We define $g_\alpha(t)$ for $\alpha>0$ by

$$g_\alpha(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}t^{\alpha-1}, & \text{if } t>0,\\ 0 & \text{, if } t<0,\end{cases} \tag{3}$$

where Γ denotes the gamma function. The function g_α has the property: $g_\alpha * g_\beta=g_{\alpha+\beta}$; here $*$ denotes the convolution defined by

$$(f*g)(t)=\int_{-\infty}^{+\infty}f(t-s)g(s)ds$$

for appropriate functions f and g.

DEFINITION 1. Weyl fractional integral of an appropriate function $f:\mathbb{R}\longrightarrow\mathbb{X}$ of order $\alpha>0$ is defined by

$$J_t^{\alpha}f(t)=\int_{-\infty}^{t}g_\alpha(t-s)f(s)ds,\ t>0, \tag{4}$$

when the integral is convergent.

DEFINITION 2. Weyl fractional derivative of an appropriate function $f:\mathbb{R}\longrightarrow\mathbb{X}$ of order $\alpha>0$ is defined by

$$D_t^{\alpha}f(t)=\frac{d^m}{dt^m}\int_{-\infty}^{t}g_{m-\alpha}(t-s)f(s)\,ds,\ t>0, \tag{5}$$

when $m=[\alpha]+1$. We have $D_t^{\alpha}J_t^{\alpha}f(t)=f(t)$ for any $\alpha>0$. Fore more details see Miller and Ross [16].

DEFINITION 3. A densely defined closed linear operator A is said to be sectorial of type ω and angle θ if there exist $\theta\in\left[0,\frac{\pi}{2}\right)$, $M>0$, $\omega\in\mathbb{R}$ such that its resolvent exists in the sector

$$\omega+\sum\nolimits_{\theta}=\left\{\omega+\lambda:\lambda\in\mathbb{C},|\arg(\lambda)|<\frac{\pi}{2}+\theta\right\}\setminus\{\omega\} \tag{6}$$

and

$$\left\|(\lambda I-A)^{-1}\right\|\leqslant\frac{M}{|\lambda-\omega|},\ \lambda\in\omega+\sum\nolimits_{\theta}. \tag{7}$$

DEFINITION 4. Let $\nu>0$ and $0<\alpha\leqslant\beta\leqslant 1$. Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $D(A)$. Then, A is said to be generator of a $(\alpha,\beta)_{\nu}$-regularized family if there exist $\omega\geqslant 0$ and a strongly continuous function $\mathscr{S}_{\alpha,\beta}(t):\mathbb{R}\longrightarrow\mathscr{B}(\mathbb{X})$ such that $\left\{\lambda^{\alpha+1}+\nu\lambda^{\beta}:\ Re(\lambda)>\omega\right\}\subset\rho(A)$ (the resolvent set of A) and

$$\begin{aligned}H(\lambda)y&=\lambda^{\nu}\left(\lambda^{\alpha+1}+\nu\lambda^{\beta}-A\right)^{-1}y\\&=\int_0^{+\infty}e^{-\lambda t}\mathscr{S}_{\alpha,\beta}(t)y\,dt,\ Re(\lambda)>\omega,\ y\in\mathbb{X}.\end{aligned} \tag{8}$$

By the uniqueness theorem for the Laplace transform, if $\nu=0$ and $\alpha=0$ this corresponds to the case of a C_0-semigroup, whereas $\nu=0$ and $\alpha=1$ corresponds to the concept of cosine family. We refer to the monograph [3] for more information about Laplace approach to semigroups and cosine families.

LEMMA 1. ([13]) *Let $\nu\geqslant 0$ and $0\leqslant\alpha,\beta\leqslant 1$. There exist Laplace transformable functions $a_{\alpha,\beta}\in C^1(\mathbb{R}^+)$ and $k_{\alpha,\beta}\in C^1(\mathbb{R}^+)$ such that $\widehat{a}_{\alpha,\beta}(\lambda)=\dfrac{1}{\lambda^{\alpha+1}+\nu\lambda^{\beta}}$ and $\widehat{k}_{\alpha,\beta}=\dfrac{\lambda^{\alpha}}{\lambda^{\alpha+1}+\nu\lambda^{\beta}}$.*

From the above result we note that $a_{\alpha,\beta}=g_{\alpha}*k_{\alpha,\beta}$. Moreover, $k_{\alpha,\beta}(0)=1$ and $k_{\alpha,\beta}$ is a differential function (see [13]).

PROPOSITION 1. ([13]) *Let $\nu\geqslant 0$ and $0\leqslant\alpha,\beta\leqslant 1$. Let $\mathscr{S}_{\alpha,\beta}(t)$ be a $(\alpha,\beta)_{\nu}$-regularized family generated by A on $\mathbb{X}$. Then , the following conditions hold true:*

i) $\mathscr{S}_{\alpha,\beta}(t)$ is strongly continuous and $\mathscr{S}_{\alpha,\beta}(0)=I$.

ii) $\mathscr{S}_{\alpha,\beta}(t)y \in D(A)$ *and* $A\mathscr{S}_{\alpha,\beta}(t)y = \mathscr{S}_{\alpha,\beta}(t)Ay$ *for all* $y \in D(A)$ *and* $t \geqslant 0$.

iii) Let $y \in \mathbb{X}$ *and* $t \geqslant 0$. *Then,*

$$\int_0^t a_{\alpha,\beta}(t-s)\mathscr{S}_{\alpha,\beta}(s)yds \in D(A)$$

and

$$\mathscr{S}_{\alpha,\beta}(t)y = k_{\alpha,\beta}(t)y + A\int_0^t \left(g_\alpha * k_{\alpha,\beta}\right)(t-s)\mathscr{S}_{\alpha,\beta}(s)yds.$$

iv) We have $\mathscr{S}_{\alpha,\beta}(t)y \in C^1(\mathbb{R}^+,\mathbb{X})$ *for some* $y \in D(A)$.

