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ON THE STABILITY AND STABILIZATION OF SOME SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we prove the existence and uniqueness of a mild solution to a class of semilinear fractional differential equation in an infinite Banach space with Caputo derivative order $0 < \alpha \leqslant 1$. Furthermore, we establish the stability conditions and then prove that the considered initial value problem is exponentially stabilizable when the stabilizer acts linearly on the control system.

1. Introduction

Let $b > 0$ and A the infinitesimal generator of a semigroup of uniformly bounded operators $(T(t))_{t\geqslant 0}$ defined on an infinite dimensional Banach space $(\mathbb{X}, \|\cdot\|)$. We consider the following system of fractional differential equations:

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = Ax(t) + f(t, x(t)),\ t \in [0,b] \\ \qquad x(0) = x_0, \end{cases} \tag{1}$$

where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha \leqslant 1$, $x_0 \in \mathbb{X}$ and $f : [0,b] \times \mathbb{X} \longrightarrow \mathbb{X}$ is a nonlinear function. We denote by $\mathscr{C} = \mathscr{C}([0,b], \mathbb{X})$ the Banach space of all continuous functions from $[0,b]$ into $\mathbb{X}$ endowed with the topology of uniform convergence

$$\|u\|_{\mathscr{C}} = \sup_{t\in[0,b]} \|u(t)\| \tag{2}$$

and let $\left(\mathscr{B}(\mathbb{X}), \|\cdot\|_{\mathscr{B}(\mathbb{X})}\right)$ be the Banach space of all linear and bounded operators from $\mathbb{X}$ to $\mathbb{X}$.

REMARK 1. Since $(T(t))_{t\geqslant o}$ is uniformly bounded, there exists $M > 0$ such that $M = \sup_{t\in[0,\infty)} \|T(t)\| < \infty$.

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Fractional derivatives provide more accurate models of real-world behaviour than standard derivatives because of their non-local nature, an intrinsic property of many complex systems. This is why in recent decades, more and more scientific researchers have become interested in using fractional differential equations to model phenomena in various branches of science. Many authors have worked on the existence of solution to initial value problems with fractional derivatives in finite and infinite dimensional space. In finte dimensional case, the existence and uniqueness of solutions to problems of type (1) are widely studied in [14] by mean of Schauder's fixed point theorem and Weissinger's fixed point theorem. Mouffack Benchohra et al. used the Banach fixed point theorem and the nonlinear alternative of Leray-Schauder type in [3] to prove the existence of solutions for fractional order functional and neutral functional differential equations with infinite delay whereas in [23], Gisèle Mophou and Gaston M. N'Guérékata investigated the existence and uniqueness of the mild solution for a semilinear fractional differential equation of neutral type with infinite delay. For more results on the existence of solutions we refer to [4, 5, 6, 7, 16, 22, 24, 25, 26] and the references therein.

Given the fact that fractional differential equations describes better the dynamics of complex systems, it became very important to study their stability and stabilization, as they arise in many scientific and engineering processes such as physics, economics, control theory, finance, etc ... This justifies the interest of several researchers in this area. For instance, in [17, 18] Yan Li et al. studied the Mittag-Leffler stability of fractional-order nonlinear dynamic system and investigated the Lyapunov direct method. In order to apply the fractional-order extension of Lyapunov direct method, Aguila-Camacho Norelys et al. also proposed in [1] a new lemma for the stability of fractional differential equations with Caputo derivative order $0 < \alpha < 1$. Meanwhile, Mihailo Lazarević used Gronwall inequality and Bellman-Gronwall inequality to present in [15] sufficient conditions for finite time stability and stabilization for nonlinear perturbated fractional order time delay systems. In [29], Roberto Triggiani proved that studying the stabilization of a general infinite dimensional system in term of its controllability need not be as informative and as general as a procedure for the finite dimensional case. More specifically, he considered the stabilizability problem of expressing the control through a bounded operator acting on the state as to make the resulting feedback system globally asymptotically stable. In both finite and infinite dimensional cases, more results can be found in [6, 8, 13, 19, 20, 27, 28, 30, 31, 32] and the references therein.

The main purpose of this paper is a generalization of [13] in an infinite Banach space. In the latter paper, Badawi Hamza Elbadawi Ibrahim et al. recently studied based on the properties of Mittag-Leffler functions, the stability and stabilization of the semilinear system (1) in a finite dimensional space, $f : [0,\infty) \times \mathbb{R}^n \longmapsto \mathbb{R}^n$ being considered as a nonlinear vector field in the n-dimensional vector space, and $A \in \mathbb{R}^{n\times n}$ a constant matrix.

The rest of this paper is organized as follows: Section 2 is devoted to some preliminary results that are useful in the sequel. In Section 3, we prove by means of the Banach contraction principle that there exists a unique mild solution to the Cauchy problem (1) whereas in Sections 4 and 5, we respectively establish conditions for exponential stability and study the stabilization of the latter system. An illustrative example for the existence and uniqueness of mild solution in an infinite Banach space is given in Section 6, as well as its stability. The last section concludes this work.

2. Preliminaries

In this section, we present recall definitions and properties of fractional calculus and semigroup theory to be used throughout this paper.

DEFINITION 2.1. The *Euler's Gamma function* is given by:

$$\Gamma(\sigma) = \int_0^\infty t^{\sigma-1} e^{-t} dt \quad \text{for} \quad \sigma > 0.$$

Furthermore, $\Gamma(1) = 1$ and $\Gamma(\sigma+1) = \sigma\Gamma(\sigma)$ for any $\sigma > 0$.

DEFINITION 2.2. The *Laplace transform* of a function f is denoted and defined by:

$$\mathscr{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t) e^{-st} dt \quad \text{for} \quad s > 0.$$

In addition, if $F(s) = \mathscr{L}\{f(t)\}(s)$ and $G(s) = \mathscr{L}\{g(t)\}(s)$, then

$$\mathscr{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\}(s) = F(s)G(s). \tag{3}$$

DEFINITION 2.3. The *left-sided Riemann-Liouville fractional integral* of order α is defined and denoted by:

$${}_aI_t^\alpha x(t) = {}_aD_t^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau \quad \text{for} \quad Re(\alpha) > 0. \tag{4}$$

Note that α could be real, integer, fraction or complex.

DEFINITION 2.4. The *left-sided Riemann-Liouville fractional derivative* of order α is denoted and defined as follow:

$${}_aD_t^\alpha x(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \left(\dfrac{d}{dt}\right)^n \displaystyle\int_a^t (t-\tau)^{n-\alpha-1} x(\tau) d\tau & \text{if } n-1 < \alpha < n, \\ \left(\dfrac{d}{dt}\right)^{n-1} x(t) & \text{if} \quad \alpha = n-1. \end{cases} \tag{5}$$

If $x(t) = c =$ constant, then

$${}_aD_t^\alpha x(t) = \frac{c}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad \alpha \neq 1, 2, \cdots \tag{6}$$

Symbolically,

$$D^n {}_aD_t^{-(n-\alpha)} = {}_aD_t^\alpha \tag{7}$$

is the left-sided Rieman Liouville fractional derivative of order α and

$$D^{-n}{}_aD_t^{\alpha} = {}_aD_t^{-(n-\alpha)} \tag{8}$$

represent the fractional integral operator.

