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THE GENERALIZED ABEL'S INTEGRAL EQUATIONS ON R^n WITH VARIABLE COEFFICIENTS

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Abstract. The convergent and stable solutions are constructed in the space of Lebesgue integrable functions for the generalized Abel's integral equations of the second kind with variable coefficients on R^n. Several applicable examples are presented, including one solving the fractional partial differential equation with the initial condition.

1. Introduction

Let $x = (x_1, x_2, \cdots, x_n)$ and I_k^α be the partial Riemann-Liouville fractional integral of order $\alpha \in R^+$ with respect to x_k, with initial point zero [10],

$$(I_k^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{x_k} (x_k - t)^{\alpha-1} u(x_1, \cdots, x_{k-1}, t, x_{k+1}, \cdots, x_n) dt$$

where $k = 1, 2, \cdots, n$.

Assuming that $\omega_i > 0$ for $i = 1, 2, \cdots, n$ and $\Omega = (0, \omega_1) \times (0, \omega_2) \times \cdots \times (0, \omega_n)$, we consider the generalized Abel's integral equation of the second kind with variable coefficients on R^n,

$$u(x_1, x_2, \cdots, x_n) - \sum_{k=1}^{m} a_k(x_1, x_2, \cdots, x_n) I_1^{\alpha_{1k}} I_2^{\alpha_{2k}} \cdots I_n^{\alpha_{nk}} u(x_1, x_2, \cdots, x_n) = g(x_1, x_2, \cdots, x_n), \quad (1)$$

where $\alpha_{ij} \geqslant 0$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m \in N$, $a_i(x)$ is Lebesgue integrable and bounded on Ω for $i = 1, 2, \cdots, m$, $g(x)$ is a given function in $L(\Omega)$ and $u(x)$ is the unknown function. Note that

$$I_1^{\alpha_{1k}} I_2^{\alpha_{2k}} \cdots I_n^{\alpha_{nk}} u(x_1, x_2, \cdots, x_n) = \frac{1}{\Gamma(\alpha_{1k}) \cdots \Gamma(\alpha_{nk})} \int_0^{x_1} \cdots \int_0^{x_n} (x_1 - t_1)^{\alpha_{1k}-1} \cdots (x_n - t_n)^{\alpha_{nk}-1} u(t_1, \cdots, t_n) dt_n \cdots dt_1,$$

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which is regarded as the partial Riemann-Liouville fractional integral with order $\alpha_{1k} + \cdots + \alpha_{nk}$ (α_{ik} th order in x_i-direction for $i = 1, 2, \cdots, n$) [13, 18]. Applying Babenko's approach [1], we establish a convergent and stable solution (note that a solution is said to be stable for equation (1) if $\forall \varepsilon > 0$ $\exists \delta > 0$, such that $\|u\| < \varepsilon$ if $\|g\| < \delta$.) for equation (1) in the space $L(\Omega)$ under certain condition on the matrix $\{\alpha_{ij}\}$. Following a similar procedure, we also construct a convergent and stable solution for the generalized Abel's integral equation

$$u(x) - \sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} u(x) = g(x) \tag{2}$$

on R^n, which cannot be reduced to a particular case of equation (1). Clearly, equation (1) turns to be

$$u(x_1) - a_1 I^{\alpha_{11}} u(x_1) = g(x_1) \tag{3}$$

if $n = m = 1$ and $a_1(x) = a_1$ (constant). Equation (3) is the classical Abel's integral equation of the second kind, with the solution given by Hille and Tamarkin [8]

$$u(x_1) = g(x_1) + a_1 \int_0^{x_1} (x_1 - t)^{\alpha_{11}-1} E_{\alpha_{11},\alpha_{11}}(a_1(x_1 - t)^{\alpha_{11}}) g(t) dt,$$

where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0$$

is the Mittag-Leffler function.