THEOREM 1. ([13]) *Let* $\nu > 0$, $\omega < 0$ *and* $0 < \alpha \leqslant \beta \leqslant 1$. *If* A *is an* ω*-sectorial operator of angle* $\dfrac{\beta\pi}{2}$, *then* A *generates a* $(\alpha,\beta)_\nu$*-regularized family* $\mathscr{S}_{\alpha,\beta}(t)$ *satisfying the estimate*

$$\left\|\mathscr{S}_{\alpha,\beta}(t)\right\| \leqslant \frac{M}{1+|\omega|\left(t^{\alpha+1}+\nu t^{\beta}\right)} := J_{\alpha,\beta}(t),\ t \geqslant 0, \tag{9}$$

where M *is a constant depending solely on* α,β.

Now, let us recall some definitions and results on almost automorphic functions.

Denote by $\mathscr{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathscr{M}$ the set of all positive mesures μ on $\mathscr{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all a, $b \in \mathbb{R}$ $(a \leqslant b)$.

DEFINITION 5. ([19, 20]) A function $f \in \mathscr{C}(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(\tau_n')_n$ there exists a subsequence $(\tau_n)_n$ such that

$$g(t) = \lim_{n\to+\infty} f(t+\tau_n) \text{ exists for all } t \in \mathbb{R} \tag{10}$$

and

$$\lim_{n\to+\infty} g(t-\tau_n) = f(t) \text{ for all } t \in \mathbb{R}. \tag{11}$$

We denote by $AA(\mathbb{R},\mathbb{X})$ the space of the almost automorphic $\mathbb{X}$-valued functions.

REMARK 1. Note that in the above limit the function g is just mesurable. If the convergence in both limits is uniform in $t \in \mathbb{R}$, then f is almost periodic. The concept of almost automorphy is then larger than almost periodicity. If f is almost automorphic, then its range is relatively compact, thus bounded in norm.

EXAMPLE 1. ([18]) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$f(t) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) \text{ for } t \in \mathbb{R}. \tag{12}$$

Then f is almost automorphic, but it is not uniformly continuous on $\mathbb{R}$. Therefore, it is not almost periodic.

PROPOSITION 2. ([19]) *$(AA(\mathbb{R},\mathbb{Y}),\|.\|_\infty)$ is a Banach space.*

DEFINITION 6. A function $f\in\mathscr{C}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is said to be almost automorphic in $t\in\mathbb{R}$ uniformly with respect to $x\in\mathbb{X}$, if the following two conditions hold:

i) for all $x\in\mathbb{X}$, $f(.,x)\in AA(\mathbb{R},\mathbb{Y})$,

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{X}$, for all $\varepsilon>0$, there exists $\delta>0$ such that for all $x_1,x_2\in K$, one has

$$\|x_1-x_2\|\leqslant\delta\Longrightarrow\sup_{t\in\mathbb{R}}\|f(t,x_1)-f(t,x_2)\|\leqslant\varepsilon. \tag{13}$$

We denote by $AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ the set of all such functions.

THEOREM 2. ([5]) *Let $f\in AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $x\in AA(\mathbb{R},\mathbb{X})$. Then $[t\longmapsto f(t,x(t))]\in AA(\mathbb{R},\mathbb{Y})$.*

DEFINITION 7. ([4]) Let $\mu\in\mathscr{M}$. A bounded continuous function $f:\mathbb{R}\longrightarrow\mathbb{X}$ is said to be μ-ergodic if

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|f(t)\|\,d\mu(t)=0. \tag{14}$$

We denote the space of all such functions by $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$.

PROPOSITION 3. ([4]) *Let $\mu\in\mathscr{M}$. Then $(\mathscr{E}(\mathbb{R},\mathbb{X},\mu),\|.\|_\infty)$ is a Banach space.*

DEFINITION 8. ([4]) Let $\mu\in\mathscr{M}$. A continuous function $f:\mathbb{R}\longrightarrow\mathbb{X}$ is said to be μ-pseudo almost automorphic if f can be written in the form:

$$f=\phi+\psi \tag{15}$$

where $\phi\in AA(\mathbb{R},\mathbb{X})$ and $\psi\in\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$. We denote the space of all such functions by $PAA(\mathbb{R},\mathbb{X},\mu)$. Then, we have

$$AA(\mathbb{R},\mathbb{X})\subset PAA(\mathbb{R},\mathbb{X},\mu)\subset\mathscr{BC}(\mathbb{R},\mathbb{X}). \tag{16}$$

REMARK 2. Without assumption on the measure μ, the decomposition (15) of the corresponding μ-pseudo almost automorphic function is not unique.

