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MULTIPLE SOLUTIONS FOR FRACTIONAL HAMILTONIAN SYSTEMS LOCALLY DEFINED NEAR THE ORIGIN

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Abstract. In this article, we are interested in the existence of infinitely many solutions for a class of fractional Hamiltonian systems

$$\begin{cases} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u)(t)+L(t)u(t)=\nabla W(t,u(t)),\ t\in\mathbb{R} \\ u\in H^{\alpha}(\mathbb{R}), \end{cases} \tag{0.1}$$

where $L(t)$ is neither uniformly positive definite nor coercive, and $W(t,x)$ is locally defined and subquadratic or superquadratic near the origin with respect to x. The proof is based on variational methods and critical point theory.

1. Introduction

In recent years, also equations including both left and right fractional derivatives are discussed. Apart from their applications in many engineering and scientific disciplines such as physics, mechanics, chemistry, biology, economics and so on, equations with left and right derivatives are an interesting and new field in fractional differential equations theory. In this topic, many results are obtained to deal with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory [3, 27], topological degree theory [4, 8] and comparison methods [13, 28].

In the previous decades, the critical point theory has attracted mathematicians and physicists as an effective tools for studying the existence and multiplicity of periodic and homoclinic solutions for differential equations with variational methods, for example, see [14, 17] and the references cited therein. Motivated by the classical works in [14, 17], for the first time, Jiao and Zhoo in [9], studied the existence of solutions for the following variational problem

$$\begin{cases} {}_tD^{\alpha}_{T}({}_{0}D^{\alpha}_t u)(t)=\nabla W(t,u(t)),\ t\in[0,T] \\ u(0)=u(T), \end{cases}$$

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via critical point theory and variational methods. Inspired by this work, Torres [19] consider the fractional Hamiltonian system

$$(\mathscr{F}\mathscr{H}\mathscr{S}) \qquad \begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t) + L(t)u(t) = \nabla W(t,u(t)),\ t \in \mathbb{R} \\ u \in H^\alpha(\mathbb{R}), \end{cases}$$

where ${}_{-\infty}D_t^\alpha$ and ${}_tD_\infty^\alpha$ are left and right Liouville-Weyl fractional derivatives of order $\frac{1}{2} < \alpha < 1$ on the whole axis respectively, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix-valued function, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$. Under some suitable conditions on L and W and using Mountain Pass Theorem, Torres [19] proved the existence of a nontrivial solution for $(\mathscr{F}\mathscr{H}\mathscr{S})$. Since then, the existence and multiplicity of solutions for problem $(\mathscr{F}\mathscr{H}\mathscr{S})$ via critical point theory, have been studied by many mathematicians, see [1, 2, 5, 6, 7, 15, 18–26]. Instead of [5, 22], in all these papers, the matrix $L(t)$ is assumed to be positive definite and the potential $W(t,x)$ is required to satisfy some kinds of growth conditions at infinity with respect to x, such as superquadratic, asymptotically quadratic or subquadratic growth condition. In [5], the author studied the existence of infinitely many solutions for $(\mathscr{F}\mathscr{H}\mathscr{S})$ when the nonlinearity $\nabla W(t,x)$ is locally defined and bounded by a constant in $\mathbb{R} \times B_\rho(0)$ for a positive constant ρ. However, in [22], Wan studied the multiplicity of solutions for system $(\mathscr{F}\mathscr{H}\mathscr{S})$ when the function L is uniformly positive definite and the potential $W(t,x)$ is superquadratic near the origin.

In this paper, we will prove the existence of infinitely many solutions for $(\mathscr{F}\mathscr{H}\mathscr{S})$ when L is neither uniformly positive definite nor coercive and the potential $W(t,x)$ is only locally defined near the origin. More precisely, Section 2 is devoted to recall some results of the Liouville-Weyl fractional calculus and formulate the variational setting. In Section 3, using a Variant Symmetric Mountain Pass Lemma due to Kajikiya [10], we are interested in the case where $W(t,x)$ is only locally defined and subquadratic near the origin with respect to x. Moreover, in Section 4, applying a Variant Fountain Theorem due to Zou [29], we are interested in the case where $W(t,x)$ is only locally defined and superquadratic near the origin with respect to x.

2. Preliminaries

In this Section, we will recall some basic definitions and results about the fractional calculus.

2.1. Liouville-Weyl fractional calculus

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as (see [11,12,16])

$$_{-\infty}I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-x)^{\alpha-1} u(x)dx, \tag{2.1}$$

and

$$_tI_\infty^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (x-t)^{\alpha-1} u(x)dx. \tag{2.2}$$

The Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [11, 12, 16])

$$ {}_{-\infty}D_t^{\alpha}u(t) = \frac{d}{dt}({}_{-\infty}I_t^{1-\alpha}u)(t), \tag{2.3} $$

and

$$ {}_tD_{\infty}^{\alpha}u(t) = -\frac{d}{dt}({}_tI_{\infty}^{1-\alpha}u)(t). \tag{2.4} $$

Denote by $L^p(\mathbb{R})$ $(1 \leqslant p < \infty)$, the Banach spaces of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norms

$$ \|u\|_{L^p} = (\int_{\mathbb{R}} |u(t)|^p\,dt)^{\frac{1}{p}}, $$

and $L^{\infty}(\mathbb{R})$ the Banach space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$ \|u\|_{L^{\infty}} = esssup\{|u(t)| : t \in \mathbb{R}\}. $$

PROPOSITION 1. 1) Let $p,q \in [1,\infty]$, $\alpha > 0$. The operators ${}_{-\infty}I_t^{\alpha}$ and ${}_tI_{\infty}^{\alpha}$ are bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if

$$ 0 < \alpha < 1,\ 1 < p < \frac{1}{\alpha},\ q = \frac{p}{1-\alpha p}, $$

2) If $\alpha > 0$, for "sufficiently good" function u, the relations

$$ ({}_{-\infty}D_t^{\alpha}({}_{-\infty}I_t^{\alpha}u))(t) = u(t),\ ({}_tD_{\infty}^{\alpha}({}_tI_{\infty}^{\alpha}u))(t) = u(t) \tag{2.5} $$

are true. In particular, these relations hold for $u \in L^1(\mathbb{R})$,

3) Let $\alpha,\beta > 0$ and $p \geqslant 1$ be such that $\alpha+\beta = \frac{1}{p}$. If $u \in L^p(\mathbb{R})$, then

$$ ({}_{-\infty}I_t^{\beta}({}_{-\infty}I_t^{\alpha}u))(t) ={}_{-\infty} I_t^{\alpha+\beta}u(t),\ ({}_tI_{\infty}^{\beta}({}_tI_{\infty}^{\alpha}u))(t) ={}_t I_{\infty}^{\alpha+\beta}u(t), \tag{2.6} $$

4) If $\alpha > \beta > 0$, then

$$ ({}_{-\infty}D_t^{\beta}({}_{-\infty}I_t^{\alpha}u))(t) ={}_{-\infty} I_t^{\alpha-\beta}u(t),\ ({}_tD_{\infty}^{\beta}({}_tI_{\infty}^{\alpha}))(t) ={}_t I_{\infty}^{\alpha-\beta}u(t). \tag{2.7} $$

PROPOSITION 2. For $\alpha > 0$, the following properties

$$ \int_{\mathbb{R}} \varphi(t)\cdot({}_{-\infty}I_t^{\alpha}\psi)(t)dt = \int_{\mathbb{R}} ({}_tI_{\infty}^{\alpha}\varphi)(t)\cdot\psi(t)dt, \tag{2.8} $$

$$ \int_{\mathbb{R}} u(t)\cdot({}_{-\infty}D_t^{\alpha}v)(t)dt = \int_{\mathbb{R}} ({}_tD_{\infty}^{\alpha}u)(t)\cdot v(t)dt, \tag{2.9} $$

are verified for "sufficiently good" functions φ,ψ,u,v. In particular, (2.8) holds for functions $\varphi \in L^p(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$, while (2.9) holds for $u \in {}_tI_{\infty}^{\alpha}(L^p(\mathbb{R}))$ and $v \in {}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R}))$ when $p > 1$, $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1+\alpha$ with

$$ {}_tI_{\infty}^{\alpha}(L^p(\mathbb{R})) = \{u : \exists\varphi \in L^p(\mathbb{R}), u = {}_tI_{\infty}^{\alpha}\varphi\}, $$

similarly, ${}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R}))$ can be defined.

2.2. Fractional derivative spaces

Let $\widehat{u}$ be the Fourier transform of u

$$\widehat{u}(s) = \int_{-\infty}^{\infty} e^{-ist} u(t) dt.$$

For $0 < \alpha < 1$, considering the semi-norm

$$|u|_{\alpha} = \left\| |s|^{\alpha} \widehat{u} \right\|_{L^2}$$

and the norm

$$\|u\|_{\alpha} = (\|u\|_{L^2}^2 + |u|_{\alpha}^2)^{\frac{1}{2}},$$

we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ as follows

$$H^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{\alpha}},$$

where $C_0^{\infty}(\mathbb{R})$ denotes the space of continuous functions u from $\mathbb{R}$ into $\mathbb{R}^N$ such that $u(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$.

Moreover, let $C(\mathbb{R})$ denotes the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. Then we obtain the following Sobolev lemma.