There have been many interesting studies on Abel's integral equation of the second kind, including its variants and generalizations in distribution [21, 24, 16, 15]. In 1930, Tamarkin obtained integrable solutions of Abel's integral equations under certain conditions by several integral operators [26]. Sumner [25] studied Abel's integral equations using the convolutional transform. Minerbo and Levy [20] found a numerical solution of Abel's integral equation by orthogonal polynomials. In 1985, Hatcher [7] worked on a nonlinear Hilbert problem of power type, solved in closed form by representing a sectionally holomorphic function by means of an integral with power kernel, and transformed the problem to one of solving a generalized Abel's integral equation. The multidimensional Abel-type hypergeometric integral equation over a pyramidal domain in R^n and its generalizations were studied in [9, 23]. Pskhu [22] considered the following generalized Abel's integral equation with constant coefficients a_k for $k = 1, 2, \cdots, n$

$$u(x) - \sum_{k=1}^{n} a_k I_k^{\alpha_k} u(x) = g(x),$$

where $\alpha_k > 0$ and $x \in \Omega$, and constructed an explicit solution based on the Wright function

$$\phi(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\alpha n + \beta)}, \quad \alpha > -1,$$

and convolution. Using fractional powers of linear multistep methods, Lubich [19] found the numerical solution for the following Abel's integral equation of the second kind

$$u(x) = g(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t,u(t))dt \quad \text{on } R^n$$

where $x \in [0,T]$ and $\alpha > 0$. The case $\alpha = 1/2$ is encountered in numerous problems in physics and chemistry [3]. Li et al. [16, 15] studied Abel's integral equations in the generalized sense based on fractional calculus of distributions, inverse convolutional operators and Babenko's approach [1]. Very recently, Li and Plowman [17] established a convergent and stable solution for the following generalized Abel's integral equation on R^n

$$u(x) - \left(\prod_{k=1}^n a_k(x) I_k^{\alpha_k} \right) u(x) = g(x),$$

where every partial Riemann-Liouville fractional integral $I_k^{\alpha_k}$ has its own weight function $a_k(x)$.

2. The main results

THEOREM 1. *Assume $\alpha_{ij} \geqslant 0$ for $i = 1,2,\cdots,n$ and $j = 1,2,\cdots,m$, and the matrix*

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \cdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nm} \end{pmatrix}$$

has a row whose elements are all positive. Then the generalized Abel's integral equation of the second kind with variable coefficients on R^n for a given function $g \in L(\Omega)$,

$$u(x) - \sum_{k=1}^m a_k(x) I_1^{\alpha_{1k}} I_2^{\alpha_{2k}} \cdots I_n^{\alpha_{nk}} u(x) = g(x), \quad x \in \Omega \subset R^n,$$

has the following convergent and stable solution in $L(\Omega)$

$$u(x) = \sum_{j=0}^\infty \left(\sum_{k=1}^m a_k(x) I_1^{\alpha_{1k}} I_2^{\alpha_{2k}} \cdots I_n^{\alpha_{nk}} \right)^j g(x), \tag{4}$$

where $a_k(x)$ is Lebesgue integrable and bounded on Ω for $k = 1,2,\cdots,m$.

Proof. Clearly,

$$u(x) = \left(1 - \sum_{k=1}^m a_k(x) I_1^{\alpha_{1k}} I_2^{\alpha_{2k}} \cdots I_n^{\alpha_{nk}} \right)^{-1} g(x) = \sum_{j=0}^\infty \left(\sum_{k=1}^m a_k(x) I_1^{\alpha_{1k}} I_2^{\alpha_{2k}} \cdots I_n^{\alpha_{nk}} \right)^j g(x)$$

$$= \sum_{j=0}^\infty \sum_{j_1+j_2+\cdots j_m=j} \left(\frac{j!}{j_1! j_2! \cdots j_m!} \right) \left(a_1(x) I_1^{\alpha_{11}} I_2^{\alpha_{21}} \cdots I_n^{\alpha_{n1}} \right)^{j_1} \cdots$$

$$\left(a_m(x) I_1^{\alpha_{1m}} I_2^{\alpha_{2m}} \cdots I_n^{\alpha_{nm}} \right)^{j_m} g(x).$$

Since $a_k(x)$ is bounded on Ω for $k=1,2,\cdots,m$, there exists $M>0$ such that

$$\sup_{x\in\Omega}|a_k(x)|\leqslant M.$$

Let $\|f\|$ be the usual norm of $f\in L(\Omega)$, given by

$$\|f\|=\int_\Omega |f(x)|dx=\int_\Omega |f(x_1,x_2,\cdots,x_n)|dx_1dx_2\cdots dx_n<\infty.$$