REMARK 3. A pseudo almost automorphic function is μ-pseudo almost automorphic function in the particular case where the measure μ is the Lebesgue measure. For more details on pseudo almost automorphic functions, we refer to [14, 15, 6].

REMARK 4. The notion of μ-pseudo almost automorphic functions is a generalization of the weighted pseudo almost automorphic functions which is due to Blot et al. [6]. Following [6], a function f is so-called *weighted pseudo almost automorphic* if f is a μ-pseudo almost automorphic function in the particular case where the measure μ is defined by $\mu(A) = \int_A \rho(t)\,dt$ for $A \in \mathscr{B}$ with $\rho(t) > 0$ a.e on $\mathbb{R}$ for the Lebesgue measure and $\int_{-\infty}^{+\infty} \rho(t)\,dt = +\infty$.

PROPOSITION 4. ([4]) *Let $\mu \in \mathscr{M}$. Then $PAA(\mathbb{R},\mathbb{X},\mu)$ is a vector space.*

DEFINITION 9. ([4]) Let μ_1 and $\mu_2 \in \mathscr{M}$. μ_1 is said to be equivalent to μ_2 $(\mu_1 \sim \mu_2)$ if there exist constants α, $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha\mu_1(A) \leqslant \mu_2(A) \leqslant \beta\mu_1(A), \text{ for } A \in \mathscr{B} \text{ satisfying } A \cap I = \emptyset.$$

REMARK 5. The relation $\sim$ is an equivalence relation on $\mathscr{M}$.

THEOREM 3. ([4]) *Let μ_1, $\mu_2 \in \mathscr{M}$. If μ_1 and μ_2 are equivalent, then $\mathscr{E}(\mathbb{R},\mathbb{X},\mu_1) = \mathscr{E}(\mathbb{R},\mathbb{X},\mu_2)$ and $PAA(\mathbb{R},\mathbb{X},\mu_1) = PAA(\mathbb{R},\mathbb{X},\mu_2)$.*

For $\mu \in \mathscr{M}$, $\tau \in \mathbb{R}$ and $A \in \mathscr{B}$, we denote μ_τ the positive measure on $(\mathbb{R},\mathscr{B})$ defined by

$$\mu_\tau(A) = \mu(\{a+\tau, a \in A\}). \tag{17}$$

From $\mu \in \mathscr{M}$, we formulate the following hypothesis:

$$\textbf{(A0)} \begin{cases} \text{For all } \tau \in \mathbb{R}, \text{ there exist } \beta > 0 \text{ and a bounded interval } I \text{ such} \\ \quad \text{that } \mu_\tau(A) \leqslant \beta\mu(A), \text{ when } A \in \mathscr{B} \text{ satisfies } A \cap I = \emptyset. \end{cases}$$

LEMMA 2. ([4]) *Let $\mu \in \mathscr{M}$. Then μ satisfies **(A0)** if and only if the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

LEMMA 3. ([4]) *Hypothesis **(A0)** implies*

$$\textit{for all } \sigma > 0, \limsup_{r\to+\infty} \frac{\mu([-r-\sigma, r+\sigma])}{\mu([-r,r])} < +\infty. \tag{18}$$

THEOREM 4. ([4]) *Let $\mu \in \mathscr{M}$ satisfying **(A0)**. Then $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ is translation invariant, therefore $PAA(\mathbb{R},\mathbb{X},\mu)$ is also translation invariant.*

THEOREM 5. ([4, Theorem 3.9]) *Let $\mu \in \mathscr{M}$ satisfy **(A0)**. If $f \in PAA(\mathbb{R},\mathbb{X},\mu)$ and $g \in L^1(\mathbb{R},\mathscr{L}(\mathbb{X}))$, then the convolution product $f * g$ is also μ-pseudo almost automorphic. In fact, if $f \in AA(\mathbb{R},\mathbb{X})$, then $f * g \in AA(\mathbb{R},\mathbb{X})$ and if $f \in \mathscr{E}(\mathbb{R},\mathbb{X},\mu)$, then $f * g \in \mathscr{E}(\mathbb{R},\mathbb{X},\mu)$.*

COROLLARY 1. ([4]) *Let* $\mu \in \mathscr{M}$ *satisfy* ***(A0)****. Assume that the* C_0-*semigroup* $(T(t))_{t\geqslant 0}$ *is exponentially stable. If* $\theta \in PAA(\mathbb{R},\mathbb{X},\mu)$, *then the function defined by*

$$h(t) = \int_{-\infty}^{t} T(t-s)\,\theta(s)\,ds \text{ for } t \in \mathbb{R}$$

is μ*-pseudo almost automorphic.*

THEOREM 6. ([4]) *Let* $\mu \in \mathscr{M}$. *Assume that* $PAA(\mathbb{R},\mathbb{X},\mu)$ *is translation invariant. Then the decomposition of a* μ*-pseudo almost automorphic function in the form* $f = \phi + \psi$ *where* $\phi \in AA(\mathbb{R},\mathbb{X})$ *and* $\psi \in \mathscr{E}(\mathbb{R},\mathbb{X},\mu)$, *is unique.*