LEMMA 1. ([21], Theorem 2.1) *If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$, and there exists a positive constant $C = C_{\alpha}$ such that*

$$\|u\|_{L^{\infty}} = \sup_{t \in \mathbb{R}} |u(t)| \leqslant C_{\alpha} \|u\|_{\alpha}, \forall u \in H^{\alpha}(\mathbb{R}). \tag{2.10}$$

REMARK 1. From Lemma 1, we know that if $u \in H^{\alpha}(\mathbb{R})$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^p(\mathbb{R})$ for all $p \in [2, \infty]$, because

$$\int_{\mathbb{R}} |u(t)|^p \, dt \leqslant \|u\|_{L^{\infty}}^{p-2} \|u\|_{L^2}^2 . \tag{2.11}$$

Let χ be the selfadjoint extension of the operator ${}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha}) + L$ with the domain $\mathscr{D}(\chi) \subset L^2(\mathbb{R})$. Let $\{E(\nu) : -\infty < \nu < +\infty\}$ and $|\chi|$ be the spectral resolution and the absolute value of χ respectively, and $|\chi|^{\frac{1}{2}}$ the square root of $|\chi|$. Set $U = I - E(0) - E(-0)$. Then U commutes with χ, $|\chi|$ and $|\chi|^{\frac{1}{2}}$, and $\chi = U|\chi|$ is the polar decomposition of χ. Let $X^{\alpha} = \mathscr{D}(|\chi|^{\frac{1}{2}})$, the domain of $|\chi|^{\frac{1}{2}}$, and denote on X^{α} the inner product

$$\langle u, v \rangle_{X^{\alpha}} = \langle |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v \rangle_{L^2} + \langle u, v \rangle_{L^2}$$

and norm $\|u\|_{X^{\alpha}} = \langle u, u \rangle_{X^{\alpha}}^{\frac{1}{2}}$, where as usual $\langle ., . \rangle_{L^2}$ denotes the inner product of $L^2(\mathbb{R})$. Then X^{α} is a Hilbert space. It is easy to check that $C_0^{\infty}(\mathbb{R})$ is dense in X^{α} and X^{α} is continuously embedded in $H^{\alpha}(\mathbb{R})$ since L is bounded from below and hence is χ in $L^2(\mathbb{R})$.

Consider the following assumptions

(L_1) the smallest eigenvalue of $L(t)$ is bounded from below;
(L_σ) There exists a constant $\sigma > 1$ such that

$$meas(\{t \in \mathbb{R} / |t|^{-\sigma} L(t) < bI_N\}) < \infty, \ \forall b > 0,$$

where *meas* denotes the Lebesgue's measure on $\mathbb{R}$. Here, for two $N \times N$ symmetric matrices M_1 and M_2, we say that $M_1 < M_2$ if

$$\min_{x \in \mathbb{R}^N, |x|=1} (M_1 - M_2)x \cdot x < 0$$

and $M_1 \geqslant M_2$ if $M_1 < M_2$ does not hold.

REMARK 2. Let $L(t) = (t^2 \sin^2 t - 1)I_N$. It is easy to see that L satisfies (L_1) and (L_σ) but it is neither uniformly positive definite nor coercive.

In the next, the following compactness embedding lemma will be needed.

LEMMA 2. *Suppose that L satisfies (L_1) and (L_σ). Then X^α is compactly embedded in $L^p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$. In particular, for all $p \in [1,\infty]$, there is a constant $\tau_p > 0$ such that*

$$\|u\|_{L^p} \leqslant \tau_p \|u\|_{X^\alpha}, \ \forall u \in X^\alpha. \tag{2.12}$$

Proof. a) First, we consider the case that $l(t) \geqslant 1$ for all $t \in \mathbb{R}$. Evidently, we have $|\chi| = \chi$ and

$$\|u\|_{X^\alpha}^2 = \int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t) \cdot u(t) + |u(t)|^2] dt.$$

By Lemma 1, we see that

$$\|u\|_{L^\infty} \leqslant C_\alpha \|u\|_{X^\alpha}, \ \forall u \in X^\alpha$$

and hence for all $p \in [2,\infty[$ and $u \in X^\alpha$, we have

$$\int_{\mathbb{R}} |u(t)|^p \, dt \leqslant \|u\|_{L^\infty}^{p-2} \int_{\mathbb{R}} |u(t)|^2 \, dt \leqslant C_\alpha^{p-2} \|u\|_{X^\alpha}^p.$$

Now, for any $\varepsilon > 0$, by condition (L_σ), we choose $r_\varepsilon \geqslant 1$ such that $meas(B_\varepsilon) \leqslant \varepsilon$, where

$$B_\varepsilon = \left\{ t \in \mathbb{R} \backslash]-r_\varepsilon, r_\varepsilon[/ |t|^{-\sigma} L(t) < \frac{1}{\varepsilon} I_N \right\}.$$

Let

$$D_\varepsilon = \mathbb{R} \setminus (B_\varepsilon \cup]-r_\varepsilon, r_\varepsilon[)$$

and

$$\mu_\varepsilon = \inf_{|\xi|=1, t \in D_\varepsilon} |t|^{-\sigma} L(t)\xi \cdot \xi.$$

Then $\frac{1}{\mu_\varepsilon} \leqslant \varepsilon$.

Let $(u_k) \subset X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in X^α weakly. The Banach-Steinhaus Theorem implies that

$$M = \sup_{k \in \mathbb{R}} \|u_k - u\|_{X^\alpha} < \infty.$$

Since $X^\alpha \subset H^\alpha(\mathbb{R}) \subset L^p(\mathbb{R})$ for $p \in [2, \infty]$ with continuous embedding, it holds

$$M_p = \sup_{k \in \mathbb{R}} \|u_k - u\|_{L^p} < \infty.$$

Sobolev's embedding Theorem implies that $u_k \longrightarrow u$ uniformly in $\bar{I}_\varepsilon = [-r_\varepsilon, r_\varepsilon]$.

Step 1. We claim that X^α is compactly embedded in $L^2(\mathbb{R})$. In fact, we have

$$\begin{aligned}
\int_{|t| \geqslant r_\varepsilon} |u_k - u|^2 \, dt &= \int_{B_\varepsilon} |u_k - u|^2 \, dt + \int_{D_\varepsilon} |u_k - u|^2 \, dt \\
&\leqslant meas(B_\varepsilon) \|u_k - u\|_{L^\infty}^2 + \int_{D_\varepsilon} |t|^\sigma |u_k - u|^2 \, dt \\
&\leqslant meas(B_\varepsilon) \|u_k - u\|_{L^\infty}^2 + \frac{1}{\mu_\varepsilon} \int_{D_\varepsilon} L(t)(u_k - u) \cdot (u_k - u) dt \\
&\leqslant \varepsilon M_\infty^2 + \varepsilon \|u_k - u\|_{X^\alpha}^2 \\
&\leqslant \varepsilon (M_\infty^2 + M^2).
\end{aligned}$$

Since $u_k \longrightarrow u$ uniformly in $\bar{I}_\varepsilon$, we get $u_k \longrightarrow u$ in $L^2(\mathbb{R})$ as $k \longrightarrow \infty$.

Step 2. $p \in]2, \infty[$. We claim that X^α is compactly embedded in $L^p(\mathbb{R})$. In fact, we have

$$\begin{aligned}
\|u_k - u\|_{L^p}^p = \int_{\mathbb{R}} |u_k - u|^p \, dt &\leqslant \|u_k - u\|_{L^\infty}^{p-2} \int_{\mathbb{R}} |u_k - u|^2 \, dt \\
&\leqslant M_\infty^{p-2} \|u_k - u\|_{L^2}^2 .
\end{aligned}$$

By Step 1, we deduce that $u_k \longrightarrow u$ in $L^p(\mathbb{R})$.

Step 3. $p \in [1, 2[$. We claim that $u_k \longrightarrow u$ in $L^p(\mathbb{R})$. Let $s = \frac{\sigma}{2-p}$. Then $p > \frac{2}{1+\sigma}$ and $sp > 1$. For $v \in L^p(\mathbb{R})$, we have

$$\begin{aligned}
\int_{|t| \geqslant r_\varepsilon} |v|^p \, dt &= \int_{B_\varepsilon} |v|^p \, dt + \int_{D_\varepsilon} |v|^p \, dt \\
&= \int_{B_\varepsilon} |v|^p \, dt + \int_{\{t \in D_\varepsilon / |t|^s |v(t)| \leqslant 1\}} |v|^p \, dt + \int_{\{t \in D_\varepsilon / |t|^s |v(t)| \geqslant 1\}} |v|^p \, dt \\
&\leqslant (meas(B_\varepsilon))^{\frac{1}{2}} \|v\|_{L^{2p}}^p + \int_{D_\varepsilon} |t|^{-sp} \, dt + \int_{\{t \in D_\varepsilon / |t|^s |v(t)| \geqslant 1\}} (|t|^s |v|)^p |t|^{-sp} \, dt \\
&\leqslant (meas(B_\varepsilon))^{\frac{1}{2}} \|v\|_{L^{2p}}^p + \int_{|t| \geqslant r_\varepsilon} |t|^{-\sigma} \, dt + \int_{D_\varepsilon} (|t|^s |v|)^2 |t|^{-sp} \, dt \\
&\leqslant \sqrt{\varepsilon} \|v\|_{L^{2p}}^p + 2 \int_{r_\varepsilon}^\infty |t|^{-\sigma} \, dt + \int_{|t| \geqslant r_\varepsilon} |t|^{(2-p)s} |v|^2 \, dt \\
&\leqslant \sqrt{\varepsilon} \|v\|_{L^{2p}}^p + \frac{2 r_\varepsilon^{1-\sigma}}{\sigma - 1} + \int_{|t| \geqslant r_\varepsilon} |t|^\sigma |v|^2 \, dt
\end{aligned}$$

$$\leqslant \sqrt{\varepsilon}\,\|v\|_{L^{2p}}^{p} + \frac{2r_{\varepsilon}^{1-\sigma}}{\sigma-1} + \frac{1}{\mu_{\varepsilon}}\int_{|t|\geqslant r_{\varepsilon}} L(t)v\cdot v dt$$
$$\leqslant \sqrt{\varepsilon}\,\|v\|_{L^{2p}}^{p} + \frac{2r_{\varepsilon}^{1-\sigma}}{\sigma-1} + \varepsilon\,\|v\|_{X^{\alpha}}^{2}.$$