Then, we have from [2] for $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$

$$\left\|I_i^{\alpha_{ij}}g\right\|=\|\Phi_{i,\alpha_{ij}}*g\|\leqslant\|\Phi_{i,\alpha_{ij}}\|\|g\|$$

where

$$\Phi_{i,\alpha_{ij}}=\frac{(x_i)_+^{\alpha_{ij}-1}}{\Gamma(\alpha_{ij})}.$$

This implies for $\alpha_{ij}>0$ that

$$\begin{aligned}\left\|I_i^{\alpha_{ij}}\right\|\leqslant\|\Phi_{i,\alpha_{ij}}\|&=\int_\Omega\frac{(x_i)_+^{\alpha_{ij}-1}}{\Gamma(\alpha_{ij})}dx_1dx_2\cdots dx_n\\&=\omega_1\cdots\omega_{i-1}\frac{\omega_i^{\alpha_{ij}}}{\Gamma(\alpha_{ij}+1)}\omega_{i+1}\cdots\omega_n\leqslant\lambda^{n-1}\frac{\omega_i^{\alpha_{ij}}}{\Gamma(\alpha_{ij}+1)}\end{aligned}$$

where

$$\lambda=\max\{\omega_1,\omega_2,\cdots,\omega_n\}>0.$$

In particular for $\alpha_{ij}=0$,

$$\|I_i^0\|\leqslant\lambda^{n-1}.$$

Therefore,

$$\begin{aligned}&\sum_{j_1+j_2+\cdots j_m=j}\binom{j!}{j_1!j_2!\cdots j_m!}\left(a_1(x)I_1^{\alpha_{11}}I_2^{\alpha_{21}}\cdots I_n^{\alpha_{n1}}\right)^{j_1}\cdots\left(a_m(x)I_1^{\alpha_{1m}}I_2^{\alpha_{2m}}\cdots I_n^{\alpha_{nm}}\right)^{j_m}\\&=M^j\sum_{j_1+j_2+\cdots+j_m=j}\binom{j!}{j_1!j_2!\cdots j_m!}\lambda^{n-1}\frac{\omega_1^{\alpha_{11}j_1+\cdots+\alpha_{1m}j_m}}{\Gamma(\alpha_{11}j_1+\cdots+\alpha_{1m}j_m+1)}\cdots\\&\quad\lambda^{n-1}\frac{\omega_n^{\alpha_{n1}j_1+\cdots+\alpha_{nm}j_m}}{\Gamma(\alpha_{n1}j_1+\cdots+\alpha_{nm}j_m+1)}\\&\leqslant\lambda^{n^2-n}M^j\sum_{j_1+j_2+\cdots+j_m=j}\binom{j!}{j_1!j_2!\cdots j_m!}\frac{\omega_1^{\alpha_{11}j_1+\cdots+\alpha_{1m}j_m}}{\Gamma(\alpha_{11}j_1+\cdots+\alpha_{1m}j_m+1)}\cdots\\&\quad\frac{\omega_n^{\alpha_{n1}j_1+\cdots+\alpha_{nm}j_m}}{\Gamma(\alpha_{n1}j_1+\cdots+\alpha_{nm}j_m+1)}.\end{aligned}$$

Without loss of generality, we assume that all elements in the first row of the matrix are positive and set

$$\alpha=\min\{\alpha_{11},\alpha_{12},\cdots,\alpha_{1m},\}>0,$$

then

$$\Gamma(\alpha_{11} j_1 + \cdots + \alpha_{1m} j_m + 1) \geqslant \Gamma(\alpha j + 1), \quad \text{and}$$
$$\Gamma(\alpha_{i1} j_1 + \cdots + \alpha_{im} j_m + 1) \geqslant 1/2$$

for $i = 2, \cdots, n$. On the other hand, we let

$$S = \max\left\{\omega_i^{\alpha_{ij}}\right\}$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. Then,

$$\omega_1^{\alpha_{11} j_1 + \cdots + \alpha_{1m} j_m} \cdots \omega_n^{\alpha_{n1} j_1 + \cdots + \alpha_{nm} j_m} \leqslant (S^n)^j.$$