THEOREM 7. ([4]) *Let* $\mu \in \mathscr{M}$. *Assume that* $PAA(\mathbb{R},\mathbb{X},\mu)$ *is translation invariant. Then* $(PAA(\mathbb{R},\mathbb{X},\mu), \|.\|_\infty)$ *is a Banach space.*

DEFINITION 10. ([4]) Let $\mu \in \mathscr{M}$. A continuous function $f:\mathbb{R}\times\mathbb{X}\longrightarrow\mathbb{Y}$ is said to be almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if the following two conditions are hold:

i) for all $x \in \mathbb{X}$, $f(.,x) \in AA(\mathbb{R},\mathbb{Y})$

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{X}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x_1, $x_2 \in K$, one has

$$\|x_1 - x_2\| \leqslant \delta \Longrightarrow \sup_{t\in\mathbb{R}} \|f(t,x_1) - f(t,x_2)\| \leqslant \varepsilon.$$

Denote by $AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ the set of all such functions.

DEFINITION 11. ([4]) Let $\mu \in \mathscr{M}$. A continuous function $f:\mathbb{R}\times\mathbb{X}\longrightarrow\mathbb{Y}$ is said to be μ-ergodic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if the following two conditions are true:

i) for all $x \in \mathbb{X}$, $f(.,x) \in \mathscr{E}(\mathbb{R},\mathbb{Y},\mu)$

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x.

Denote by $\mathscr{E}U(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ the set of all such functions.

DEFINITION 12. ([4]) Let $\mu \in \mathscr{M}$. A continuous function $f:\mathbb{R}\times\mathbb{X}\longrightarrow\mathbb{Y}$ is said to be μ-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if f can be written in the form $f = \phi + \psi$ where $\phi \in AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $\psi \in \mathscr{E}U(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$.

$PAAU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ denotes the set of all such functions.

REMARK 6. We have $AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y}) \subset PAAU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$.

THEOREM 8. ([4, Theorem 5.7]) *Let* $\mu \in \mathscr{M}$, $f \in PAAU(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$ *and* $x \in PAA(\mathbb{R},\mathbb{X},\mu)$. *Assume that for all bounded subset* B *of* $\mathbb{X}$, f *is bounded on* $\mathbb{R}\times B$. *Then* $[t \longmapsto f(t,x(t))] \in PAA(\mathbb{R},\mathbb{Y},\mu)$.

3. Main results

We consider the two time fractional order linear differential equation

$$D^{\alpha+1}y(t)+\nu D^{\beta}y(t)=Ay(t)+D^{\alpha}f(t). \tag{19}$$

The following theorem established in [2] guarantees the existence of mild solutions of equation (19).

THEOREM 9. *Let* A *be a generator of the* $(\alpha,\beta)_\nu$*-regularized family* $\{\mathscr{S}_{\alpha,\beta}(t)\}_{t\geqslant 0}$. *Then, equation* (19) *admits a mild solution given by*

$$y(t)=\int_{-\infty}^{t}\mathscr{S}_{\alpha,\beta}(t-s)f(s)\,ds,\ t\in\mathbb{R}, \tag{20}$$

provided that $\{\mathscr{S}_{\alpha,\beta}(t)\}_{t\geqslant 0}$ *exists and is integrable.*

The following definition is inspired by the above representation of mild solutions for the problem (19) establish by Alvarez-Pardo and Lizama in [2].

DEFINITION 13. Let A be a generator of the $(\alpha,\beta)_\nu$-regularized family $\{\mathscr{S}_{\alpha,\beta}(t)\}_{t\geqslant 0}$. A function $y:\mathbb{R}\longrightarrow\mathbb{X}$ defined by

$$y(t)=\int_{-\infty}^{t}\mathscr{S}_{\alpha,\beta}(t-s)F(s,y(s))\,ds,\ t\in\mathbb{R}, \tag{21}$$

is called a mild solution of (1).

DEFINITION 14. Let A be a generator of the $(\alpha,\beta)_\nu$-regularized family $\{\mathscr{S}_{\alpha,\beta}(t)\}_{t\geqslant 0}$. A function $y:\mathbb{R}\longrightarrow\mathbb{X}$ defined by

$$y(t)=\int_{-\infty}^{t}\mathscr{S}_{\alpha,\beta}(t-s)G\left(s,\int_{-\infty}^{s}\mathscr{K}(s-\xi)h(\xi,y(\xi))\,d\xi\right)ds,\ t\in\mathbb{R}, \tag{22}$$

is called a mild solution of (2).

3.1. The linear case

The following theorem is the main result for the linear case.