Choose r_{ε} large enough such that $r_{\varepsilon}^{1-\sigma} \leqslant \sqrt{\varepsilon}$, we obtain

$$\int_{|t|\geqslant r_{\varepsilon}} |v|^{p} \leqslant \sqrt{\varepsilon}\Big(\|v\|_{L^{2p}}^{p}\,dt + \frac{2}{\sigma-1} + \sqrt{\varepsilon}\,\|v\|_{X^{\alpha}}^{2}\Big).$$

Since $2p \geqslant 2$, we have $\|u_k - u\|_{L^{2p}} \leqslant M_{2p}$ for all $k \in \mathbb{N}$ and

$$\int_{|t|\geqslant r_{\varepsilon}} |u_k - u|^{p}\,dt \leqslant \sqrt{\varepsilon}\Big(M_{2p}^{p} + \frac{2}{\sigma-1} + \sqrt{\varepsilon}M^{2}\Big),\ \forall k \in \mathbb{N}.$$

As above, we have $\int_{\overline{I}_{\varepsilon}} |u_k - u|^{p}\,dt \longrightarrow 0$ as $k \longrightarrow \infty$. Hence $u_k \longrightarrow u$ in $L^{p}(\mathbb{R})$.

b) Now, we consider the general case which does not need the condition $l(t) \geqslant 1$ for all $t \in \mathbb{R}$. It follows from (L_1) that there is a positive constant a such that $l(t)+a \geqslant 1$ for all $t \in \mathbb{R}$. Since $X^{\alpha} = \mathscr{D}((\chi + aI)^{\frac{1}{2}})$, we can introduce a norm on it

$$\|u\|_{a}^{2} = \left\|(\chi + aI)^{\frac{1}{2}}u\right\|_{L^{2}}^{2} + \|u\|_{L^{2}}^{2}.$$

From the first case a) it follows that $(X^{\alpha}, \|.\|_{a})$ is compactly embedded in $L^{p}(\mathbb{R})$ for $p \in [1,\infty]$, so it suffices to show that $\|.\|_{X^{\alpha}}$ is equivalent to $\|.\|_{a}$. In fact, for $u \in \mathscr{D}(\chi) = \mathscr{D}(\chi + aI)$, we have

$$\begin{aligned}\left\|(\chi + aI)^{\frac{1}{2}}u\right\|_{L^{2}}^{2} &= \langle \chi u, u\rangle_{L^{2}} + a\,\|u\|_{L^{2}}^{2}\\ &= \langle |\chi| U u, u\rangle_{L^{2}} + a\,\|u\|_{L^{2}}\\ &= \langle U\,|\chi|^{\frac{1}{2}}\,u, |\chi|^{\frac{1}{2}}\,u\rangle_{L^{2}} + a\,\|u\|_{L^{2}}^{2}\\ &\leqslant \left\||\chi|^{\frac{1}{2}}\,u\right\|_{L^{2}}^{2} + a\,\|u\|_{L^{2}}^{2}\\ &\leqslant \sup(1,a)\,\|u\|_{X^{\alpha}}^{2}.\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\left\||\chi|^{\frac{1}{2}}\,u\right\|_{L^{2}}^{2} = \langle |\chi|\,u, u\rangle_{L^{2}} &= \langle (\chi + aI)Uu, u\rangle_{L^{2}} - a\langle Uu, u\rangle_{L^{2}}\\ &\leqslant \left\|(\chi + aI)^{\frac{1}{2}}u\right\|_{L^{2}}^{2} + a\,\|u\|_{L^{2}}^{2}\\ &\leqslant \sup(1,a)\,\|u\|_{a}^{2}.\end{aligned} \tag{2.14}$$

Since $\mathscr{D}(\chi)$ is dense in X^{α}, it follows from (2.13) and (2.14) that $\|.\|_{X^{\alpha}}$ and $\|.\|_{a}$ are equivalent. The proof of Lemma 2 is completed.

Since the selfadjoint operator χ is bounded from below in $L^{2}(\mathbb{R})$, then Lemma 2 implies that it has a compact resolvent. Consequently, the spectrum $\sigma(\chi)$ consists of eigenvalues numbered in $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \longrightarrow +\infty$ (counted in their multiplicities), and a corresponding system of eigenfunctions $(e_j)_{j\in\mathbb{N}}$, $(\chi e_j = \lambda_j e_j)$, forms an orthonormal basis in $L^2(\mathbb{R})$. Let k^- (resp. k^0) be the number of $\lambda_j < 0$ (resp. $\lambda_j = 0$), $\bar{k} = k^- + k^0$ and let $X^- = span\{e_1, , \dots, e_{k^-}\}$, $X^- = span\left\{e_{k^-+1}, , \dots, e_{\bar{k}}\right\}$ and $X^+ = Cl_{X^\alpha} span\left\{e_{\bar{k}+1}, , \dots\right\}$, where $Cl_{X^\alpha} S$ is the closure of the set S in X^α. Then $X^\alpha = X^- \oplus X^0 \oplus X^+$. We introduce on X^α the following inner product

$$\langle u, v\rangle = \langle |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v\rangle_{L^2} + \langle u^0, v^0\rangle_{L^2}$$

and the associated norm

$$\|u\|^2 = \left\| |\chi|^{\frac{1}{2}} u \right\|_{L^2}^2 + \left\|u^0\right\|_{L^2}^2,$$

where $u = u^- + u^0 + u^+, v = v^- + v^0 + v^+ \in X^- \oplus X^0 \oplus X^+$. Clearly, $\|u\|_{L^2}^2 \leqslant \lambda \|u\|^2$ for all $u \in X^\alpha$, where $\lambda = \max\left\{1, \lambda_{\bar{k}+1}^{-1}, |\lambda_{k^-}|^{-1}\right\}$. Since $\|u\|_{X^\alpha}^2 = \left\|u^- + u^0\right\|_{L^2}^2 + \|u\|^2$ for all $u \in X^\alpha$, one has $\|u\|^2 \leqslant \|u\|_{X^\alpha}^2 \leqslant (1+\lambda)\|u\|^2$, i.e. the norms $\|.\|_{X^\alpha}$ and $\|.\|$ are equivalent. From now on the norm $\|.\|$ on X^α will be used. By Lemma 2, for all $p \in [1, \infty]$, there is a positive constant η_p such that

$$\|u\|_{L^p} \leqslant \eta_p \|u\|, \ \forall u \in X^\alpha. \tag{2.15}$$

For later use, let

$$a(u, v) = \langle |\chi|^{\frac{1}{2}} Uu, |\chi|^{\frac{1}{2}} v\rangle_{L^2}, \ \forall u, v \in X^\alpha$$

be the bilinear form associated with χ. For all $u \in \mathscr{D}(\chi)$, $v \in X^\alpha$, one has

$$a(u, v) = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t) . {}_{-\infty}D_t^\alpha v(t) + L(t)u(t) \cdot v(t))dt \tag{2.16}$$

and since $\mathscr{D}(\chi)$ is dense in X^α, (2.16) is verified for all $u \in X^\alpha$. Moreover, we have

$$a(u, u) = \left\|u^+\right\|^2 - \left\|u^-\right\|^2 \tag{2.17}$$

for all $u = u^- + u^0 + u^+ \in X^\alpha = X^- \oplus X^0 \oplus X^+$.

3. Local subquadratic conditions

In this Section, we are interested in the existence of infinitely many solutions of $(\mathscr{FHS})$ when the potential $W(t, x)$ is only locally defined and subquadratic in x. More precisely, let $W : \mathbb{R} \times B_\delta(0) \longrightarrow \mathbb{R}$ be a continuous function, differentiable with respect to the second variable with continuous derivative $\nabla W(t, x) = \frac{\partial W}{\partial x}(t, x)$, where δ is a positive constant and $B_\delta(0)$ is the open ball in $\mathbb{R}^N$ centered at zero with radius δ, we make the following conditions
(W_1) $W(t, 0) = 0$ and there is constants $\nu \in]0, 1[$, $\gamma \in [\frac{1}{\nu}, \infty[$, $b \geqslant 0$ and $a \in L^\gamma(\mathbb{R}, \mathbb{R}^+)$ such that

$$|\nabla W(t, x)| \leqslant a(t) + b|x|^\nu, \ \forall (t, x) \in \mathbb{R} \times B_\delta(0);$$

(W_2)
$$\lim_{|x|\to 0} \frac{|W(t,x)|}{|x|^2} = +\infty, \text{ uniformly for all } t \in \mathbb{R};$$

(W_3)
$$W(t,-x) = W(t,x),\ \forall (t,x) \in \mathbb{R} \times B_\delta(0).$$

Our main result in this Section reads as follows.

THEOREM 1. *If* (L_1), (L_σ) *and* (W_1)–(W_3) *hold, then* $(\mathscr{F}\mathscr{H}\mathscr{S})$ *admits infinitely many nontrivial solutions* (u_k) *satisfying* $\max_{t\in\mathbb{R}} |u_k(t)| \longrightarrow 0$ *as* $k \longrightarrow \infty$.

REMARK 3. Theorem 1 generalizes Theorem 1 in [5].