DEFINITION 2.5. The *left-sided Caputo fractional derivative* of order α is denoted and defined as follow:

$$ {}_a^CD_t^{\alpha}x(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau & \text{if } n-1<\alpha<n \\ x^{(n-1)}(t) & \text{if } \quad \alpha = n-1 \end{cases} \tag{9}$$

where $x^{(n)}(t)$ is the n^{th} integer order derivative of x with respect to t.

If $x(t)$ is a constant, then

$$ {}_a^CD_t^{\alpha}x(t) = 0. \tag{10}$$

REMARK 2. For $Re(\alpha) \geqslant 0$, the left-sided Caputo fractional derivative ${}_a^CD_t^{\alpha}x(t)$ and the left-sided Riemann-Liouville fractional derivative ${}_aD_t^{\alpha}x(t)$ are connected by the following relation (see [14], p. 91):

$$ {}_a^CD_t^{\alpha}x(t) = {}_aD_t^{\alpha}x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha} \quad (n = [Re(\alpha)]+1). \tag{11}$$

REMARK 3. For any $t \in [a,b]$, if $g(t) \in L^1(a,b)$ then the equality

$$ {}_aI_t^{\alpha}{}_aD_t^{\alpha}g(t) = g(t) - \sum_{j=1}^{n}\frac{\left({}_aI_t^{1-\alpha}g\right)^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j} \tag{12}$$

holds almost everywhere on $[a,b]$ (see [14], p. 75).

REMARK 4. (See [14], p. 91) If $0<\alpha\leqslant 1$, then the left-sided Caputo fractional derivative of a function g coincides with the left-sided Riemann-Liouville derivative if $g(a)=0$. That is,

$$ {}_a^CD_t^{\alpha}\, g(t) = {}_aD_t^{\alpha}\, g(t) \quad \text{if} \quad g(a)=0. \tag{13}$$

REMARK 5. The Laplace transform of the *one-sided stable probability density*

$$\rho_\alpha(\eta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\eta^{-\alpha n-1}\frac{\Gamma(\alpha n+1)}{n!}\sin(n\pi\alpha), \quad \eta \in (0,\infty)$$

is given by

$$\int_0^{\infty} e^{-s\eta}\rho_\alpha(\eta)d\eta = e^{-s^{\alpha}}, \quad \text{where} \quad 0<\alpha\leqslant 1 \text{ and } s>0. \tag{14}$$

Furthermore, for any $0\leqslant\delta\leqslant 1$, we have (see [9]):

$$\int_0^{\infty}\frac{1}{\eta^{\delta}}\rho_\alpha(\eta)d\eta = \frac{\Gamma(1+\dfrac{\delta}{\alpha})}{\Gamma(1+\delta)}. \tag{15}$$

For more details on the above preliminaries, we refer to [10, 21].

3. Existence and uniqueness of the mild solution to the Cauchy problem (1)

We first of all introduce the Banach contraction principle, which we further use to prove the existence and uniqueness result.

LEMMA 3.1. *Assume* (U,d) *to be a non-empty complete metric space, let* $0 \leqslant K < 1$ *and let the mapping* $F : U \longrightarrow U$ *satisfy the inequality*

$$d(Fu,Fv) \leqslant Kd(u,v) \quad \text{for every } u,v \in U.$$

Then, F *has a uniquely determined fixed point* u^*. *Furthermore, for any* $u_0 \in U$, *the sequence* $\left(F^j u_0\right)_{j=1}^{\infty}$ *converges to this fixed point* u^*.

Proof. The proof of this lemma can be found in [12]. □

LEMMA 3.2. *The Cauchy problem* (1) *is equivalent to the volterra integral equation*

$$x(t) = x_0 + \frac{A}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau,x(\tau)) d\tau, \ \text{for } t \in [0,b]. \tag{16}$$

Proof. Let us assume that $x \in \mathscr{C}$ *satisfies the Cauchy problem* (1) *and prove that* (16) *holds.*

Applying the operator ${}_0I_t^{\alpha}$ to both sides of the first equation of (1), we have:

$${}_0I_t^{\alpha} \, {}_0^C D_t^{\alpha} x(t) = A \, {}_0I_t^{\alpha} x(t) + {}_0I_t^{\alpha} f(t,x(t)). \tag{17}$$

In order to evaluate the left hand side of (17), we note from (11) that

$$\begin{aligned} {}_0^C D_t^{\alpha} x(t) &= {}_0D_t^{\alpha} x(t) - \frac{x(0)}{\Gamma(1-\alpha)} t^{-\alpha} \\ &= D^1 \, {}_aD_t^{-(1-\alpha)} x(t) - \frac{x_0}{\Gamma(1-\alpha)} t^{-\alpha} \\ &= \frac{d}{dt} \, {}_0I_t^{1-\alpha} x(t) - \frac{x_0}{\Gamma(1-\alpha)} t^{-\alpha}. \end{aligned} \tag{18}$$

We have also:

$$\begin{aligned} {}_0I_t^{1-\alpha} x_0 &= x_0 \, {}_0I_t^{1-\alpha} 1 \\ &= \frac{x_0}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} d\tau \\ &= \frac{x_0}{(1-\alpha)\Gamma(1-\alpha)} t^{1-\alpha} \end{aligned}$$

and then, $\dfrac{d}{dt} \, {}_0I_t^{1-\alpha} x_0 = \dfrac{x_0}{\Gamma(1-\alpha)} t^{-\alpha}$.

Equation (18) becomes:

$$\begin{aligned} {}_0^C D_t^\alpha x(t) &= \frac{d}{dt}\, {}_0 I_t^{1-\alpha} x(t) - \frac{d}{dt}\, {}_0 I_t^{1-\alpha} x_0 \\ &= {}_0 D_t^\alpha (x(t) - x_0). \end{aligned} \tag{19}$$

If we set $g(t) = x(t) - x_0$, then we obtain from (12) the following:

$$ {}_0 I_t^\alpha \; {}_0 D_t^\alpha x(t) = x(t) - x_0. \tag{20}$$

In addition,

$$ {}_0 I_t^\alpha \; x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau \tag{21}$$

and

$$ {}_0 I_t^\alpha \; f(t, x(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{22}$$

Substituting (20), (21) and (22) into (17), we obtain the Volterra integral equation (16).