THEOREM 10. *Let* $\mu\in\mathscr{M}$. *Let* $f=\phi+\psi$ *be in* $PAA(\mathbb{R},\mathbb{X},\mu)$ *with* $\phi\in AA(\mathbb{R},\mathbb{X})$ *and* $\psi\in\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$. *Then* (19) *admits a* μ*-pseudo almost automorphic mild solution.*

We need the following lemma:

LEMMA 4. *Let* $\nu > 0$, $\omega < 0$ *and* $0 < \alpha \leqslant \beta \leqslant 1$. *Let* $\left\{\mathscr{S}_{\alpha,\beta}(t)\right\}_{t\geqslant 0}$ *be a* $(\alpha,\beta)_\nu$*-regularized family generated by* A *an* ω*-sectorial operator of angle* $\frac{\beta\pi}{2}$. *Then, for each* $f \in AA(\mathbb{R},\mathbb{X})$,

$$F(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s) f(s)\, ds \in AA(\mathbb{R},\mathbb{X}).$$

Proof. Since $\left\{\mathscr{S}_{\alpha,\beta}(t)\right\}_{t\geqslant 0}$ is a $(\alpha,\beta)_\nu$-regularized family and $f \in AA(\mathbb{R},\mathbb{X})$ then $\mathscr{S}_{\alpha,\beta}(t)$ and f are continuous, so

$$F(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s) f(s)\, ds = \int_{0}^{+\infty} \mathscr{S}_{\alpha,\beta}(s) f(t-s)\, ds \in \mathscr{C}(\mathbb{R},\mathbb{X}).$$

Let $(\tau_n')_n$ be a sequence of real numbers. Since $f \in AA(\mathbb{R},\mathbb{X})$, there exists a subsequence $(\tau_n)_n$ such that

$$g(t) = \lim_{n\to+\infty} f(t+\tau_n) \text{ exists for all } t \in \mathbb{R}$$

and

$$\lim_{n\to+\infty} g(t-\tau_n) = f(t) \text{ for all } t \in \mathbb{R}.$$

Therefore, for a fixed $t \in \mathbb{R}$, it follows that

$$\mathscr{S}_{\alpha,\beta}(s)\, g(t-s) = \lim_{n\to+\infty} \mathscr{S}_{\alpha,\beta}(s) f(t-s+\tau_n) \text{ exists for all } s \in \mathbb{R} \tag{23}$$

and

$$\lim_{n\to+\infty} \mathscr{S}_{\alpha,\beta}(s)\, g(t-s-\tau_n) = \mathscr{S}_{\alpha,\beta}(s) f(t-s) \text{ for all } s \in \mathbb{R}. \tag{24}$$

We have

$$\begin{aligned} \left\|\mathscr{S}_{\alpha,\beta}(s) f(t-s+\tau_n)\right\| &\leqslant \left\|\mathscr{S}_{\alpha,\beta}(s)\right\| \|f\|_\infty \\ &\leqslant \frac{M\|f\|_\infty}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)} \text{ for all } s \in \mathbb{R} \end{aligned}$$

with $\frac{M\|f\|_\infty}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)} \in L^1(\mathbb{R})$. By using the Lebesgue dominated theorem, we obtain

$$\begin{aligned} \lim_{n\to+\infty} F(t+\tau_n) &= \lim_{n\to+\infty} \int_0^{+\infty} \mathscr{S}_{\alpha,\beta}(s) f(t-s+\tau_n)\, ds \\ &= \int_0^{+\infty} \mathscr{S}_{\alpha,\beta}(s)\, g(t-s)\, ds = G(t) \text{ exists for all } t \in \mathbb{R} \end{aligned}$$

and

$$\lim_{n\to+\infty} G(t-\tau_n) = \lim_{n\to+\infty}\int_0^{+\infty} \mathscr{S}_{\alpha,\beta}(s)\, g(t-s-\tau_n)\, ds$$
$$= \int_0^{+\infty} \mathscr{S}_{\alpha,\beta}(s)\, f(t-s)\, ds = F(t) \text{ for all } t\in\mathbb{R}.$$

Thus, $F(t)=\int_{-\infty}^{t}\mathscr{S}_{\alpha,\beta}(t-s)\, f(s)\, ds \in AA(\mathbb{R},\mathbb{X})$.
Next, we prove Theorem 10.

Proof of Theorem 10.
Let us define a map $\Lambda f(t)$ by

$$\Lambda f(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\, f(s)\, ds. \tag{25}$$

In view of Theorem 9, $\Lambda f(t)$ is a mild solution of (19). Now, we will prove that $\Lambda f(t) \in PAA(\mathbb{R},\mathbb{X},\mu)$.

Since $f=\phi+\psi\in PAA(\mathbb{R},\mathbb{X},\mu)$, we have

$$\begin{aligned}\Lambda f(t) &= \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\, f(s)\, ds\\ &= \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\,\phi(s)\, ds + \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\,\psi(s)\, ds\\ &= \Lambda\phi(t)+\Lambda\psi(t)\end{aligned}$$

where

$$\Lambda\phi(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\,\phi(s)\, ds \text{ and } \Lambda\psi(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\,\psi(s)\, ds$$

with $\phi\in AA(\mathbb{R},\mathbb{X})$ and $\psi\in\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$. From Lemma 4, it follows that $\Lambda\phi(t)\in AA(\mathbb{R},\mathbb{X})$.