REMARK 4. Let $0 < \nu < 1$ and $b \geqslant 0$ be some given constants, define

$$W(t,x) = \Big(\frac{1}{t^2+1}\Big)^{\frac{\nu}{2}} \ln(1+|x|^2) + \frac{b}{\nu+1}|x|^{\nu+1},\ t \in \mathbb{R},\ |x| < 1.$$

It is easy to see that W satisfies (W_2) and (W_3) with $\delta = 1$. An easy computation shows that

$$|\nabla W(t,x)| \leqslant a(t) + b|x|^\nu,\ \forall t \in \mathbb{R},\ |x| < 1$$

where $a(t) = (\frac{1}{t^2+1})^{\frac{\nu(\omega+2)}{4}}$. Moreover, for $\gamma = \frac{2}{\nu} > \frac{1}{\nu}$, we have

$$\int_{\mathbb{R}} (a(t))^\gamma = \int_{\mathbb{R}} \frac{1}{t^2+1} < \infty.$$

Hence assumption (W_1) is satisfied. Therefore by Theorem 1, the corresponding system $(\mathscr{F}\mathscr{H}\mathscr{S})$ possesses infinitely many solutions.

Proof of Theorem 1. Condition (L_1) implies the existence of a constant $b_0 > 0$ such that $L(t) + 2b_0 I_N \geqslant I_N$ for all $t \in \mathbb{R}$. Let $\overline{L}(t) = L(t) + 2b_0 I_N$ and $\overline{W}(t,x) = W(t,x) + b_0|x|^2$. Set

$(\overline{\mathscr{F}\mathscr{H}\mathscr{S}})$
$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t) + \overline{L}(t)u(t) = \nabla\overline{W}(t,u(t)),\ t \in \mathbb{R} \\ u \in H^\alpha(\mathbb{R}), \end{cases}$$

then $(\overline{\mathscr{F}\mathscr{H}\mathscr{S}})$ is equivalent to $(\mathscr{F}\mathscr{H}\mathscr{S})$. Moreover, it is clear that $\overline{W}$ satisfies (W_1)–(W_3) with b in (W_1) is replaced by $\overline{b} = b + 2b_0\delta^{1-\nu}$, as soon as W checks them, and $\overline{L}$ satisfies the conditions (L_σ) and

(L_0)
$$l(t) = \inf_{|x|=1} \overline{L}(t)x\cdot x \geqslant 1.$$

Hence, in this Section, we will always assume without loss of generality that L satisfies (L_0) and (L_σ).

Let $r \in]0, \frac{\delta}{2}[$ be a constant and choose a cut-off function $\rho \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $\rho(s) = 1$ for $0 \leqslant s \leqslant r$, $\rho(s) = 0$ for $s \geqslant 2r$ and $-\frac{2}{r} \leqslant \rho'(s) < 0$ for $r < s < 2r$. Let

$$\widetilde{W}(t,x) = \rho(|x|)W(t,x),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}}. \tag{3.1}$$

Combining (W_1) and the definition of ρ yields

$$\left|\widetilde{W}(t,x)\right| \leqslant a(t)\,|x| + b\,|x|^{\nu+1},\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{3.2}$$

and

$$\left|\nabla\widetilde{W}(t,x)\right| \leqslant 5(a(t) + b\,|x|^{\nu}),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{3.3}$$

Consider the following modified fractional Hamiltonian system

$$(\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}}) \qquad \begin{cases} {}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha}u)(t) + L(t)u(t) = \nabla\widetilde{W}(t,u(t)),\ t \in \mathbb{R} \\ u \in H^{\alpha}(\mathbb{R}), \end{cases}$$

and the associateed variational functional

$$\begin{aligned} f(u) &= \frac{1}{2}\int_{\mathbb{R}} (|{}_{-\infty}D_t^{\alpha}u(t)|^2 + L(t)u(t)\cdot u(t))dt - \int_{\mathbb{R}} \widetilde{W}(t,u)dt \\ &= \frac{1}{2}\|u\|^2 - g(u) \end{aligned} \tag{3.4}$$

for all $u \in X^{\alpha}$, where

$$g(u) = \int_{\mathbb{R}} \widetilde{W}(t,u)dt,\ u \in X^{\alpha}.$$

To prove Theorem 1, the following Variant Symmetric Mountain Pass Lemma will be needed.

Let X be a Banach space and let A be a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k for which there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If such a k does not exist, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\phi) = 0$. Let

$$\Gamma_k = \{A \subset X/A \textit{ is a closed symmetric subset},\ 0 \notin A,\ \gamma(A) \geqslant k\}.$$

LEMMA 3. [10] *Let A and B be closed symmetric subsets of X that do not contain the origin. Then the following hold.*

a) If $A \subset B$, then $\gamma(A) \leqslant \gamma(B)$.

b) The N-dimensional sphere S^N has a genus of $N+1$ by the Borsuk-Ulam theorem.

LEMMA 4. [10] *Let X be an infinite-dimensional Banach space and $f \in C^1(X,\mathbb{R})$ satisfies the following*

(f_1) $f(0) = 0$, f is even and bounded from below and f satisfies the (PS)-condition;

(f_2) *For each* $k \in \mathbb{N}$*, there exists* $A_k \subset \Gamma_k$ *such that*

$$\sup_{u \in A_k} f(u) < 0.$$

Then f *possesses a sequence of critical points* (u_k) *such that*

$$f(u_k) \leqslant 0,\ u_k \neq 0,\ \forall k \in \mathbb{N} \ \text{and} \ \lim_{k \longrightarrow \infty} u_k = 0.$$

LEMMA 5. *Suppose that* (L_0), (L_σ) *and* (W_1) *hold. If* $u_n \rightharpoonup u$ *in* X^α*, then* $\nabla \widetilde{W}(t,u_n) \longrightarrow \nabla \widetilde{W}(t,u)$ *in* $L^\gamma(\mathbb{R})$.

Proof. By negation, Lemma 2 implies that there is a subsequence (u_{n_k}) such that

$$u_{n_k} \longrightarrow u \ \text{in} \ L^{\nu\gamma}(\mathbb{R}) \ \text{and} \ u_{n_k} \longrightarrow u \ a.e. \ \text{in} \ \mathbb{R} \ \text{as} \ k \longrightarrow \infty \tag{3.5}$$

and

$$\int_{\mathbb{R}} \left| \nabla \widetilde{W}(t,u_{n_k}) - \nabla \widetilde{W}(t,u) \right|^\gamma dt \geqslant \varepsilon_0,\ \forall k \in \mathbb{N} \tag{3.6}$$

for some positive constant ε_0. By (3.5) and up to a subsequence if necessary, we can assume that $\sum_{k=1}^{\infty} \left\| u_{n_k} - u \right\|_{L^{\nu\gamma}} < \infty$. Let $w(t) = \sum_{k=1}^{\infty} \left| u_{n_k}(t) - u(t) \right|$ for all $t \in \mathbb{R}$, then $w \in L^{\nu\gamma}(\mathbb{R})$. By (W_1), there holds for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$

$$\begin{aligned}
&\left| \nabla \widetilde{W}(t,u_{n_k}(t)) - \nabla \widetilde{W}(t,u(t)) \right|^\gamma \\
&\leqslant \left(\left| \nabla \widetilde{W}(t,u_{n_k}(t)) \right| + \left| \nabla \widetilde{W}(t,u(t)) \right| \right)^\gamma \\
&\leqslant 2^{\gamma-1} \left(\left| \nabla W(t,u_{n_k}(t)) \right|^\gamma + |\nabla W(t,u(t))|^\gamma \right) \\
&\leqslant 2^{\gamma-1} \left[(a(t) + b\left|u_{n_k}(t)\right|^\nu)^\gamma + (a(t) + b|u(t)|^\nu)^\gamma \right] \\
&\leqslant 2^{2(\gamma-1)} \left[2(a(t))^\gamma + b^\gamma \left|u_{n_k}(t)\right|^{\nu\gamma} + b^\gamma |u(t)|^{\nu\gamma} \right] \\
&\leqslant 2^{2(\gamma-1)} \left[2(a(t))^\gamma + b^\gamma (\left|u_{n_k}(t) - u(t)\right| + |u(t)|)^{\nu\gamma} + b^\gamma |u(t)|^{\nu\gamma} \right] \\
&\leqslant 2^{2(\gamma-1)} \left[2(a(t))^\gamma + b^\gamma 2^{\nu\gamma-1} (\left|u_{n_k}(t) - u(t)\right|^{\nu\gamma} + |u(t)|^{\nu\gamma}) + b^\gamma |u(t)|^{\nu\gamma} \right] \\
&\leqslant c_1 \left[(a(t))^\gamma + |w(t)|^{\nu\gamma} + |u(t)|^{\nu\gamma} \right] \in L^1(\mathbb{R})
\end{aligned}$$

where c_1 is a positive constant. Combining this with (3.5), the Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k \longrightarrow \infty} \int_{\mathbb{R}} \left| \nabla \widetilde{W}(t,u_{n_k}(t)) - \nabla \widetilde{W}(t,u(t)) \right|^\gamma dt = 0$$

which contradicts to (3.6). Hence $\nabla \widetilde{W}(t,u_{n_k}) \longrightarrow \nabla \widetilde{W}(t,u)$ in $L^\gamma(\mathbb{R})$ and the proof of Lemma 5 is completed.