Using the identity

$$\sum_{j_1 + j_2 + \cdots + j_m = j} \binom{j!}{j_1! j_2! \cdots j_m!} = m^j,$$

we derive that

$$\|u(x)\| \leqslant \lambda^{n^2 - n} 2^{n-1} \|g\| \sum_{j=0}^{\infty} \frac{(MmS^n)^j}{\Gamma(\alpha j + 1)} < \infty$$

by the Mittag-Leffler function. Furthermore, the solution

$$u(x) = \sum_{j=0}^{\infty} \left(\sum_{k=1}^{m} a_k(x) I_1^{\alpha_{1k}} I_2^{\alpha_{2k}} \cdots I_n^{\alpha_{nk}} \right)^j g(x)$$

is stable from the above inequality. This completes the proof of Theorem 1. □

Note that we can follow a similar approach to construct a convergent and stable solution in $L(\Omega)$ for the generalized Abel's integral equation of the second kind

$$u(x) - \sum_{k=1}^{m} (a_{1k}(x) I_1^{\alpha_{1k}})(a_{2k}(x) I_2^{\alpha_{2k}}) \cdots (a_{nk}(x) I_n^{\alpha_{nk}}) u(x) = g(x), \quad x \in \Omega \subset R^n,$$

where every partial Riemann-Liouville fractional integral $I_i^{\alpha_{ik}}$ has its own weight function $a_{ik}(x)$.

REMARK 1. In particular, if the dimension $n = 1$, and $\alpha_{1k} > 0$, $\alpha_{2k} = \cdots = \alpha_{nk} = 0$ for $k = 1, 2, \cdots, m$, then equation (1) becomes

$$u(x_1) - \sum_{k=1}^{m} a_k(x_1) I_1^{\alpha_{1k}} u(x_1) = g(x_1), \tag{5}$$

where every element in the first row of the matrix in Theorem 1 is positive. Applying a modification of the Mikusinski operational calculus and the Mittag-Leffler function of several variables, Gorenflo and Luchko [6] obtained an explicit solution of the generalized Abel's integral equation of the second kind with constant coefficients λ_i

$$u(x_1) - \sum_{i=1}^{m} \lambda_i (I_1^{\alpha_i \mu} u)(x_1) = g(x_1), \ \ \alpha_i > 0, m \geqslant 1, \mu > 0, x > 0,$$

which is clearly a special case of equation (5).

Using Babenko's method, we can also derive Theorem 2 below, with the corresponding matrix

$$\begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

that does not have a row whose elements are all positive.

THEOREM 2. *Let $\alpha_k > 0$ and $a_k(x)$ be Lebesgue integrable and bounded on Ω for $k = 1,2,\cdots,n$. Then the generalized Abel's integral equation*

$$u(x) - \sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} u(x) = g(x), \quad x \in \Omega \subset R^n, \tag{6}$$

has the following convergent and stable solution

$$u(x) = \sum_{m=0}^{\infty} \left(\sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} \right)^m g(x),$$

where $g(x) \in L(\Omega)$.

Proof. Clearly,

$$u(x) - \sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} u(x) = \left(1 - \sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} \right) u(x) = g(x).$$

This implies by Babenko's method that

$$u(x) = \left(1 - \sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} \right)^{-1} g(x) = \sum_{m=0}^{\infty} \left(\sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} \right)^m g(x).$$

It remains to show that the above is convergent in $L(\Omega)$. Let

$$W = \left(\sum_{k=1}^{n} a_k(x) I_k^{\alpha_k} \right)^m = \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!} \left(a_1(x) I_1^{\alpha_1} \right)^{m_1} \cdots \left(a_n(x) I_n^{\alpha_n} \right)^{m_n}.$$

Since $a_k(x)$ is bounded on Ω for $k = 1,2,\cdots,n$, there exists $M > 0$ such that

$$\sup_{x\in\Omega} |a_k(x)| \leqslant M.$$

Hence,

$$\begin{aligned} \|W\| &\leqslant M^m \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!} \|I_1^{m_1\alpha_1}\| \cdots \|I_n^{m_n\alpha_n}\| \\ &\leqslant M^m \lambda^{n^2-n} \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!} \frac{\omega_1^{m_1\alpha_1}}{\Gamma(m_1\alpha_1+1)} \cdots \frac{\omega_n^{m_n\alpha_n}}{\Gamma(m_n\alpha_n+1)}, \end{aligned}$$

since for $i = 1, 2, \cdots, n$,

$$\left\|I_i^{m_i\alpha_i}\right\| \leqslant \lambda^{n-1}\frac{\omega_i^{m_i\alpha_i}}{\Gamma(m_i\alpha_i+1)}$$

where

$$\lambda = \max\{\omega_1, \omega, \cdots, \omega_n\} > 0$$

in Theorem 1. Let

$$A = \max\{\omega_1^{\alpha_1}, \omega_2^{\alpha_2}, \cdots, \omega_n^{\alpha_n}\} > 0.$$