Now, it remains to show that $\Lambda\psi(t)\in\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$. We have

$$\Lambda\psi(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\,\psi(s)\, ds = \int_0^{+\infty} \mathscr{S}_{\alpha,\beta}(s)\,\psi(t-s)\, ds.$$

So,

$$\|\Lambda\psi(t)\| \leqslant \int_0^{+\infty} \frac{M}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}\,\|\psi(t-s)\|\, ds.$$

Therefore

$$\begin{aligned}&\frac{1}{\mu([-n,n])}\int_{-n}^{n}\|\Lambda\psi(t)\|\, d\mu(t)\\ &\leqslant \frac{1}{\mu([-n,n])}\int_{-n}^{n}\left[\int_0^{+\infty}\frac{M\,\|\psi(t-s)\|}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}ds\right]d\mu(t)\end{aligned}$$

$$\leqslant \int_0^{+\infty}\left[\frac{1}{\mu([-n,n])}\int_{-n}^{n}\frac{M\|\psi(t-s)\|}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}d\mu(t)\right]ds$$
$$\leqslant \int_0^{+\infty}\psi_n(s)\,ds \tag{26}$$

where

$$\psi_n(s)=\frac{1}{\mu([-n,n])}\int_{-n}^{n}\frac{M\|\psi(t-s)\|}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}d\mu(t).$$

Since $\psi(t)\in\mathscr{BC}(\mathbb{R},\mathbb{X})$ then

$$\begin{aligned}|\psi_n(s)| &\leqslant \sup_{t\in\mathbb{R}}\|\psi(t)\|\frac{M}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}\frac{1}{\mu([-n,n])}\int_{-n}^{n}d\mu(t)\\ &\leqslant \sup_{t\in\mathbb{R}}\|\psi(t)\|\frac{M}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}\in L^1(\mathbb{R})\end{aligned}$$

because $0<\alpha\leqslant\beta\leqslant 1$. On the other hand,

$$\begin{aligned}\psi_n(s) &= \frac{M}{\mu([-n,n])}\int_{-n}^{n}\frac{\|\psi(t-s)\|}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}d\mu(t)\\
&= J_{\alpha,\beta}(s)\frac{1}{\mu([-n,n])}\int_{-n}^{n}\|\psi(t-s)\|\,d\mu(t)\\
&= J_{\alpha,\beta}(s)\frac{1}{\mu_s([-n-s,n-s])}\int_{-n-s}^{n-s}\|\psi(t)\|\,d\mu_s(t)\\
&\leqslant \frac{\mu_s([-n-s,n+s])}{\mu_s([-n-s,n-s])}\frac{J_{\alpha,\beta}(s)}{\mu_s([-n-s,n+s])}\int_{-n-s}^{n-s}\|\psi(t)\|\,d\mu_s(t)\\
&\leqslant \frac{\mu_s([-n-2s,n+s])}{\mu_s([-n-s,n-s])}\frac{J_{\alpha,\beta}(s)}{\mu_s([-n-s,n+s])}\int_{-n-s}^{n-s}\|\psi(t)\|\,d\mu_s(t)\end{aligned}$$

where $J_{\alpha,\beta}(s)$ is defined by (9). Since $\psi(t)\in\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$, we deduce from Lemma 2 and Theorem 3 that $\psi(t)\in\mathscr{E}(\mathbb{R},\mathbb{X},\mu_s)$, so

$$\lim_{n\to+\infty}\frac{1}{\mu_s([-n-s,n+s])}\int_{-n-s}^{n-s}\|\psi(t)\|\,d\mu_s(t)=0$$

and by Lemma 3 we have

$$\lim_{n\to+\infty}\frac{\mu_s([-n-2s,n+s])}{\mu_s([-n-s,n-s])}=\lim_{n\to+\infty}\frac{\mu_s([-n-s,n+s])}{\mu_s([-n,n])}<+\infty.$$

Thus,

$$\lim_{n\to+\infty}\psi_n(s)=0.$$

Therefore, by Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\to+\infty}\int_0^{+\infty}\psi_n(s)\,ds=0. \tag{27}$$

From (26) and (27) we deduce that

$$\lim_{n\to+\infty}\frac{1}{\mu([-n,n])}\int_{-n}^{n}\|\Lambda\psi(t)\|\,d\mu(t)=0.$$

This ends the proof.

3.2. The nonlinear case

Let us make the following assumptions:

(A1): $h=\phi_h+\psi_h\in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ is such that $\phi_h\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ is uniformly continuous in a bounded set $K\subset\mathbb{X}$ for all $t\in\mathbb{R}$ and $\psi_h\in\mathscr{E}(\mathbb{R}\times\mathbb{X},L^p([0,1],\mathbb{X}),\mu)$. Moreover, there exists a positive constant L_h such that for all y, $z\in\mathbb{X}$, $t\in\mathbb{R}$,

$$\|h(t,y)-h(t,z)\|\leqslant L_h\|y-z\|.$$

(A2): $F=\phi_F+\psi_F\in PAAU(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ with $\phi_F\in AAU(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $\psi_F\in\mathscr{E}U(\mathbb{R}\times\mathbb{X},L^p([0,1],\mathbb{X}),\mu)$. Moreover, assume that there exist positive constants L_F, L_{ϕ_F} such that for all y, $z\in\mathbb{X}$, $t\in\mathbb{R}$ we have

$$\|F(t,y)-F(t,z)\|\leqslant L_F\|y-z\|,$$

$$\|\phi_F(t,y)-\phi_F(t,z)\|\leqslant L_{\phi_F}\|y-z\|.$$

We now establish the existence and uniqueness of weighted pseudo automorphic mild solutions for (1) and (2) when the forcing terms F, G and h satisfy Lipschitz type conditions.