Conversely, let us assume that $x \in \mathscr{C}$ satisfies the Volterra integral equation (16) *and prove that the Cauchy problem* (1) *holds.*

Equation (16) can also be written as

$$ x(t) = x_0 + A\, {}_0 I_t^\alpha \; x(t) + {}_0 I_t^\alpha \; f(t, x(t)). \tag{23}$$

Applying the operator ${}_0^C D_t^\alpha$ on both sides of the equality (23) while taking into account (10), we have:

$$ {}_0^C D_t^\alpha \; x(t) = A\, {}_0^C D_t^\alpha \; {}_0 I_t^\alpha \; x(t) \; + \; {}_0^C D_t^\alpha \; {}_0 I_t^\alpha \; f(t, x(t)). \tag{24}$$

Now let us evaluate the right hand side. We have the following estimate:

$$\begin{aligned} \| {}_0 I_t^\alpha \; x(t) \| &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau \right\| \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\infty (t-\tau)^{\alpha-1} \|x(\tau)\| d\tau \\ &\leqslant \frac{1}{\Gamma(\alpha)} \sup_{\tau \in [0,b]} \|x(\tau)\| \int_0^\infty (t-\tau)^{\alpha-1} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \|x\|_{\mathscr{C}} \left[-\frac{1}{\alpha} (t-\tau)^\alpha \right]_{\tau=0}^t \\ &= \frac{\|x\|_{\mathscr{C}}}{\alpha \Gamma(\alpha)} t^\alpha \end{aligned}$$

Therefore,

$$ ({}_0 I_t^\alpha \; x)(0) = 0. \tag{25}$$

Taking into account (25), if we set $g(t) = {}_0I_t^\alpha\, x(t)$ in Remark 4, we have:

$$\begin{aligned} {}_0^C D_t^\alpha\, {}_0I_t^\alpha\, x(t) &= {}_0D_t^\alpha\, {}_0I_t^\alpha\, x(t) \\ &= {}_0D_t^\alpha\, {}_0D_t^{-\alpha}\, x(t) \\ &= x(t) \end{aligned} \tag{26}$$

We have also:

$$\begin{aligned} \|{}_0I_t^\alpha\, f(t,x(t))\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau,x(\tau))d\tau \right\| \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^\infty (t-\tau)^{\alpha-1} \|f(\tau,x(\tau))\| d\tau \\ &\leqslant \frac{1}{\Gamma(\alpha)} \sup_{\tau\in[0,b]} \|f(\tau,x(\tau))\| \int_0^\infty (t-\tau)^{\alpha-1} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \|f(\cdot,x(\cdot))\|_{\mathscr{C}} \left[-\frac{1}{\alpha}(t-\tau)^\alpha \right]_{\tau=0}^t \\ &= \frac{\|f(\cdot,x(\cdot))\|_{\mathscr{C}}}{\alpha\Gamma(\alpha)} t^\alpha \end{aligned}$$

Therefore,

$$\left({}_0I_t^\alpha\, f(t,x(t))\right)(0) = 0. \tag{27}$$

Taking into account (27), if we set $g(t) = {}_0I_t^\alpha\, f(t,x(t))$ in Remark 4, we have:

$$\begin{aligned} {}_0^C D_t^\alpha\, {}_0I_t^\alpha\, f(t,x(t)) &= {}_0D_t^\alpha\, {}_0I_t^\alpha\, f(t,x(t)) \\ &= {}_0D_t^\alpha\, {}_0D_t^{-\alpha}\, f(t,x(t)) \\ &= f(t,x(t)) \end{aligned} \tag{28}$$

Hence substituting (26) and (28) into (24), we deduce that

$${}_0^C D_t^\alpha\, x(t) = Ax(t) + f(t,x(t)). \tag{29}$$

In addition, for $t = 0$ in (23), considering (25) and (27), we obtain:

$$x(0) = x_0. \tag{30}$$

Combining (29) and (30), we conclude that if $x \in \mathscr{C}$ satisfies the Volterra integral equation (16), then x is solution to the Cauchy problem (1). This ends the proof of Lemma 3.2. □

LEMMA 3.3. *If (16) holds, then we have the following integral equation:*

$$x(t) = Q(t)x_0 + \int_0^t (t-\tau)^{\alpha-1} R(t-\tau) f(\tau,x(\tau))d\tau \quad \textit{for } t \in [0,b],$$

where the operators $\{Q(t)\}_{t\in[0,b]}$ *and* $\{R(t)\}_{t\in[0,b]}$ *are defined by*

$$Q(t)x = \int_0^\infty \zeta_\alpha(\eta) T(t^\alpha \eta) x d\eta \quad \forall\ x \in \mathbb{X} \tag{31}$$

and

$$R(t)x = \alpha \int_0^\infty \eta \zeta_\alpha(\eta) T(t^\alpha \eta) x d\eta \quad \forall \; x \in \mathbb{X}, \tag{32}$$

where

$$\zeta_\alpha(\eta) = \frac{1}{\alpha} \eta^{-\frac{1}{\alpha}-1} \rho_\alpha \left(\eta^{-\frac{1}{\alpha}} \right) \tag{33}$$

is the probability density function defined on $(0,\infty)$*, that is,* $\zeta_\alpha(\eta) \geqslant 0$ *for* $\eta \in (0,\infty)$ *and* $\int_0^\infty \zeta_\alpha(\eta) d\eta = 1$.

Proof. Let $s > 0$. Applying the Laplace transform on both sides of (16), we have:

$$\begin{aligned} \mathscr{L}\{x(t)\}(s) &= x_0 \mathscr{L}\{1\}(s) + \frac{A}{\Gamma(\alpha)} \mathscr{L}\left\{ \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau \right\}(s) \\ &\quad + \frac{1}{\Gamma(\alpha)} \mathscr{L}\left\{ \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \right\}(s) \end{aligned}$$

which is equivalent to

$$X(s) = \frac{1}{s} x_0 + \frac{A}{\Gamma(\alpha)} \frac{(\alpha-1)!}{s^\alpha} X(s) + \frac{1}{\Gamma(\alpha)} \frac{(\alpha-1)!}{s^\alpha} F(s), \tag{34}$$

where $X(s) = \mathscr{L}\{x(t)\}(s)$ and $F(s) = \mathscr{L}\{f(t,x(t))\}(s)$.

Equation (34) is equivalent to

$$X(s) \left(\frac{s^\alpha I - A}{s^\alpha} \right) = \frac{1}{s} x_0 + \frac{1}{s^\alpha} F(s)$$

which implies that

$$\begin{aligned} X(s) &= (s^\alpha I - A)^{-1} s^{\alpha-1} x_0 + (s^\alpha I - A)^{-1} F(s) \\ &= s^{\alpha-1} \mathscr{L}\{T(\tau)\}(s^\alpha) x_0 + \mathscr{L}\{T(\tau)\}(s^\alpha) F(s) \\ &= s^{\alpha-1} \int_0^\infty e^{-s^\alpha \tau} T(\tau) x_0 d\tau + \int_0^\infty e^{-s^\alpha \tau} T(\tau) F(s) d\tau. \end{aligned} \tag{35}$$

Now we consider the following change of variable: $\tau = t^\alpha$.