LEMMA 6. *Let* (L_0), (L_σ) *and* (W_1) *hold. Then* $g \in C^1(X^\alpha,\mathbb{R})$ *and* $g' : X^\alpha \longrightarrow (X^\alpha)^*$ *is compact, and hence* $f \in C^1(X^\alpha,\mathbb{R})$*. Moreover*

$$g'(u)v = \int_{\mathbb{R}} \nabla \widetilde{W}(t,u) \cdot v dt \tag{3.7}$$

and

$$f'(u)v = \int_{\mathbb{R}} ({}_{-\infty}D_t^{\alpha}u(t) \cdot {}_{-\infty}D_t^{\alpha}v(t) + L(t)u(t) \cdot v(t))dt - \int_{\mathbb{R}} \nabla\widetilde{W}(t,u) \cdot vdt$$
$$= \langle u, v \rangle - \int_{\mathbb{R}} \nabla\widetilde{W}(t,u) \cdot vdt \tag{3.8}$$

for $u, v \in X^{\alpha}$ and nontrivial critical points of f on X^{α} are solutions of $(\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}})$.

Proof. For any $u \in X^{\alpha}$, Set $K(u) : X^{\alpha} \longrightarrow \mathbb{R}$ the linear operator defined by

$$\langle K(u), v \rangle = \int_{\mathbb{R}} \nabla\widetilde{W}(t,u) \cdot vdt, \ v \in X^{\alpha}. \tag{3.9}$$

By (2.15), (3.3) and Hölder's inequality, one has

$$\begin{aligned}
|\langle K(u), v \rangle| &\leqslant \int_{\mathbb{R}} \left|\nabla\widetilde{W}(t,u)\right| |v|\, dt \\
&\leqslant 5 \int_{\mathbb{R}} [a(t) + b\,|u|^{\nu}]\, |v|\, dt \\
&\leqslant 5 \int_{\mathbb{R}} a(t)\, |v|\, dt + 5b \int_{\mathbb{R}} |u|^{\nu}\, |v|\, dt \\
&\leqslant 5 (\int_{\mathbb{R}} (a(t))^{\gamma} dt)^{\frac{1}{\gamma}} (\int_{\mathbb{R}} |v(t)|^{\frac{\gamma}{\gamma-1}}\, dt)^{\frac{\gamma-1}{\gamma}} \\
&\quad + 5b (\int_{\mathbb{R}} |u(t)|^{\nu+1}\, dt)^{\frac{\nu}{\nu+1}} (\int_{\mathbb{R}} |v(t)|^{\nu+1}\, dt)^{\frac{1}{\nu+1}} \\
&\leqslant 5\, \|a\|_{L^{\gamma}}\, \|v\|_{L^{\frac{\gamma}{\gamma-1}}} + 5b\, \|u\|_{L^{\nu+1}}^{\nu}\, \|v\|_{L^{\nu+1}} \\
&\leqslant 5[\eta_{\frac{\gamma}{\gamma-1}}\, \|a\|_{L^{\gamma}} + b\eta_{\nu+1}^{\nu+1}\, \|u\|^{\nu}]\, \|v\|, \ \forall v \in X^{\alpha}.
\end{aligned}$$

Hence $K(u)$ is bounded.

By (3.3), for any $s \in [0,1]$, $t \in \mathbb{R}$ and $u, v \in X^{\alpha}$, there holds

$$\begin{aligned}
\left|\nabla\widetilde{W}(t, u+sv)v\right| &\leqslant 5[a(t)\, |v| + b\, |u+sv|^{\nu}\, |v|] \\
&\leqslant 5[a(t)\, |v| + b(|u|^{\nu}\, |v| + |v|^{\nu+1})]
\end{aligned}$$

which is integrable in $\mathbb{R}$. Consequently, for all $u, v \in X^{\alpha}$, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, there holds

$$\begin{aligned}
\lim_{s \longrightarrow 0} \frac{g(u+sv) - g(u)}{s} &= \lim_{s \longrightarrow 0} \int_{\mathbb{R}} \int_0^1 \nabla\widetilde{W}(t, u+\theta sv) \cdot vd\theta dt \\
&= \int_{\mathbb{R}} \nabla\widetilde{W}(t,u) \cdot vdt = \langle K(u), v \rangle.
\end{aligned} \tag{3.10}$$

This implies that g is Gâteaux differentiable on X^{α} and $K(u)$ is its Gâteaux derivative at u.

Next, we prove that K is compact. Suppose $u_n \rightharpoonup u$ in X^α, then by Lemma 3, $\nabla\widetilde{W}(t,u_n) \longrightarrow \nabla\widetilde{W}(t,u)$ in $L^\gamma(\mathbb{R})$. By Hölder's inequality and (2.15), it holds

$$\begin{aligned}\|K(u_n)-K(u)\|_{(X^\alpha)^*} &= \sup_{\|v\|=1}\int_{\mathbb{R}}(\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u))\cdot v dt\\ &\leqslant \sup_{\|v\|=1}\Big(\int_{\mathbb{R}}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right|^\gamma dt\Big)^{\frac{1}{\gamma}}\Big(\int_{\mathbb{R}}|v|^{\frac{\gamma}{\gamma-1}}dt\Big)^{\frac{\gamma-1}{\gamma}}\\ &\leqslant \eta_{\frac{\gamma}{\gamma-1}}\Big(\int_{\mathbb{R}}\left|\nabla\widetilde{W}(t,u_n)-\nabla\widetilde{W}(t,u)\right|^\gamma dt\Big)^{\frac{1}{\gamma}} \longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned}$$

This means that K is compact and weakly continuous and then continuous. Thus $g \in C^1(X^\alpha,\mathbb{R})$ and (3.7) holds with $g' = K$ is compact. In addition, due to the form of f, it is clear to see that $f \in C^1(X^\alpha,\mathbb{R})$ and

$$f'(u)v = \int_{\mathbb{R}}[{}_{-\infty}D_t^\alpha u(t)\cdot{}_{-\infty}D_t^\alpha v(t)+L(t)u(t)\cdot v(t)]dt-\int_{\mathbb{R}}\nabla\widetilde{W}(t,u)\cdot v dt,\ \forall u,v\in X^\alpha.$$

Finally, if u is a critical point of f, for any $v\in X^\alpha\subset C(\mathbb{R})$, we have

$$\int_{\mathbb{R}}({}_{-\infty}D_t^\alpha u(t)\cdot{}_{-\infty}D_t^\alpha v(t)dt = -\int_{\mathbb{R}}[L(t)u(t)\cdot v(t)-\nabla\widetilde{W}(t,u(t))\cdot v(t)]dt$$

which with Proposition 2 (2.9) implies for all $v \in X^\alpha$

$$\int_{\mathbb{R}}({}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t))\cdot v(t)dt = -\int_{\mathbb{R}}[L(t)u(t)\cdot v(t)-\nabla\widetilde{W}(t,u(t))\cdot v(t)]dt.$$

Since $C_0^\infty(\mathbb{R})$ is dense in X^α, then we get

$${}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t) = -L(t)u(t)+\nabla\widetilde{W}(t,u(t)),$$

i.e., u is a solution of $(\widetilde{\mathscr{FHS}})$.

LEMMA 7. *If* (L_0), (L_σ) *and* (W_1) *hold, then* f *is bounded from below and satisfies the* (PS)*-condition.*

Proof. Firstly, by (2.15), (3.2) and the Hölder's inequality, it holds for all $u\in X^\alpha$

$$\begin{aligned}\int_{\mathbb{R}}\left|\widetilde{W}(t,u)\right|dt &\leqslant \int_{\mathbb{R}}(a(t)|u|+|u|^{\nu+1})dt\\ &\leqslant \|a\|_{L^\gamma}\|u\|_{L^{\frac{\gamma}{\gamma-1}}}+b\|u\|_{L^{\nu+1}}^{\nu+1}\\ &\leqslant \eta_{\frac{\gamma}{\gamma-1}}\|a\|_{L^\gamma}\|u\|+b\eta_{\nu+1}^{\nu+1}\|u\|^{\nu+1}.\end{aligned}$$

Thus

$$\begin{aligned}f(u) &\geqslant \frac{1}{2}\|u\|^2-\int_{\mathbb{R}}\left|\widetilde{W}(t,u)\right|dt\\ &\geqslant \frac{1}{2}\|u\|^2-\eta_{\frac{\gamma}{\gamma-1}}\|a\|_{L^\gamma}\|u\|-b\eta_{\nu+1}^{\nu+1}\|u\|^{\nu+1},\ \forall u\in X^\alpha.\end{aligned}\tag{3.11}$$

Since $\nu < 1$, it follows that f is bounded from below.

Let $(u_n) \subset X^\alpha$ be a (PS)-sequence, that is

$$|f(u_n)| \leqslant M,\ \forall n \in \mathbb{N},\ f'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{3.12}$$

for some constant $M > 0$. By (3.11) and (3.12), we get

$$M \geqslant \frac{1}{2}\|u_n\|^2 - \eta_{\frac{\gamma}{\gamma-1}}\|a\|_{L^\gamma}\|u_n\| - b\eta_{\nu+1}^{\nu+1}\|u_n\|^{\nu+1},$$

which implies that (u_n) is bounded in X^α since $\nu < 1$. Hence, taking a subsequence if necessary, we may suppose that

$$u_n \rightharpoonup u \text{ in } X^\alpha \text{ and } u_n \longrightarrow u \text{ in } L^{\frac{\gamma}{\gamma-1}} \text{ as } n \longrightarrow \infty, \tag{3.13}$$

for some $u \in X^\alpha$. Next, we have

$$\|u_n - u\|^2 = (f'(u_n) - f'(u))(u_n - u) + \int_{\mathbb{R}} (\nabla\widetilde{W}(t,u_n) - \nabla\widetilde{W}(t,u)) \cdot (u_n - u)dt. \tag{3.14}$$

It is clear that

$$(f'(u_n) - f'(u))(u_n - u) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.15}$$

By Hölder's inequality, (2.15) and Lemma 5, one has

$$\begin{aligned}&\left|\int_{\mathbb{R}} (\nabla\widetilde{W}(t,u_n) - \nabla\widetilde{W}(t,u)) \cdot (u_n - u)dt\right| \\ &\leqslant \left|\nabla\widetilde{W}(t,u_{n_k}) - \nabla\widetilde{W}(t,u)\right|\Big\|_{L^\gamma}\|u_n - u\|_{L^{\frac{\gamma}{\gamma-1}}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned} \tag{3.16}$$

Combining (3.14)–(3.16), we deduce that $u_n \longrightarrow u$ in X^α and completes the proof of Lemma 7.