LEMMA 5. *Let $\mu\in\mathscr{M}$. Assume that h satisfies* ***(A1)****. Then, for $y\in PAA(\mathbb{R},\mathbb{X},\mu)$, we have*

$$\int_{-\infty}^{t}\mathscr{K}(t-s)\,h(s,y(s))\,ds\in PAA(\mathbb{R},\mathbb{X},\mu).$$

Proof. Let $y\in PAA(\mathbb{R},\mathbb{X},\mu)$. Since $h=\phi_h+\psi_h\in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ with $\phi_h\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $\psi_h\in\mathscr{E}(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ then, by Theorem 8 we have $[H:t\longmapsto H(y)(t)=h(t,y(t))]\in PAA(\mathbb{R},\mathbb{X},\mu)$. Setting

$$\widetilde{\mathscr{K}}(t)=\begin{cases}\mathscr{K}(t) & \text{if } t>0\\ 0 & \text{if } t\leqslant 0\end{cases},$$

by assumption $\mathscr{K}(t)\in L^1(\mathbb{R})$ we have $\widetilde{\mathscr{K}}(t)\in L^1(\mathbb{R})$. By Theorem 5, we can conclude that $\left(\widetilde{\mathscr{K}}*H\right)(y)(t)=\int_{-\infty}^{t}\mathscr{K}(t-s)\,h(s,y(s))\,ds\in PAA(\mathbb{R},\mathbb{X},\mu)$.

THEOREM 11. *Let $\mu\in\mathscr{M}$, $\nu>0$, $\omega<0$ and $0<\alpha\leqslant\beta\leqslant 1$. Let $\{\mathscr{S}_{\alpha,\beta}(t)\}_{t\geqslant 0}$ be a $(\alpha,\beta)_\nu$-regularized family generated by A an ω-sectorial operator of angle $\dfrac{\beta\pi}{2}$.*

Suppose that μ *and* $F \in PAAU(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ *satisfy the assumptions* ***(A0)*** *and* ***(A2)*** *respectively. We assume that*

$$L_F I_{\alpha,\beta} < 1 \tag{28}$$

where

$$I_{\alpha,\beta} = \int_0^{+\infty} \frac{M}{1+|\omega|\left(t^{\alpha+1}+\nu t^{\beta}\right)}dt.$$

Then the problem (1) *admits a unique* μ*-pseudo almost automorphic mild solution.*

Proof. Since $\mu \in \mathscr{M}$ verifies assumption **(A0)** then, from Theorem 7 we have that $PAA(\mathbb{R},\mathbb{X},\mu)$ is a Banach space.

We now consider the map Λ_F defined on $PAA(\mathbb{R},\mathbb{X},\mu)$ by

$$\Lambda_F(y(t)) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)F(s,y(s))\,ds.$$

Using condition **(A2)** and Theorem 8 we deduce that

$F(s,y(s)) \in PAA(\mathbb{R},\mathbb{X},\mu)$. Thus, it follows from Theorem 10 that $\Lambda_F(y(t)) \in PAA(\mathbb{R},\mathbb{X},\mu)$ for all $y(t) \in PAA(\mathbb{R},\mathbb{X},\mu)$.

On the other hand, for all $y,z \in PAA(\mathbb{R},\mathbb{X},\mu)$ we have

$$\begin{aligned}
&\|\Lambda_F(y)(t)-\Lambda_F(z)(t)\| \\
&= \left\|\int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)F(s,y(s))\,ds - \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)F(s,z(s))\,ds\right\| \\
&\leqslant \int_{-\infty}^{t} \left\|\mathscr{S}_{\alpha,\beta}(t-s)\right\| \|F(s,y(s))-F(s,z(s))\|\,ds \\
&\leqslant L_F \int_{-\infty}^{t} \left\|\mathscr{S}_{\alpha,\beta}(t-s)\right\| \|y(s)-z(s)\|\,ds \\
&\leqslant L_F \|y-z\|_\infty \int_{-\infty}^{t} \frac{M}{1+|\omega|\left[(t-s)^{\alpha+1}+\nu(t-s)^{\beta}\right]}ds \\
&\leqslant L_F \|y-z\|_\infty \int_0^{+\infty} \frac{M}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)}ds \\
&\leqslant L_F.I_{\alpha,\beta}.\|y-z\|_\infty.
\end{aligned}$$

So,

$$\|\Lambda_F(y)-\Lambda_F(z)\|_\infty \leqslant L_F.I_{\alpha,\beta}.\|y-z\|_\infty$$

with $L_F.I_{\alpha,\beta} < 1$. This implies that Λ_F is a contration mapping from the Banach space $PAA(\mathbb{R},\mathbb{X},\mu)$ into itself. By the Banach contration principle, Λ_F has a unique fixed point. Hence, there exists a unique $y \in PAA(\mathbb{R},\mathbb{X},\mu)$ such that $y(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)F(s,y(s))\,ds$. This end the proof.