Equation (35) becomes:

$$\begin{aligned} X(s) &= s^{\alpha-1} \int_0^\infty \alpha t^{\alpha-1} e^{-(st)^\alpha} T(t^\alpha) x_0 dt + \int_0^\infty \alpha t^{\alpha-1} e^{-(st)^\alpha} T(t^\alpha) F(s) dt \\ &= \int_0^\infty \alpha (st)^{\alpha-1} e^{-(st)^\alpha} T(t^\alpha) x_0 dt \\ &\quad + \int_0^\infty \alpha t^{\alpha-1} e^{-(st)^\alpha} T(t^\alpha) \left(\int_0^\infty e^{-s\tau} f(\tau, x(\tau)) d\tau \right) dt \\ &= \int_0^\infty -\frac{1}{s} T(t^\alpha) x_0 \frac{d}{dt} \left(e^{-(st)^\alpha} \right) dt \\ &\quad + \int_0^\infty \int_0^\infty \alpha t^{\alpha-1} e^{-(st)^\alpha} T(t^\alpha) e^{-s\tau} f(\tau, x(\tau)) d\tau dt. \end{aligned} \tag{36}$$

Taking into account (14), equation (36) becomes:

$$\begin{aligned}
X(s) &= \int_0^\infty -\frac{1}{s}T(t^\alpha)x_0\left(\int_0^\infty \frac{d}{dt}\left(e^{-st\sigma}\right)\rho_\alpha(\sigma)d\sigma\right)dt \\
&\quad+\int_0^\infty\int_0^\infty \alpha t^{\alpha-1}\left(\int_0^\infty e^{-st\sigma}\rho_\alpha(\sigma)d\sigma\right)T(t^\alpha)e^{-s\tau}f(\tau,x(\tau))d\tau dt \\
&= \int_0^\infty\int_0^\infty \sigma T(t^\alpha)x_0e^{-st\sigma}\rho_\alpha(\sigma)d\sigma dt \\
&\quad+\int_0^\infty\int_0^\infty\int_0^\infty \alpha t^{\alpha-1}e^{-st\sigma}\rho_\alpha(\sigma)T(t^\alpha)e^{-s\tau}f(\tau,x(\tau))d\sigma d\tau dt. \qquad (37)
\end{aligned}$$

Now we consider the change of variable $t\sigma=\theta \implies dt=\frac{1}{\sigma}d\theta$.

(37) becomes

$$\begin{aligned}
X(s) &= \int_0^\infty\int_0^\infty T\left(\frac{\theta^\alpha}{\sigma^\alpha}\right)x_0e^{-s\theta}\rho_\alpha(\sigma)d\sigma d\theta \\
&\quad+\int_0^\infty\int_0^\infty\int_0^\infty \alpha\frac{\theta^{\alpha-1}}{\sigma^\alpha}e^{-s(\theta+\tau)}\rho_\alpha(\sigma)T\left(\frac{\theta^\alpha}{\sigma^\alpha}\right)f(\tau,x(\tau))d\sigma d\tau d\theta.
\end{aligned} \qquad (38)$$

Considering the new change of variable $\theta+\tau=t$, we have:

$$\begin{cases}\theta\to 0 \implies t\to\tau\\ \theta\to\infty\implies t\to\infty\end{cases}, \quad \begin{cases}0<\tau<\infty\\ \tau<t<\infty\end{cases}\implies 0<\tau<t$$

and

$$\begin{cases}0<\tau<\infty\\ \tau<t<\infty\end{cases}\implies 0<t<\infty$$

Then, (38) becomes:

$$\begin{aligned}
X(s) &= \int_0^\infty e^{-s\theta}\left(\int_0^\infty T\left(\frac{\theta^\alpha}{\sigma^\alpha}\right)\rho_\alpha(\sigma)x_0d\sigma\right)d\theta \\
&\quad+\int_{t=0}^\infty e^{-st}\left(\int_{\sigma=0}^\infty\int_{\tau=0}^t \alpha\frac{(t-\tau)^{\alpha-1}}{\sigma^\alpha}\rho_\alpha(\sigma)T\left(\frac{(t-\tau)^\alpha}{\sigma^\alpha}\right)f(\tau,x(\tau))d\tau d\sigma\right)dt. \\
&= \int_{t=0}^\infty e^{-st}\left\{\int_{\sigma=0}^\infty T\left(\frac{t^\alpha}{\sigma^\alpha}\right)\rho_\alpha(\sigma)x_0d\sigma\right. \\
&\quad\left.+\alpha\int_{\sigma=0}^\infty\int_{\tau=0}^t \frac{(t-\tau)^{\alpha-1}}{\sigma^\alpha}\rho_\alpha(\sigma)T\left(\frac{(t-\tau)^\alpha}{\sigma^\alpha}\right)f(\tau,x(\tau))d\tau d\sigma\right\}dt. \qquad (39)
\end{aligned}$$

Applying the inverse Laplace transform on (39), we deduce that

$$\begin{aligned}
x(t) &= \int_0^\infty T\left(\frac{t^\alpha}{\sigma^\alpha}\right)\rho_\alpha(\sigma)x_0d\sigma \\
&\quad+\alpha\int_0^\infty\int_0^t \frac{(t-\tau)^{\alpha-1}}{\sigma^\alpha}\rho_\alpha(\sigma)T\left(\frac{(t-\tau)^\alpha}{\sigma^\alpha}\right)f(\tau,x(\tau))d\tau d\sigma. \qquad (40)
\end{aligned}$$

Now we consider the following change of variable:

$$\frac{1}{\sigma^\alpha}=\eta \implies \sigma=\eta^{-\frac{1}{\alpha}} \text{ and } d\sigma=-\frac{1}{\alpha}\eta^{-\frac{1}{\alpha}-1}d\eta.$$

Equation (40) becomes

$$\begin{aligned} x(t) &= \int_0^\infty T(t^\alpha\eta)\left(\frac{1}{\alpha}\eta^{-\frac{1}{\alpha}-1}\rho_\alpha\left(\eta^{-\frac{1}{\alpha}}\right)\right)x_0 d\eta \\ &\quad +\alpha\int_0^\infty\int_0^t \eta(t-\tau)^{\alpha-1}T\left((t-\tau)^\alpha\eta\right)\left(\frac{1}{\alpha}\eta^{-\frac{1}{\alpha}-1}\rho_\alpha\left(\eta^{-\frac{1}{\alpha}}\right)\right)f(\tau,x(\tau))d\tau d\eta \\ &= \int_0^\infty T(t^\alpha\eta)\zeta_\alpha(\eta)x_0 d\eta \\ &\quad +\alpha\int_0^\infty\int_0^t \eta(t-\tau)^{\alpha-1}T\left((t-\tau)^\alpha\eta\right)\zeta_\alpha(\eta)f(\tau,x(\tau))d\tau d\eta, \end{aligned}$$

where

$$\zeta_\alpha(\eta) = \frac{1}{\alpha}\eta^{-\frac{1}{\alpha}-1}\rho_\alpha\left(\eta^{-\frac{1}{\alpha}}\right), \quad \eta\in(0,\infty).$$

Therefore,

$$x(t)=Q(t)x_0+\int_0^t(t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau \quad \text{for } t\in[0,b], \tag{41}$$

where the operators $\{Q(t)\}_{t\in[0,b]}$ and $\{R(t)\}_{t\in[0,b]}$ are defined by (31) and (32) and the proof of Lemma 3.3 is complete. □

Motivated by Lemma 3.3, we give the following definition of the mild solution of the Cauchy problem (1).

DEFINITION 3.1. A function $x\in\mathscr{C}$ is said to be a mild solution to the initial value problem (1) if x satisfies

$$x(t)=Q(t)x_0+\int_0^t(t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau, \quad \text{for } t\in[0,b], \tag{42}$$

where the operators $\{Q(t)\}_{t\in[0,b]}$ and $\{R(t)\}_{t\in[0,b]}$ are defined by (31) and (32) respectively.