LEMMA 8. *If (L_0), (L_σ) and (W_2) hold, then for each $k \in \mathbb{N}$, there exists an $A_k \subset X^\alpha$ with $\gamma(A_k) = k$ satisfying* $\sup_{u \in A_k} f(u) < 0$.

Proof. Let (e_n) be an orthonormal basis of X^α. For any $k \in \mathbb{N}$, let

$$X_k = \oplus_{m=1}^k E_m,\ E_m = \mathbb{R}e_m.$$

Since X_k is with finite-dimensional, there is a positive constant β_k such that

$$\|u\| \leqslant \beta_k\|u\|_{L^2},\ \forall u \in X_k. \tag{3.17}$$

By (W_2), there is a constant $R > 0$ such that

$$\widetilde{W}(t,x) \geqslant \beta_k^2|x|^2,\ \forall t \in \mathbb{R},\ |x| \leqslant R. \tag{3.18}$$

Let $u \in X^\alpha$ with $\|u\| \leqslant \frac{R}{\eta_\infty}$. By (2.15), we know that $|u(t)| \leqslant R$ for all $t \in \mathbb{R}$, thus by (3.18), it holds

$$\widetilde{W}(t,u(t)) \geqslant \beta_k^2|u(t)|^2,\ \forall t \in \mathbb{R}. \tag{3.19}$$

Therefore, by (3.17) and (3.19), for all $u \in X_k$ with $0 < \|u\| = \tau_k = \min\{r, R\} \frac{1}{\eta_\infty}$, one has

$$\begin{aligned} f(u) =& \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widetilde{W}(t,u)dt \\ \leqslant& \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \beta_k^2 |u(t)|^2 dt \\ \leqslant& \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\tau_k^2, \end{aligned}$$

which implies

$$\{u \in X_k / \|u\| = \tau_k\} \subset A_k = \left\{u \in X^\alpha / f(u) \leqslant -\frac{1}{2}\tau_k^2\right\}. \tag{3.20}$$

Thus by Lemma 3, (3.20) implies

$$\gamma(A_k) \geqslant \gamma(\{u \in X_k / \|u\| = \tau_k\}) \geqslant k,$$

hence, by the definition of Γ_k, we have $A_k \subset \Gamma_k$. Moreover, the definition of Γ_k implies

$$\sup_{u \in A_k} f(u) \leqslant -\frac{1}{2}\tau_k^2 < 0,$$

which ends the proof of Lemma 8.

Finally, assumptions (W_1) and (W_3) imply that $f(0) = 0$ and f is even. It follows from this and Lemma 7 that f verifies the condition (f_1) of Lemma 4. Lemma 8 shows that the condition (f_2) of Lemma 4 is verified. Therefore, by Lemma 4, f admits a nontrival sequence $(u_k) \in X^\alpha$ satisfying $f'(u_k) = 0$ for all $k \in \mathbb{N}$ and $u_k \longrightarrow 0$ as $k \longrightarrow \infty$. Lemma 6 implies that u_k is a nontrivial solution of $(\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}})$ for all positive integer k. By (2.15), it follows that $\sup_{t\in\mathbb{R}} |u_k(t)| \longrightarrow 0$ as $k \longrightarrow \infty$. Hence, there is a constant $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$, $\sup_{t\in\mathbb{R}} |u_k(t)| \leqslant r$, where r is defined as above. Therefore for all $k \geqslant k_0$, u_k is a solution of $(\mathscr{F}\mathscr{H}\mathscr{S})$. The proof of Theorem 1 is finished. □

4. Locally superquadratic conditions

In this Section, we are interested in the existence of infinitely many solutions of $(\mathscr{F}\mathscr{H}\mathscr{S})$ when the potential $W(t,x)$ is only locally defined and superquadratic near the origin with respect to x. More precisely, let $W : \mathbb{R} \times B_\delta(0) \longrightarrow \mathbb{R}$ be a continuous function, differentiable in x with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$, where δ is a positive constant and $B_\delta(0)$ is the open ball in $\mathbb{R}^N$ centered at zero with radius δ, we consider the following conditions

(W_1) $W(t,0) = 0$ and there exist constants $c > 0$ and $0 < \theta < 1$ such that

$$|\nabla W(t,x)| \leqslant c|x|^\theta, \ \forall (t,x) \in \mathbb{R} \times B_\delta(0);$$

(W_2) There is a constant $\gamma > 2$ such that

$$\lim_{|x|\longrightarrow 0} \frac{W(t,x)}{|x|^{\gamma}} = 0 \text{ uniformly for } t \in \mathbb{R};$$

$$2W(t,x) - \nabla W(t,x)\cdot x < 0,\ \forall (t,x) \in \mathbb{R} \times (B_{\delta}(0) \setminus \{0\}); \tag{W_3}$$

(W_4) There is a constant $\beta > 2$ such that

$$\lim_{|x|\longrightarrow 0} \frac{W(t,x)}{|x|^{\beta}} = +\infty \text{ uniformly for } t \in \mathbb{R}.$$

(W_5) $W(t,-x) = W(t,x)$, $\forall (t,x) \in \mathbb{R} \times B_{\delta}(0)$.

Our main result in this Section reads as follows.

THEOREM 2. *Suppose that (L_1), (L_{σ}) and (W_1)–(W_5) hold. Then the fractional Hamiltonian system $(\mathscr{FHS})$ possesses a sequence of solutions (u_k) such that $\sup_{t\in\mathbb{R}} |u_k(t)| \longrightarrow 0$ as $k \longrightarrow \infty$.*

REMARK 5. Theorem 1 generalizes Theorem 1 in [22].

REMARK 6. There are functions $W(t,x)$ that satisfy all assumptions in Theorem 2. For example, $W(t,x) = |x|^3$ for $|x| < 1$ with $\theta = \frac{1}{2}$, $\gamma = \frac{5}{2}$ and $\beta = \frac{7}{2}$.

Proof of Theorem 2. Choose $0 < b < \frac{1}{4\eta_{\gamma}^{\gamma}}$. By (W_2), there is a constant $r \in]0, \frac{\delta}{2}[$ such that

$$W(t,x) \leqslant b\,|x|^{\gamma},\ \forall t \in \mathbb{R},\ |x| \leqslant 2r. \tag{4.1}$$

Let

$$\widetilde{W}(t,x) = \rho(|x|)W(t,x) + (1-\rho(|x|))b\,|x|^{\gamma} \tag{4.2}$$

where $\rho \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ is a cut-off function satisfying $\rho(s) = 1$ for $0 \leqslant s \leqslant r$, $\rho(s) = 0$ for $s \geqslant 2r$ and $-\frac{2}{r} \leqslant \rho'(s) < 0$ for $r < s < 2r$.

For later use, the following lemma will be needed.

LEMMA 9. *If (W_1) and (W_3) hold, t hen $\widetilde{W}(t,x)$ is continuously differentiable in x with continuous derivative such that*

$$\left|\nabla\widetilde{W}(t,x)\right| \leqslant c_1(|x|^{\theta} + |x|^{\gamma-1}),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N; \tag{4.3}$$

$$\widehat{W}(t,x) = 2\widetilde{W}(t,x) - \nabla\widetilde{W}(t,x)\cdot x \leqslant 0,\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N; \tag{4.4}$$

$$\widehat{W}(t,x) = 0 \Leftrightarrow x = 0. \tag{4.5}$$

Proof. Note first that by the Mean Value Theorem, (W_1) implies

$$|W(t,x)| \leqslant c\,|x|^{\theta+1}, \ \forall (t,x) \in \mathbb{R} \times B_\delta(0). \tag{4.6}$$

By direct computation, we get

$$\begin{aligned} \nabla \widetilde{W}(t,x) &= \rho(|x|)\nabla W(t,x) + \rho'(|x|)W(t,x)\frac{x}{|x|} \\ &\quad + (1-\rho(|x|))b\gamma|x|^{\gamma-2}x - \rho'(|x|)b\,|x|^\gamma \frac{x}{|x|} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \widehat{W}(t,x) = {} & \rho(|x|)[2W(t,x) - \nabla W(t,x)\cdot x] + (2-\gamma)(1-\rho(|x|))b\,|x|^\gamma \\ & - \rho'(|x|)(W(t,x) - b\,|x|^\gamma)\,|x| \end{aligned} \tag{4.8}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$. Besides, it is easy to see that (W_1) implies

$$\nabla \widetilde{W}(t,0) = 0, \ \widehat{W}(t,0) = 0, \ \forall t \in \mathbb{R}. \tag{4.9}$$

By (4.6), (4.7) and the choice of the cut-off function ρ, we have

$$\left|\nabla \widetilde{W}(t,x)\right| \leqslant b\gamma|x|^{\gamma-1}, \ \forall t \in \mathbb{R}, \ |x| \geqslant 2r,$$

and

$$\begin{aligned} \left|\nabla \widetilde{W}(t,x)\right| &\leqslant |\nabla W(t,x)| + \frac{2}{r}\,|W(t,x| + b\gamma|x|^{\gamma-1} + \frac{2}{r}b\,|x|^\gamma \\ &\leqslant c\,|x|^\theta + \frac{2}{r}c\,|x|^{\theta+1} + 4b\,|x|^{\gamma-1} + b\gamma|x|^{\gamma-1} \\ &\leqslant c\,|x|^\theta + 4c\,|x|^\theta + 4b\,|x|^{\gamma-1} + b\gamma|x|^{\gamma-1} \\ &\leqslant 5c\,|x|^\theta + (4+\gamma)b\,|x|^{\gamma-1}, \ \forall t \in \mathbb{R}, \ |x| \leqslant 2r. \end{aligned}$$

Hence (4.3) is satisfied with $c_1 = \max\{5c, (4+\gamma)b\}$. Using the fact that $\rho'(|x|) \leqslant 0$ for $|x| \leqslant 2r$ and $\rho'(|x|) = 0$ for $|x| \geqslant 2r$, then (W_3), (4.1) and (4.8) imply (4.4).