Then,

$$\|W\| \leqslant \lambda^{n^2-n} M^m A^m \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!}\frac{1}{\Gamma(m_1\alpha_1+1)\cdots\Gamma(m_n\alpha_n+1)}.$$

Note that

$$\sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!} = n^m, \quad \text{and}$$
$$\Gamma(m_1\alpha_1+1)\Gamma(m_2\alpha_2+1)\cdots\Gamma(m_n\alpha_n+1) \geqslant \frac{1}{2^{n-1}}\Gamma\left(\alpha\frac{m}{n}+1\right),$$

where

$$\alpha = \min\{\alpha_1, \alpha_2, \cdots, \alpha_n\},$$

and since for any nonnegative numbers $m_1, m_2, \cdots, m_n$ satisfying $m_1+m_2+\cdots+m_n = m$, there exists an index $1 \leqslant i \leqslant n$ such that $m_i \geqslant m/n$, and

$$\Gamma(m_j\alpha_j+1) \geqslant 1/2$$

for $j = 1, \cdots, i-1, i+1, \cdots, n$. Therefore,

$$\|W\| \leqslant \lambda^{n^2-n} 2^{n-1}\frac{(MAn)^m}{\Gamma\left(\frac{\alpha}{n}m+1\right)},$$

and

$$\|u\| \leqslant \lambda^{n^2-n} 2^{n-1}\|g\| \sum_{m=0}^{\infty} \frac{(MAn)^m}{\Gamma\left(\frac{\alpha}{n}m+1\right)} < \infty$$

by the Mittag-Leffler function. Furthermore, the solution

$$u(x) = \sum_{m=0}^{\infty}\left(\sum_{k=1}^{n} a_k(x) I_k^{\alpha_k}\right)^m g(x)$$

is stable from the above inequality. This completes the proof of Theorem 2. □

3. Examples

Let α and β be arbitrary real numbers. Then it follows from [5]

$$\Phi_\alpha * \Phi_\beta = \Phi_{\alpha+\beta}$$

where

$$\Phi_\alpha(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}.$$

EXAMPLE 1. Abel's integral equation with variable coefficients

$$u(x_1,x_2) - x_1 I_1^{0.5} u(x_1,x_2) - x_2^{1.5} I_2^{1.5} u(x_1,x_2) = \frac{1}{2} x_1 x_2^2$$

has the following convergent and stable solution

$$u(x_1,x_2) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} A_k B_{m-k} \frac{x_1^{1.5k+1}\, x_2^{3(m-k)+2}}{(3(m-k)+2)!\,\Gamma(1.5k+2)},$$

where the coefficients A_k and B_{m-k} are given as

$$A_k = \begin{cases} 1 & \text{if } k = 0, \\ (2.5)\cdot 4 \cdots (1.5k+1) & \text{if } k \geqslant 1 \end{cases}$$

and

$$B_{m-k} = \begin{cases} 1 & \text{if } m = k, \\ \dfrac{\Gamma(6)\cdots\Gamma(3(m-k+1))}{\Gamma(4.5)\cdots\Gamma(3(m-k)+1.5)} & \text{if } m \geqslant k. \end{cases}$$

Proof. Indeed, we have from Theorem 1 that

$$\begin{aligned} u(x_1,x_2) &= \frac{1}{2} x_1 x_2^2 + \sum_{m=1}^{\infty} \left(x_1 I_1^{0.5} + x_2^{1.5} I_2^{1.5}\right)^m \frac{1}{2} x_1 x_2^2 \\ &= \frac{1}{2} x_1 x_2^2 + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \left(x_1 I_1^{0.5}\right)^k \left(x_2^{1.5} I_2^{1.5}\right)^{m-k} \frac{1}{2} x_1 x_2^2 \\ &= \frac{1}{2} x_1 x_2^2 + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \left(x_1 I_1^{0.5}\right)^k x_1 \left(x_2^{1.5} I_2^{1.5}\right)^{m-k} \frac{1}{2} x_2^2. \end{aligned}$$