LEMMA 3.4. *For any fixed $t\in[0,b]$, $Q(t)$ and $R(t)$ are linear bounded operators.*

Proof. For any fixed $t\in[0,b]$, since $T(t)$ is a linear operator, we can easily see that $Q(t)$ and $R(t)$ are also linear operators.

Furthermore, for any $0\leqslant\delta\leqslant 1$ we have the following:

$$\begin{aligned} \int_0^\infty \eta^\delta\zeta_\alpha(\eta)d\eta &= \int_0^\infty \eta^\delta\frac{1}{\alpha}\eta^{-\frac{1}{\alpha}-1}\rho_\alpha\left(\eta^{-\frac{1}{\alpha}}\right)d\eta \\ &= \int_0^\infty \frac{1}{\alpha}\eta^{-\frac{1}{\alpha}-1+\delta}\rho_\alpha\left(\eta^{-\frac{1}{\alpha}}\right)d\eta. \end{aligned} \tag{43}$$

Now we consider the following change of variable:

$$\sigma = \eta^{-\frac{1}{\alpha}} \implies \eta = \sigma^{-\alpha} \implies d\eta = -\alpha\sigma^{-\alpha-1}d\sigma.$$

So, equation (43) becomes:

$$\begin{aligned}\int_0^\infty \eta^\delta \zeta_\alpha(\eta)d\eta &= \int_0^\infty \sigma^{-\alpha\delta}\rho_\alpha(\sigma)d\sigma \\ &= \int_0^\infty \frac{1}{\sigma^{\alpha\delta}}\rho_\alpha(\sigma)d\sigma. \end{aligned} \tag{44}$$

But $\begin{cases} 0 \leqslant \delta \leqslant 1 \\ 0 < \alpha \leqslant 1 \end{cases} \implies 0 \leqslant \alpha\delta \leqslant 1.$

Then, from (15), we deduce that

$$\int_0^\infty \eta^\delta \zeta_\alpha(\eta)d\eta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}.$$

In particular, for $\delta = 1$, we deduce the following:

$$\int_0^\infty \eta\zeta_\alpha(\eta)d\eta = \frac{1}{\Gamma(\alpha+1)}. \tag{45}$$

For any $x \in \mathbb{X}$, we have:

$$\begin{aligned} ||Q(t)x|| &= \left|\left|\int_0^\infty \zeta_\alpha(\eta)T(t^\alpha\eta)xd\eta\right|\right| \\ &\leqslant \underbrace{\sup_{\tau\in[0,\infty)} \|T(\tau)\|}_{=M} \|x\| \underbrace{\int_0^\infty \zeta_\alpha(\eta)d\eta}_{=1} \\ &= M\|x\| \end{aligned}$$

$$\implies \frac{||Q(t)x||}{\|x\|} \leqslant M$$

which implies that

$$\|Q(t)\|_{\mathscr{B}(\mathbb{X})} \leqslant M. \tag{46}$$

In addition, we have for any $x \in \mathbb{X}$ the following:

$$\begin{aligned} ||R(t)x|| &= \left|\left|\alpha\int_0^\infty \eta\zeta_\alpha(\eta)T(t^\alpha\eta)xd\eta\right|\right| \\ &\leqslant \alpha \underbrace{\sup_{\tau\in[0,\infty)} \|T(\tau)\|}_{=M} \|x\| \underbrace{\int_0^\infty \eta\zeta_\alpha(\eta)d\eta}_{=\frac{1}{\Gamma(\alpha+1)}} \\ &= \frac{\alpha M}{\Gamma(\alpha+1)}\|x\| \end{aligned}$$

$$\implies \frac{||R(t)x||}{\|x\|} \leqslant \frac{\alpha M}{\Gamma(\alpha+1)}$$

which implies that

$$\|R(t)\|_{\mathscr{B}(\mathbb{X})} \leqslant \frac{\alpha M}{\Gamma(\alpha+1)}. \tag{47}$$

From inequalities (46) and (47), we deduce that the operators $\{Q(t)\}_{t\in[0,b]}$ and $\{R(t)\}_{t\in[0,b]}$ are linear and bounded. □

We assume that

H_1 : $f(t,x)$ is of Caratheodory; that is, for any $x\in\mathbb{X}$, $f(t,x)$ is strongly measurable with respect to $t\in[0,b]$ and for any $t\in[0,b]$, $f(t,x)$ is continuous with respect to $x\in\mathbb{X}$,

H_2 : $\exists\, L>0\ :\ \|f(t,x)-f(t,y)\|\leqslant L\|x-y\|\ \ \forall\ x,y\in\mathbb{X},\ \ \forall\ t\in[0,b]$.

Let us consider the operator G from $\mathscr{C}$ to $\mathscr{C}$ defined by:

$$Gx(t) \ = \ Q(t)x_0+\int_0^t (t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau \quad \text{for } \ t\in[0,b]. \tag{48}$$

By Lemma 3.4, our hypothesis H_1 and the fact that a function f is strongly measurable if $\|f\|$ is Lebesgue integrable, we deduce that G is well defined on $\mathscr{C}$.

THEOREM 3.1. *Under the assumptions* $(H_1)-(H_2)$*, the Cauchy problem* (1) *has a unique mild solution provided that the constant*

$$\Omega_\alpha = \frac{MLb^\alpha}{\Gamma(\alpha+1)}$$

satisfies

$$0\leqslant\Omega_\alpha<1. \tag{49}$$

Proof. Consider $x,y\in\mathscr{C}$ and let $t\in[0,b]$. We have:

$$\begin{aligned}
&\|Gx(t)-Gy(t)\|\\
&=\left\|\int_0^t (t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau-\int_0^t (t-\tau)^{\alpha-1}R(t-\tau)f(\tau,y(\tau))d\tau\right\|\\
&\leqslant \int_0^t (t-\tau)^{\alpha-1}\|R(t-\tau)\|_{\mathscr{B}(\mathbb{X})}\|f(\tau,x(\tau))-f(\tau,y(\tau))\|d\tau\\
&\leqslant \int_0^t (t-\tau)^{\alpha-1}\|R(t-\tau)\|_{\mathscr{B}(\mathbb{X})}\|x(\tau)-y(\tau)\|d\tau
\end{aligned}$$

which implies from Lemma 3.4 that

$$\begin{aligned}
\|Gx(t)-Gy(t)\| &\leqslant \frac{\alpha ML}{\Gamma(\alpha+1)}\sup_{\tau\in[0,b]}\|x(\tau)-y(\tau)\|\int_0^t (t-\tau)^{\alpha-1}d\tau \\
&= \frac{\alpha ML}{\Gamma(\alpha+1)}\sup_{\tau\in[0,b]}\|x(\tau)-y(\tau)\|\left[-\frac{1}{\alpha}(t-\tau)^{\alpha}\right]_{\tau=0}^{t} \\
&= \frac{\alpha ML}{\Gamma(\alpha+1)}\|x-y\|_{\mathscr{C}}\frac{t^{\alpha}}{\alpha} \\
&\leqslant \frac{MLb^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|_{\mathscr{C}} \\
&= \Omega_{\alpha}\|x-y\|_{\mathscr{C}}
\end{aligned}$$

which implies that

$$\sup_{t\in[0,b]}\|Gx(t)-Gy(t)\| \leqslant \Omega_{\alpha}\|x-y\|_{\mathscr{C}};$$

that is,

$$\|Gx-Gy\|_{\mathscr{C}} \leqslant \Omega_{\alpha}\|x-y\|_{\mathscr{C}}. \tag{50}$$