It follows from (4.9) that $\widehat{W}(t,0) = 0$. Conversely, by (W_3) and (4.1), we have for all $0 < |x| < 2r$

$$2W(t,x) - \nabla W(t,x)\cdot x < 0, \ (2-\gamma)b\,|x|^\gamma < 0 \ \text{ and } \ W(t,x) - b\,|x|^\gamma \leqslant 0$$

therefore; by (4.8) and the definition of the cut-off function ρ, we obtain

$$\widehat{W}(t,x) < 0, \ \forall t \in \mathbb{R}, \ 0 < |x| < 2r.$$

For $|x| \geqslant 2r$, (4.8) implies

$$\widehat{W}(t,x) = (2-\gamma)b\,|x|^\gamma < 0.$$

Hence $\widehat{W}(t,x) < 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$ and then (4.5) is verified. The proof of Lemma 9 is completed.

Now, consider the following modified system

$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u)(t) + L(t)u(t) = \nabla\widetilde{W}(t,u(t)),\ t\in\mathbb{R} \\ u\in H^\alpha(\mathbb{R}), \end{cases} \qquad (\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}})$$

and the associated continuously differentiable functional

$$f(u) = \frac{1}{2}\int_{\mathbb{R}}(|{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t)\cdot u(t))dt - \int_{\mathbb{R}}\widetilde{W}(t,u)dt.$$

Then by (2.17), $f(u)$ can be rewritten

$$f(u) = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - g(u)$$

for all $u = u^- + u^0 + u^+ \in X^\alpha = X^- \oplus X^0 \oplus X^+$, where

$$g(u) = \int_{\mathbb{R}}\widetilde{W}(t,u)dt.$$

Next, the following critical point theorem due to Zou [26] will be needed to prove our Theorem 2.

LEMMA 10. [29] *Let* $X = \oplus_{j\in\mathbb{N}}X(j)$ *where* $X(j)$ *are all finite-dimensional subspaces. Let* $f\in C^1(X,\mathbb{R})$ *be an even functional satisfying*

(A_1) *There is* $k_0\in\mathbb{N}$ *such that for every* $k\geqslant k_0$*, there exists* $R_k>0$ *such that* $f(u)\geqslant 0$ *for every* $u\in X_k = \oplus_{j\geqslant k}X(j)$ *with* $\|u\| = R_k$*, and* $b_k = \inf_{u\in B_k} f(u)\longrightarrow 0$ *as* $k\longrightarrow\infty$*. Here,* $B_k = \{u\in X_k/\ \|u\|\leqslant R_k\}$*.*

(A_2) *For every* $k\in\mathbb{N}$*, there exist* $r_k\in]0,R_k[$ *and* $d_k<0$ *such that* $f(u)\leqslant d_k$ *for every* $u\in X^k = \oplus_{j\leqslant k}X(j)$ *with* $\|u\| = r_k$*.*

(A_3) f *satisfies the* $(PS)^*$*-condition, with respect to* $\left\{X^k/k\in\mathbb{N}\right\}$*, i.e., every sequence* (u_k) *such that* $u_k\in X^k$ *with* $f(u_k)<0$ *for all* $k\in\mathbb{N}$ *and* $(f_{|X^k})'(u_k)\longrightarrow 0$ *as* $k\longrightarrow\infty$ *admits a convergent subsequence.*

Then, for each $k\geqslant k_0$*,* f *has a critical value* $\xi_k\in[b_k,d_k]$*, hence* $\xi_k<0$ *and* $\xi_k\longrightarrow 0$ *as* $k\longrightarrow\infty$*.*

LEMMA 11. *If* (L_1)*,* (L_σ)*,* (W_1) *and* (W_3) *hold, then* 0 *is the only critical point of* f *such that* $f(0)=0$*.*

Proof. By (W_1) and Lemma 9, we know that $f'(0)=0$ and $f(0)=0$. Let u be such that $f'(u)=0$ and $f(u)=0$, then we have

$$0 = 2f(u) - f'(u)u = -\int_{\mathbb{R}}\widehat{W}(t,u)dt,$$

which together with (4.4) and (4.5) implies that $u(t)=0$ for all $t\in\mathbb{R}$. The proof of Lemma 11 is completed.

Now, let $X(j)=\mathbb{R}e_j$ for each $j\in\mathbb{N}$, where $\left\{e_j, j\in\mathbb{N}\right\}$ is the system of eigenfunctions given in Section 2. In the following, we will show that the functional f satisfies the geometry properties (A_1)–(A_3) of Lemma 10.

LEMMA 12. *Suppose that* (L_1)*,* (L_σ) *and* (W_1) *hold. Then there is* $k_0 \in \mathbb{N}$ *and a sequence* $R_k \longrightarrow 0^+$ *as* $k \longrightarrow \infty$ *such that*

$$\inf_{u\in X_k, \|u\|=R_k} f(u) \geqslant 0, \ \forall k \geqslant k_0$$

and

$$\inf_{u\in B_k} f(u) \longrightarrow 0, \ \ as \ \ k \longrightarrow \infty$$

where $X_k = \oplus_{j\geqslant k} X(j)$ *and* $B_k = \{u \in X_k, \|u\| \leqslant R_k\}$ *for all* $k \in \mathbb{N}$.

Proof. By the Mean Value Theorem, (4.6) and the definitions of $\widetilde{W}(t,x)$ and the cut-off function ρ, we obtain

$$\left|\widetilde{W}(t,x)\right| = \left|\int_0^1 \nabla\widetilde{W}(t,sx)\cdot x ds\right| \leqslant c\,|x|^{\theta+1} + b\,|x|^{\gamma} \tag{4.10}$$

Note that $X_k \subset X^+$ for all $k \geqslant \bar{k}+1$, by definition of $\bar{k}$ and X^+. Thus for each $k \geqslant \bar{k}+1$, (4.10) implies

$$\begin{aligned} f(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \widetilde{W}(t,u)dt \\ &\geqslant \frac{1}{2}\|u\|^2 - c\,\|u\|_{L^{\theta+1}}^{\theta+1} - b\,\|u\|_{L^\gamma}^{\gamma}, \ \forall u \in X_k. \end{aligned} \tag{4.11}$$

Set

$$l_k = \sup_{u\in X_k, \|u\|=1} \|u\|_{L^{\theta+1}}, \ for\ all\ k \geqslant \bar{k}+1. \tag{4.12}$$

Since X^α is compactly embedded in $L^{\theta+1}(\mathbb{R})$, then it is well known that

$$l_k \longrightarrow 0 \ \text{ as } \ k \longrightarrow \infty. \tag{4.13}$$

It follows from (2.15), (4.11), (4.12) and the choice of b that for all $k \geqslant \bar{k}+1$ and $u \in X_k$ with $\|u\| \leqslant 1$

$$\begin{aligned} f(u) &\geqslant \frac{1}{2}\|u\|^2 - c l_k^{\theta+1}\|u\|^{\theta+1} - \frac{1}{4}\|u\|^{\gamma} \\ &\geqslant \frac{1}{4}\|u\|^2 - c l_k^{\theta+1}\|u\|^{\theta+1}. \end{aligned} \tag{4.14}$$

Choose $R_k = \min\left\{1, (4cl_k^{\theta+1})^{\frac{1}{1-\theta}}\right\}$. Then by (4.13), $R_k \longrightarrow 0^+$ as $k \longrightarrow \infty$. By virtue of (4.14) and the definition of R_k, we have

$$\inf_{u\in X_k, \|u\|=R_k} f(u) \geqslant 0, \ \forall k \geqslant \bar{k}+1. \tag{4.15}$$

Furthermore, since $f(0) = 0$ and

$$f(u) \geqslant -c l_k^{\theta+1}\|u\|^{\theta+1} \geqslant -c l_k^{\theta+1} R_k^{\theta+1}, \ \forall u \in B_k, \ \forall k \geqslant \bar{k}+1,$$

then

$$b_k = \inf_{u \in B_k} f(u) \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

The proof of Lemma 12 is completed.