Clearly,

$$\left(x_1 I_1^{0.5}\right) x_1 = (x_1 \Phi_1(x_1)) * x_1 = \frac{x_1^{2.5}}{\Gamma(2.5)} = \frac{\Gamma(3.5)}{\Gamma(2.5)} \frac{x_1^{2.5}}{\Gamma(3.5)},$$

$$\left(x_1 I_1^{0.5}\right) \frac{\Gamma(3.5)}{\Gamma(2.5)} \frac{x_1^{2.5}}{\Gamma(3.5)} = \frac{\Gamma(3.5)\Gamma(5)}{\Gamma(2.5)\Gamma(4)} \frac{x_1^4}{\Gamma(5)},$$

$\cdots,$

$$\left(x_1 I_1^{0.5}\right)^k x_1 = \frac{\Gamma(3.5)\Gamma(5)\cdots\Gamma(1.5k+2)}{\Gamma(2.5)\Gamma(4)\cdots\Gamma(1.5k+1)} \frac{x_1^{1.5k+1}}{\Gamma(1.5k+2)} = A_k \frac{x_1^{1.5k+1}}{\Gamma(1.5k+2)}.$$

On the other hand,

$$\left(x_2^{1.5}I_2^{1.5}\right)^{m-k}\frac{1}{2}x_2^2 = \left(x_2^{1.5}\Phi_{1.5}(x_2)\right)^{m-k} * \Phi_3(x_2)$$
$$= \frac{\Gamma(6)\cdots\Gamma(3(m-k+1))}{\Gamma(4.5)\cdots\Gamma(3(m-k)+1.5)}\frac{x_2^{3(m-k)+2}}{(3(m-k)+2)!}$$
$$= B_{m-k}\frac{x_2^{3(m-k)+2}}{(3(m-k)+2)!}. \quad \square$$

EXAMPLE 2. Let $a_k(x_k)$ be continuous and bounded on Ω for $k=1,2,\cdots,n$, and let

$$g(x_1,x_2,\cdots,x_n) = \sum_{n_1=0}^{\infty}\cdots\sum_{n_n=0}^{\infty}\frac{x_1^{n_1}\cdots x_n^{n_n}}{n_1!\cdots n_n!}\frac{\partial^{n_1+\cdots+n_n}}{\partial x_1^{n_1}\cdots\partial x_n^{n_n}}g(0,\cdots,0).$$

Then Abel's integral equation

$$u(x) - \sum_{k=1}^{n} a_k(x_k)I_k^{\alpha_k}u(x) = g(x), \quad x\in\Omega\subset R^n$$

has the following convergent and stable solution

$$u(x) = \sum_{m=0}^{\infty}\sum_{k_1+\cdots+k_n=m}\frac{m!}{k_1!\cdots k_n!}\cdot$$
$$\sum_{n_1=0}^{\infty}\left(a_1(x_1)I_1^{\alpha_1}\right)^{k_1}\frac{x_1^{n_1}}{n_1!}\cdots\sum_{n_n=0}^{\infty}\left(a_n(x_n)I_n^{\alpha_n}\right)^{k_n}\frac{x_n^{n_n}}{n_n!}\frac{\partial^{n_1+\cdots+n_n}}{\partial x_1^{n_1}\cdots\partial x_n^{n_n}}g(0,\cdots,0)$$

in $L(\Omega)$.