Hence, taking into account the condition (49) and the inequality (50), we deduce by the Banach's contraction principle (Lemma 3.1) that $\mathscr{C}$ has a unique fixed point $x\in\mathscr{C}$, and

$$x(t) = Q(t)x_0+\int_0^t (t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau \quad \text{for } t\in[0,b], \tag{51}$$

which is the mild solution of (1). □

4. Stability

So far we have been concerned with the existence and uniqueness of the mild solution to the Cauchy problem (1), and we have quoted one theorem which guarantees that there exists a unique solution for any $t\in[0,b]$ and all initial states within $\mathbb{X}$. In this section, we aim to find the conditions under which the system (1) is stable.

We begin by giving the definition of exponential stability and introducing a Lemma which will be used thereafter.

DEFINITION 4.1. System (1) is said to be exponentially stable if for every initial state $x_0\in\mathbb{X}$, there exist two constants $K>0$ and $\omega>0$ such that

$$\|x(t)\|\leqslant Ke^{-\omega t}\|x_0\| \quad \forall\, t\in[0,b].$$

LEMMA 4.1. *Let $u(t)$ be a continuous function which, for $t>t_0$, satisfies the inequality*

$$0<u(t)<k+\int_{t_0}^t (l+\beta u(t))\,dt,$$

where k, l *and* β *are constants such that* $k,l \geqslant 0$ *and* $\beta > 0$. *Then the following inequality holds*

$$u(t) < \frac{l}{\beta}\left(e^{\beta(t-t_0)} - 1\right) + ke^{\beta(t-t_0)} \quad \forall\, t > t_0.$$

Proof. The proof of this lemma can be found in [2], p. 16. □

From Section 3, we easily deduce that the Cauchy operator of the equation

$$ {}_0^C D_t^{\alpha} x(t) = Ax(t) \tag{52}$$

subject to the initial condition

$$x(0) = x_0 \tag{53}$$

is given by:

$$Q(t)x = \int_0^{\infty} \zeta_\alpha(\eta) T(t^\alpha \eta) x d\eta \quad \forall\, x \in \mathbb{X}. \tag{54}$$

Let us suppose that the inequality

$$\|T(t^\alpha \eta)\| \leqslant \tilde{\mathscr{M}} e^{-\tilde{\mu} t} \quad \forall\, t \in [0,b], \ \forall\, \eta \in (0,\infty) \tag{55}$$

holds, where $\tilde{\mu}$ and $\tilde{\mathscr{M}}$ are strictly positive constants.

We have from (54) and (55), the following:

$$\begin{aligned} \|Q(t)\|_{\mathscr{B}(\mathbb{X})} &\leqslant \int_0^{\infty} \zeta_\alpha(\eta) \|T(t^\alpha \eta)\| d\eta \\ &\leqslant \int_0^{\infty} \zeta_\alpha(\eta) \tilde{\mathscr{M}} e^{-\tilde{\mu} t} d\eta \\ &= \tilde{\mathscr{M}} e^{-\tilde{\mu} t} \underbrace{\int_0^{\infty} \zeta_\alpha(\eta) d\eta}_{=1} \\ &= \tilde{\mathscr{M}} e^{-\tilde{\mu} t}. \end{aligned}$$

So,

$$\|Q(t)\|_{\mathscr{B}(\mathbb{X})} \leqslant \tilde{\mathscr{M}} e^{-\tilde{\mu} t} \quad \forall\ t \in [0,b]. \tag{56}$$

The following theorem establishes the stability of the solution to system (1), with respect to the first approximation, as relation (56) is the condition for exponential stability of the solution to (52)–(53).

THEOREM 4.1. *If* (55) *and the Lipschitz condition* (H_2) *are fulfilled, and if in addition the constants* $\tilde{\mu}$, $\tilde{\mathscr{M}}$ *and* L *satisfy the inequality*

$$\tilde{\mu} - \frac{L\tilde{\mathscr{M}} b^{\alpha-1}}{\Gamma(\alpha)} > 0, \tag{57}$$

then the solution of the Cauchy problem (1) *is exponentially stable.*

Proof. We have proven in the previous section that if a function $x \in \mathscr{C}$ satisfies (1), then x can be written as follow:

$$x(t) = Q(t)x_0 + \int_0^t (t-\tau)^{\alpha-1} R(t-\tau) f(\tau, x(\tau)) d\tau \quad \text{for } t \in [0,b]. \tag{58}$$

From (32) and (55), we have:

$$\begin{aligned}
\|R(t)\|_{\mathscr{B}(\mathbb{X})} &\leqslant \alpha \int_0^\infty \eta \zeta_\alpha(\eta) \|T(t^\alpha \eta)\| d\eta \\
&\leqslant \alpha \int_0^\infty \eta \zeta_\alpha(\eta) \tilde{\mathscr{M}} e^{-\tilde{\mu}t} d\eta \\
&= \alpha \tilde{\mathscr{M}} e^{-\tilde{\mu}t} \underbrace{\int_0^\infty \eta \zeta_\alpha(\eta) d\eta}_{=\frac{1}{\Gamma(\alpha+1)}} \\
&= \frac{\tilde{\mathscr{M}}}{\Gamma(\alpha)} e^{-\tilde{\mu}t}.
\end{aligned}$$

So,

$$\|R(t)\|_{\mathscr{B}(\mathbb{X})} \leqslant \frac{\tilde{\mathscr{M}}}{\Gamma(\alpha)} e^{-\tilde{\mu}t} \quad \forall\ t \in [0,b]. \tag{59}$$

In addition, the Lipschitz condition (H_2) with $y = 0$ gives for any $x \in \mathbb{X}$ and any $t \in [0,b]$, the following:

$$\|f(t,x)\| \leqslant L\|x\| \tag{60}$$

Then, by (56), (59) and (60), we obtain from (58) the estimate:

$$\begin{aligned}
\|x(t)\| &\leqslant \tilde{\mathscr{M}} e^{-\tilde{\mu}t} \|x_0\| + \frac{L\tilde{\mathscr{M}}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|x(\tau)\| e^{-\tilde{\mu}(t-\tau)} d\tau \\
&\leqslant \tilde{\mathscr{M}} e^{-\tilde{\mu}t} \|x_0\| + \frac{L\tilde{\mathscr{M}}}{\Gamma(\alpha)} \sup_{\tau \in [0,b]} |t-\tau|^{\alpha-1} \int_0^t e^{-\tilde{\mu}(t-\tau)} \|x(\tau)\| d\tau,
\end{aligned}$$

since

$$0 \leqslant \tau \leqslant t \implies -t \leqslant -\tau \leqslant 0 \implies 0 \leqslant t-\tau \leqslant t \leqslant b \implies (t-\tau)^{\alpha-1} \leqslant b^{\alpha-1};$$

we deduce:

$$\|x(t)\| \leqslant \tilde{\mathscr{M}} e^{-\tilde{\mu}t} \|x_0\| + \frac{L\tilde{\mathscr{M}} b^{\alpha-1}}{\Gamma(\alpha)} \int_0^t e^{-\tilde{\mu}(t-\tau)} \|x(\tau)\| d\tau. \tag{61}$$