LEMMA 13. *If (L_1), (L_σ) and (W_4) hold, then for every $k \in \mathbb{N}$, there are constants $r_k \in]0, R_k[$ and $d_k < 0$ such that*

$$f(u) \leqslant d_k \text{ for all } u \in X^k = \oplus_{j \leqslant k} X(j) \text{ with } \|u\| = R_k.$$

Proof. Let $k \in \mathbb{N}$. Since X^k is finite dimensional, there is a constant $a_k > 0$ such that

$$a_k \|u\|^\beta \leqslant \|u\|_{L^\beta}^\beta, \ \forall u \in X^k. \tag{4.16}$$

By (W_4), there exists a constant $0 < \nu_k < r$ such that

$$\widetilde{W}(t,x) = W(t,x) \geqslant m_k |x|^\beta, \ \forall t \in \mathbb{R}, \ |x| \leqslant \nu_k; \tag{4.17}$$

where $m_k = \frac{1}{a_k R_k^{\beta-2}}$. Now, by (2.15), (4.16) and (4.17), for $u \in X^k$ with $\|u\| \leqslant \frac{\nu_k}{\eta_\infty}$, we obtain

$$\begin{aligned} f(u) &= \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \int_{\mathbb{R}} \widetilde{W}(t,u)dt \\ &\leqslant \frac{1}{2}\|u\|^2 - m_k \|u\|_\beta^\beta \leqslant \frac{1}{2}\|u\|^2 - m_k a_k \|u\|^\beta \\ &= \frac{1}{2}\|u\|^2 (1 - 2m_k a_k \|u\|^{\beta-2}). \end{aligned}$$

Choose $r_k = (\frac{3}{4m_k a_k})^{\frac{1}{\beta-2}} < R_k$ and $d_k = -\frac{r_k^2}{4} < 0$. If $u \in X^k$ with $\|u\| = r_k$, we have

$$f(u) = \frac{1}{2} r_k^2 (1 - 2m_k a_k r_k^{\beta-2}) = -\frac{r_k^2}{4} = d_k < 0,$$

which completes the proof of Lemma 13.

LEMMA 14. *Suppose that (L_1), (L_σ), (W_1) and (W_4) hold. Then f satisfies the $(PS)^*$-condition with respect to $(X^k)_{k \in \mathbb{N}}$.*

Proof. Let $u_k \in X^k$ be a $(PS)^*$-sequence, that is

$$(f(u_k)) \text{ is bounded and } (f_{|X^k})'(u_k) \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{4.18}$$

We claim that (u_k) is bounded. Otherwise, by going to a subsequence if necessary, we can assume that

$$\|u_k\| \longrightarrow \infty \text{ as } k \longrightarrow \infty. \tag{4.19}$$

From (4.8), (W_3) and the definition of the cut-off function ρ, we have

$$\begin{aligned}2f(u_k)-(f_{|X^k})'(u_k)u_k &= -\int_{\mathbb{R}}\widehat{W}(t,u_k)dt\\ &= -\int_{\{t\in\mathbb{R}/|u_k(t)|\leqslant 2r\}}\widehat{W}(t,u_k)dt-\int_{\{t\in\mathbb{R}/|u_k(t)|\geqslant 2r\}}\widehat{W}(t,u_k)dt\\ &\geqslant (\gamma-2)b\int_{\{t\in\mathbb{R}/|u_k(t)|\geqslant 2r\}}|u_k|^{\gamma}dt,\ \forall k\in\mathbb{N}.\end{aligned} \tag{4.20}$$

By (4.18)–(4.20), we have

$$\frac{1}{\|u_k\|}\int_{\{t\in\mathbb{R}/|u_k(t)|\geqslant 2r\}}|u_k|^{\gamma}dt \longrightarrow 0 \text{ as } k\longrightarrow\infty. \tag{4.21}$$

Let

$$v_k(t)=\begin{cases}u_k(t), \ if\ |u_k(t)|<2r\\ 0,\ if\ |u_k(t)|\geqslant 2r\end{cases} \tag{4.22}$$

and

$$w_k(t)=u_k(t)-v_k(t),\ \forall k\in\mathbb{N},\ \forall t\in\mathbb{R}. \tag{4.23}$$

It follows from (4.18), (4.20) and (4.23) that

$$\|w_k\|_{L^\gamma}^{\gamma}\leqslant c_3(1+\|u_k\|),\ \forall k\in\mathbb{N}. \tag{4.24}$$

Since $X^-\oplus X^0$ is of finite dimension, we obtain by Hölder's inequality

$$\begin{aligned}\left\|u_k^-+u_k^0\right\|_{L^2}^2 &= \langle u_k^-+u_k^0,u_k\rangle_{L^2}\\ &= \langle u_k^-+u_k^0,v_k\rangle_{L^2}+\langle u_k^-+u_k^0,w_k\rangle_{L^2}\\ &\leqslant \left\|u_k^-+u_k^0\right\|_{L^1}\|v_k\|_{L^\infty}+\left\|u_k^-+u_k^0\right\|_{L^{\gamma'}}\|w_k\|_{L^\gamma}\\ &\leqslant c_4\left\|u_k^-+u_k^0\right\|(1+\|w_k\|_{L^\gamma}),\ \forall k\in\mathbb{N},\end{aligned} \tag{4.25}$$

where $\frac{1}{\gamma'}+\frac{1}{\gamma}=1$. By the equivalence of the norms on the finite dimensional subspace $X^-\oplus X^0$, (4.24) and (4.25), we have

$$\left\|u_k^-+u_k^0\right\|\leqslant c_5\left\|u_k^-+u_k^0\right\|_{L^2}\leqslant c_4c_5(1+\|w_k\|_{L^\gamma})\leqslant c_6(1+\|u_k\|^{\frac{1}{\gamma}})$$

for all $k\in\mathbb{N}$. Therefore

$$\frac{\left\|u_k^-+u_k^0\right\|}{\|u_k\|}\longrightarrow 0 \text{ as } k\longrightarrow\infty. \tag{4.26}$$

From (4.3), it follows that

$$\left|\nabla\widetilde{W}(t,x)\right|\leqslant c_7(1+|x|^{\gamma-1}),\ \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N,$$

which with (2.15) implies

$$
\begin{aligned}
(f_{|X^k})'(u_k)u_k^+ &\geqslant \left\|u_k^+\right\|^2 - \int_{\mathbb{R}} \left|\nabla\widetilde{W}(t,u_k)\right| \left|u_k^+\right| dt \\
&\geqslant \left\|u_k^+\right\|^2 - c_7 \int_{\mathbb{R}} |u_k|^{\gamma-1} \left|u_k^+\right| dt - c_7 \int_{\mathbb{R}} \left|u_k^+\right| dt \\
&\geqslant \left\|u_k^+\right\|^2 - c_7 \int_{\{t\in\mathbb{R}/|u_k(t)|\geqslant 2r\}} |u_k|^{\gamma-1}\, dt \left\|u_k^+\right\|_{L^\infty} \\
&\quad - c_7(2r)^{\gamma-1} \int_{\{t\in\mathbb{R}/|u_k(t)|<2r\}} \left|u_k^+\right| dt - c_7 \left\|u_k^+\right\|_{L^1} \\
&\geqslant \left\|u_k^+\right\|^2 - c_7(2r)^{-1} \int_{\{t\in\mathbb{R}/|u_k(t)|\geqslant 2r\}} |u_k|^{\gamma}\, dt \left\|u_k^+\right\|_{L^\infty} \\
&\quad - c_7(2r)^{\gamma-1} \left\|u_k^+\right\|_{L^1} - c_7 \left\|u_k^+\right\|_{L^1} \\
&\geqslant \left\|u_k^+\right\|^2 - c_7(2r)^{-1}\eta_\infty \int_{\{t\in\mathbb{R}/|u_k(t)|\geqslant 2r\}} |u_k|^{\gamma}\, dt \left\|u_k^+\right\| \\
&\quad - c_7(2r)^{\gamma-1}\eta_1 \left\|u_k^+\right\| - c_7\eta_1 \left\|u_k^+\right\|,
\end{aligned}
$$

which combined with (4.21) implies

$$
\frac{\left\|u_k^+\right\|}{\|u_k\|} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{4.27}
$$

We deduce from (4.26) and (4.27) that

$$
1 = \frac{\|u_k\|}{\|u_k\|} \leqslant \frac{\left\|u_k^+\right\| + \left\|u_k^- + u_k^0\right\|}{\|u_k\|} \longrightarrow 0 \text{ as } k \longrightarrow \infty
$$

which is a contradiction. Hence (u_k) is bounded.

By a standard argument, (u_k) possesses a convergent subsequence, which completes the proof of Lemma 14.

Finally, from Lemmas 12-14, it follows that f satisfies the conditions (A_1)–(A_3) of Lemma 10. By (W_5), f is even. Thus, Lemma 10 implies that f has a sequence of critical values $\xi_k < 0$ with $\xi_k \longrightarrow 0$ as $k \longrightarrow \infty$. Let (u_k) be such that $f(u_k) = \xi_k$ and $f'(u_k) = 0$ for all $k \in \mathbb{N}$. Then (u_k) is a sequence of solutions of system $(\widetilde{\mathscr{F}\mathscr{H}\mathscr{S}})$. Moreover, (u_k) is a $(PS)^*$-sequence in X^α. By Lemma 14, f satisfies the $(PS)^*$ condition and hence (u_k) admits a subsequence, noted again by (u_k), satisfying $u_k \longrightarrow u$ in X^α as $k \longrightarrow \infty$. Evidently, we have $f'(u) = 0$ and $f(u) = 0$. Then by Lemma 11, u must be 0. Thus $u_k \longrightarrow 0$ in X^α as $k \longrightarrow \infty$. By (2.15) we have $u_k \longrightarrow 0$ in $L^\infty(\mathbb{R})$ as $k \longrightarrow \infty$. Consequently, for k large enough, u_k is a solution of $(\mathscr{F}\mathscr{H}\mathscr{S})$. The proof of Theorem 2 is completed. □

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