If $\alpha\in(0,1)$, then the partial Riemann-Liouville derivative ${}_{RL}D_{x_1}^{\alpha}u(x_1,x_2)$ is defined as [13]

$${}_{RL}D_{x_1}^{\alpha}u(x_1,x_2) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x_1}\int_0^{x_1}(x_1-t)^{-\alpha}u(t,x_2)dt.$$

EXAMPLE 3. Let $\lambda_1(x_2)$ and $\lambda_2(x_2)$ be Lebesgue integrable and bounded on $\Omega\subset R^2$. Then the integro-differential equation for a given function $g(x_1,x_2)\in L(\Omega)$

$${}_{RL}D_{x_1}^{\alpha}u(x_1,x_2) - \lambda_1(x_2)I_2^{\beta_1}u(x_1,x_2) - \lambda_2(x_2)I_2^{\beta_2}u(x_1,x_2) = g(x_1,x_2) \ \ (x_1,x_2)\in\Omega, \quad (7)$$

with the initial condition

$${}_{RL}\partial_{x_1}^{\alpha-1}(0,x_2) = 0,$$

has the following convergent and stable solution in $L(\Omega)$

$$u(x_1,x_2) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\lambda_1(x_2)I_1^{\alpha}I_2^{\beta_1}\right)^k\left(\lambda_2(x_2)I_1^{\alpha}I_2^{\beta_2}\right)^{m-k}I_1^{\alpha}g(x_1,x_2),$$

where $\beta_1,\ \beta_2>0$.

Proof. In fact, we derive that

$$I_1^{\alpha}\,{}_{RL}D_{x_1}^{\alpha} u(x_1,x_2) = u(x_1,x_2)$$

by the initial condition. Equation (7) turns to be

$$u(x_1,x_2) - I_1^{\alpha}\lambda_1(x_2)I_2^{\beta_1}u(x_1,x_2) - I_1^{\alpha}\lambda_2(x_2)I_2^{\beta_2}u(x_1,x_2) = I_1^{\alpha}g(x_1,x_2).$$

Clearly,

$$I_1^{\alpha}\lambda_1(x_2)I_2^{\beta_1}u(x_1,x_2) = \lambda_1(x_2)I_1^{\alpha}I_2^{\beta_1}u(x_1,x_2),$$
$$I_1^{\alpha}\lambda_2(x_2)I_2^{\beta_2}u(x_1,x_2) = \lambda_2(x_2)I_1^{\alpha}I_2^{\beta_2}u(x_1,x_2).$$

Furthermore, $I_1^{\alpha}g(x_1,x_2)$ is Lebesgue integrable on $\Omega \subset R^2$, which is a bounded domain. From Theorem 1, equation

$$u(x_1,x_2) - \lambda_1(x_2)I_1^{\alpha}I_2^{\beta_1}u(x_1,x_2) - \lambda_2(x_2)I_1^{\alpha}I_2^{\beta_2}u(x_1,x_2) = I_1^{\alpha}g(x_1,x_2)$$

has the following convergent and stable solution in $L(\Omega)$

$$\begin{aligned} u(x_1,x_2) &= \sum_{m=0}^{\infty}\left(\lambda_1(x_2)I_1^{\alpha}I_2^{\beta_1} + \lambda_2(x_2)I_1^{\alpha}I_2^{\beta_2}\right)^m I_1^{\alpha}g(x_1,x_2) \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\lambda_1(x_2)I_1^{\alpha}I_2^{\beta_1}\right)^k\left(\lambda_2(x_2)I_1^{\alpha}I_2^{\beta_2}\right)^{m-k} I_1^{\alpha}g(x_1,x_2). \quad \square \end{aligned}$$

REMARK 2. In seeking solutions to fractional partial differential or integral equations, integral transforms play an important role, especially to constant coefficient equations that possess time and spatial dependence, since after applying Laplace transform in the time variable and Fourier (Mellin or Hankel) transform in the spatial variable, we get an algebraic equation. After solving this algebraic equation, we find original solutions by means of the corresponding inverse transforms. However, it seems impossible to obtain solutions for equations (1) and (6) by any existing integral transform as they involve almost arbitrary variable coefficients, according to the author's knowledge. The operational calculus obtained from Mellin or Hankel transforms only works on certain types of differential or integral equations with variable coefficients [4, 11, 12, 27].

4. Conclusion

We constructed convergent and stable solutions for the generalized Abel's integral equations with variable coefficients

$$u(x) - \sum_{k=1}^{m} a_k(x)I_1^{\alpha_{1k}}I_2^{\alpha_{2k}}\cdots I_n^{\alpha_{nk}}u(x) = g(x),$$
$$u(x) - \sum_{k=1}^{n} a_k(x)I_k^{\alpha_k}u(x) = g(x)$$

in $L(\Omega)$, and presented several illustrative examples, including one solving the fractional partial differential equation with the initial condition.

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