THEOREM 12. *Let $\mu \in \mathscr{M}$ verifies **(A0)**, $\nu > 0$, $\omega < 0$ and $0 < \alpha \leqslant \beta \leqslant 1$. Let $\{\mathscr{S}_{\alpha,\beta}(t)\}_{t\geqslant 0}$ be a $(\alpha,\beta)_\nu$-regularized family generated by A an ω-sectorial operator of angle $\frac{\beta\pi}{2}$. Suppose that $h \in PAA(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ and $G \in PAAU(\mathbb{R}\times\mathbb{X},\mathbb{X},\mu)$ satisfy the assumptions **(A1)** and **(A2)** respectively. If*

$$\frac{L_G C_{\mathscr{K}} L_h I_{\alpha,\beta}}{b} < 1$$

then, the problem (2) *admits a unique μ-pseudo almost automorphic mild solution.*

Proof. Since $\mu \in \mathscr{M}$ verifies assumption **(A0)** then, from Theorem 7 we have that $PAA(\mathbb{R},\mathbb{X},\mu)$ is a Banach space.

We consider the map Λ_G defined on $PAA(\mathbb{R},\mathbb{X},\mu)$ by

$$\Lambda_G(y)(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\, G\left(s,\widetilde{h}(y)(s)\right) ds$$

where

$$\widetilde{h}(y)(s) = \int_{-\infty}^{s} \mathscr{K}(t-\tau)\, h(\tau,y(\tau))\, d\tau.$$

Since G satisfies assumption **(A2)**, then using Lemma 5 and Theorem 8 we get that $G\left(s,\widetilde{h}(y)(s)\right) \in PAA(\mathbb{R},\mathbb{X},\mu)$ for all $y(s) \in PAA(\mathbb{R},\mathbb{X},\mu)$. Further, it follows from Theorem 10 that $\Lambda_G(y)(t) \in PAA(\mathbb{R},\mathbb{X},\mu)$ for all $y(s) \in PAA(\mathbb{R},\mathbb{X},\mu)$. Let $y,z \in PAA(\mathbb{R},\mathbb{X},\mu)$. We have

$$\begin{aligned}
&\|\Lambda_G(y)(t) - \Lambda_G(z)(t)\| \\
&= \left\| \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\, G\left(s,\widetilde{h}(y)(s)\right) ds - \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\, G\left(s,\widetilde{h}(z)(s)\right) ds \right\| \\
&\leqslant \int_{-\infty}^{t} \|\mathscr{S}_{\alpha,\beta}(t-s)\| \left\| G\left(s,\widetilde{h}(y)(s)\right) - G\left(s,\widetilde{h}(z)(s)\right) \right\| ds \\
&\leqslant L_G \int_{-\infty}^{t} \|\mathscr{S}_{\alpha,\beta}(t-s)\| \left\| \widetilde{h}(y)(s) - \widetilde{h}(z)(s) \right\| ds \\
&\leqslant L_G \int_{-\infty}^{t} \left[\|\mathscr{S}_{\alpha,\beta}(t-s)\| \left\| \int_{-\infty}^{s} \mathscr{K}(s-\tau)\, h(\tau,y(\tau))\, d\tau - \int_{-\infty}^{s} \mathscr{K}(s-\tau)\, h(\tau,z(\tau))\, d\tau \right\| \right] \\
&\leqslant L_G \int_{-\infty}^{t} \left[\|\mathscr{S}_{\alpha,\beta}(t-s)\| \int_{-\infty}^{s} |\mathscr{K}(s-\tau)|\, \|h(\tau,y(\tau)) - h(\tau,z(\tau))\|\, d\tau \right] ds \\
&\leqslant L_G L_h C_{\mathscr{K}} \int_{-\infty}^{t} \left[\|\mathscr{S}_{\alpha,\beta}(t-s)\| \int_{-\infty}^{s} e^{-b(s-\tau)}\, \|y(\tau) - z(\tau)\|\, d\tau \right] ds \\
&\leqslant \frac{L_G L_h C_{\mathscr{K}}}{b} \|y-z\|_\infty \int_{-\infty}^{t} \left[\frac{M}{1+|\omega|\left[(t-s)^{\alpha+1} + \nu(t-s)^{\beta}\right]} \right] ds
\end{aligned}$$

$$\leqslant \frac{L_G L_h C_{\mathscr{K}}}{b}\|y-z\|_\infty \int_0^{+\infty} \frac{M}{1+|\omega|\left(s^{\alpha+1}+\nu s^{\beta}\right)} ds$$
$$\leqslant \frac{L_G.L_h.C_{\mathscr{K}}.I_{\alpha,\beta}}{b}\|y-z\|_\infty .$$

So,

$$\|\Lambda_G(y)(t)-\Lambda_G(z)(t)\|_\infty \leqslant \frac{L_G.L_h.C_{\mathscr{K}}.I_{\alpha,\beta}}{b}\|y-z\|_\infty$$

with $\frac{L_G.L_h.C_{\mathscr{K}}.I_{\alpha,\beta}}{b} < 1$. This implies that Λ_G is a contration mapping from the Banach space $PAA(\mathbb{R},\mathbb{X},\mu)$ into itself. Using again Banach contration principle we obtain that Λ_G has a unique fixed point. Hence, there exists a unique $y \in PAA(\mathbb{R},\mathbb{X},\mu)$ such that

$y(t) = \int_{-\infty}^{t} \mathscr{S}_{\alpha,\beta}(t-s)\, G\left(s,\widetilde{h}(y)(s)\right) ds$. The proof is complete.

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