Now, let us consider the function ψ below defined on $[0,b]$.

$$\psi(t) = e^{\tilde{\mu}t} \|x(t)\|. \tag{62}$$

We obtain from the inequality (61), the following:

$$\psi(t) \leqslant \tilde{\mathscr{M}}\|x_0\| + \frac{L\tilde{\mathscr{M}}b^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \psi(\tau)d\tau. \tag{63}$$

By construction, the function $\psi(t)$ is continuous on $[0,b]$. So, taking into account Lemma 4.1 we deduce from (63) that

$$\psi(t) \leqslant \tilde{\mathscr{M}}e^{\left(\frac{L\tilde{\mathscr{M}}b^{\alpha-1}}{\Gamma(\alpha)}\right)t}\|x_0\| \quad \forall\, t \in [0,b]$$

which, by (62) implies

$$\|x(t)\| \leqslant \tilde{\mathscr{M}}e^{\left(\frac{L\tilde{\mathscr{M}}b^{\alpha-1}}{\Gamma(\alpha)}-\tilde{\mu}\right)t}\|x_0\| \quad \forall\, t \in [0,b]. \tag{64}$$

Therefore, according to (57) we conclude that the solution of the Cauchy problem (1) is exponentially stable. □

5. Stabilization

Let U (control space) be a separable reflexive Banach space. Under conditions of Problem (1), we consider the semilinear control system:

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = Ax(t) + f(t,x(t)) + Bu(t),\ t \in [0,b] \\ \qquad x(0) = x_0, \end{cases} \tag{65}$$

where $B:\ U \longrightarrow \mathbb{X}$ is a bounded linear operator and $u \in U$.

As we have seen previously, the mild solution of (65) is well defined for every integrable control $u(t)$, $t \in [0,b]$ and is given by:

$$\begin{aligned} x(t) = Q(t)x_0 &+ \int_0^t (t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau \\ &+ \int_0^t (t-\tau)^{\alpha-1}R(t-\tau)Bu(\tau)d\tau \quad \forall\ t \in [0,b]. \end{aligned} \tag{66}$$

Let us assume that our stabilizer (u) acts linearly on (65); that is $u = Vx$, where $V:\ \mathbb{X} \longrightarrow U$ is a bounded linear operator. Then, (66) becomes:

$$\begin{aligned} x(t) = Q(t)x_0 &+ \int_0^t (t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau \\ &+ \int_0^t (t-\tau)^{\alpha-1}R(t-\tau)BVx(\tau)d\tau \quad \forall\ t \in [0,b]. \end{aligned} \tag{67}$$

DEFINITION 5.1. The Cauchy problem (1) is said to be stabilizable if there exists a bounded linear operator $V:\ \mathbb{X} \longrightarrow U$ satisfying $u(t) = Vx(t)$ for any $t \in [0,b]$ such that the control system (65) is stable.

The following lemma will be useful for the rest of this paper.

LEMMA 5.1. *Let* $\beta \geqslant 0$ *be a constant and* $b > 0$. *If* $v,k: [0,b] \longrightarrow [0,\infty)$ *are two bounded nonnegatives continuous functions satisfying*

$$v(t) \leqslant \beta + \int_0^t k(\tau)v(\tau)d\tau \ \ \forall \ t \in [0,b],$$

then

$$v(t) \leqslant \beta \ e^{\int_0^t k(\tau)d\tau} \ \ \forall \ t \in [0,b]$$

Proof. The proof of this lemma can be found in [11], p. 371. □

THEOREM 5.1. *If the constants* $\tilde{\mu}$, L, $\tilde{\mathscr{M}}$ *and the bounded linear operators* B, V *satisfy the inequality*

$$\tilde{\mu} - \left(L + \frac{\|BV\|_{\mathscr{B}(\mathbb{X})}}{\Gamma(\alpha)}\right)\tilde{\mathscr{M}} b^{\alpha-1} > 0 \tag{68}$$

and

$$\alpha\eta \leqslant 1 \tag{69}$$

for $\eta \in (0,\infty)$, *then the Cauchy problem* (1) *is exponentially stabilizable.*

Proof. Let us consider the following operator, for $x_0 \in \mathbb{X}$:

$$Z(t)x_0 = Q(t)x_0 + \int_0^t (t-\tau)^{\alpha-1}R(t-\tau)BVZ(\tau)x_0 d\tau \quad \forall \ t \in [0,b]. \tag{70}$$

By construction, $\{Z(t)\}_{t\in[0,b]}$ is a linear bounded operator.

Similarly to the estimation (61) in the previous section, we can easily find that:

$$\|Z(t)\|_{\mathscr{B}(\mathbb{X})} \leqslant \tilde{\mathscr{M}} e^{-\tilde{\mu}t} + \frac{\|BV\|_{\mathscr{B}(\mathbb{X})}\tilde{\mathscr{M}} b^{\alpha-1}}{\Gamma(\alpha)} \int_0^t e^{-\tilde{\mu}(t-\tau)}\|Z(\tau)\|d\tau$$

and then, proceed the same way as on pages 15–16 to deduce that

$$\|Z(t)\|_{\mathscr{B}(\mathbb{X})} \leqslant \tilde{\mathscr{M}} e^{\left(\frac{\|BV\|_{\mathscr{B}(\mathbb{X})}\tilde{\mathscr{M}} b^{\alpha-1}}{\Gamma(\alpha)} - \tilde{\mu}\right)t} \quad \forall \ t \in [0,b]. \tag{71}$$

Furthermore, for any $t \in [0,b]$ we have:

$$\begin{aligned} x(t) &= Q(t)x_0 + \int_0^t (t-\tau)^{\alpha-1}R(t-\tau)f(\tau,x(\tau))d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}R(t-\tau)BVx(\tau)d\tau + Z(t)x_0 - Z(t)x_0 \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}Z(t-\tau)f(\tau,x(\tau))d\tau - \int_0^t (t-\tau)^{\alpha-1}Z(t-\tau)f(\tau,x(\tau))d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau, x(\tau)) d\tau + Z(t)x_0 \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} R(t-\tau) BV[x(\tau) - Z(\tau)x_0] d\tau \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} [R(t-\tau) - Z(t-\tau)] f(\tau, x(\tau)) d\tau \\
&= \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau, x(\tau)) d\tau + Z(t)x_0 \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} R(t-\tau) BV[x(\tau) - Z(\tau)x_0] d\tau \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} [R(t-\tau) - Q(t-\tau)] f(\tau, x(\tau)) d\tau \\
&\quad - \int_0^t (t-\tau)^{\alpha-1} \left[\int_0^{t-\tau} (t-(\tau+r))^{\alpha-1} R(t-(\tau+r)) BVZ(r) dr \right] f(\tau, x(\tau)) d\tau
\end{aligned}
$$

Considering the change of variable $\theta = \tau + r$, we obtain:

$$
\begin{aligned}
x(t) &= \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau, x(\tau)) d\tau + Z(t)x_0 \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} R(t-\tau) BV[x(\tau) - Z(\tau)x_0] d\tau \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} \left[\int_0^{\infty} (\alpha\eta - 1) \zeta_\alpha(\eta) T((t-\tau)^\alpha \eta) d\eta \right] f(\tau, x(\tau)) d\tau \\
&\quad - \int_0^t (t-\tau)^{\alpha-1} \left[\int_\tau^t (t-\theta)^{\alpha-1} R(t-\theta) BVZ(\theta-\tau) d\theta \right] f(\tau, x(\tau)) d\tau \\
&\leqslant \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau, x(\tau)) d\tau + Z(t)x_0 \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} R(t-\tau) BV[x(\tau) - Z(\tau)x_0] d\tau \\
&\quad - \int_0^t (t-\tau)^{\alpha-1} \left[\int_\tau^t (t-\theta)^{\alpha-1} R(t-\theta) BVZ(\theta-\tau) d\theta \right] f(\tau, x(\tau)) d\tau
\end{aligned}
$$

But we know that

$$
\begin{cases} \tau \leqslant \theta \leqslant t \\ 0 \leqslant \tau \leqslant t \end{cases} \Longrightarrow 0 \leqslant \tau \leqslant \theta
$$

and

$$
\begin{cases} \tau \leqslant \theta \leqslant t \\ 0 \leqslant \tau \leqslant t \end{cases} \Longrightarrow 0 \leqslant \theta \leqslant t.
$$

So,

$$x(t) \leqslant \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau,x(\tau))d\tau + Z(t)x_0 + \int_0^t (t-\tau)^{\alpha-1} R(t-\tau)BV[x(\tau)-Z(\tau)x_0]d\tau - \int_0^t (t-\theta)^{\alpha-1} R(t-\theta)BV\left[\int_0^\theta (\theta-\tau)^{\alpha-1} Z(\theta-\tau) f(\tau,x(\tau))d\tau\right]d\theta$$

which can also be written as:

$$x(t) \leqslant \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau,x(\tau))d\tau + Z(t)x_0 + \int_0^t (t-\theta)^{\alpha-1} R(t-\theta)BV\ \left[x(\theta)-Z(\theta)x_0 - \int_0^\theta (\theta-\tau)^{\alpha-1} Z(\theta-\tau) f(\tau,x(\tau))d\tau\right]d\theta \tag{72}$$

Let us define the function Φ on $[0,b]$ as follow:

$$\Phi(t) = x(t) - Z(t)x_0 - \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau,x(\tau))d\tau. \tag{73}$$

Then, (72) is equivalent to:

$$\Phi(t) \leqslant \int_0^t (t-\theta)^{\alpha-1} R(t-\theta)BV\Phi(\theta)d\theta$$

which by (47) implies that

$$\|\Phi(t)\| \leqslant \int_0^t \frac{M\|BV\|_{\mathscr{B}(\mathbb{X})}}{\Gamma(\alpha)}(t-\theta)^{\alpha-1}\|\Phi(\theta)\|d\theta. \tag{74}$$

By Lemma 5.1, we deduce that

$$\Phi(t) = 0 \quad \forall\ t \in [0,b];$$

that is,

$$x(t) = Z(t)x_0 + \int_0^t (t-\tau)^{\alpha-1} Z(t-\tau) f(\tau,x(\tau))d\tau. \tag{75}$$

Taking into account (60) and (71), we have the following estimate:

$$\|x(t)\| \leqslant \tilde{\mathscr{M}} e^{\left(\frac{\|BV\|_{\mathscr{B}(\mathbb{X})}\tilde{\mathscr{M}} b^{\alpha-1}}{\Gamma(\alpha)} - \tilde{\mu}\right)t}\|x_0\| + L\tilde{\mathscr{M}} b^{\alpha-1}\int_0^t e^{\left(\frac{\|BV\|_{\mathscr{B}(\mathbb{X})}\tilde{\mathscr{M}} b^{\alpha-1}}{\Gamma(\alpha)} - \tilde{\mu}\right)(t-\tau)}\|x(\tau)\|d\tau.$$

Now, let us consider the function Ψ below, defined on $[0,b]$:

$$\Psi(t) = e^{\left(\tilde{\mu} - \frac{\|BV\|_{\mathscr{B}(\mathbb{X})}\tilde{\mathscr{M}}b^{\alpha-1}}{\Gamma(\alpha)}\right)t}\|x(t)\|. \tag{76}$$

From (75), we obtain:

$$\Psi(t) \leqslant \tilde{\mathscr{M}}\|x_0\| + L\tilde{\mathscr{M}}b^{\alpha-1}\int_0^t \Psi(\tau)d\tau. \tag{77}$$

By construction, the function $t \longmapsto \Psi(t)$ is continuous on $[0,b]$. So, using Lemma 4.1 we deduce from (76) that

$$\Psi(t) \leqslant \tilde{\mathscr{M}}e^{L\tilde{\mathscr{M}}b^{\alpha-1}t}\|x_0\| \quad \forall\ t \in [0,b]$$

which, by (76) implies:

$$\|x(t)\| \leqslant \tilde{\mathscr{M}}e^{\left(L\tilde{\mathscr{M}}b^{\alpha-1} + \frac{\|BV\|_{\mathscr{B}(\mathbb{X})}\tilde{\mathscr{M}}b^{\alpha-1}}{\Gamma(\alpha)} - \tilde{\mu}\right)t}\|x_0\| \quad \forall\ t \in [0,b];$$

that is,

$$\|x(t)\| \leqslant \tilde{\mathscr{M}}e^{\left(\left[L + \frac{\|BV\|_{\mathscr{B}(\mathbb{X})}}{\Gamma(\alpha)}\right]\tilde{\mathscr{M}}b^{\alpha-1} - \tilde{\mu}\right)t}\|x_0\| \quad \forall\ t \in [0,b]. \tag{78}$$

By (68) we conclude that the initial value problem (1) is exponentially stabilizable. □

6. Example

We consider the following semilinear Cauchy problem:

$$\begin{cases} {}_0^C D_t^{\frac{9}{10}} x(t) = Ax(t) + f(t, x(t)),\ t \in [0,1],\ x \in l^\infty(\mathbb{N}) \\ \\ \qquad x(0) = x_0, \end{cases} \tag{79}$$

where $\mathbb{X} = l^\infty(\mathbb{N})$ is the space of bounded sequences with the norm

$$\|(x_1, x_2, x_3, \cdots)\|_\infty = \sup_{i\in\mathbb{N}} |x_i|, \tag{80}$$

$A = (a_{ij})_{i,j=1}^\infty$ is an infinite matrix defined from $l^\infty(\mathbb{N})$ to itself by $(Ax)_i = \sum_{j=1}^\infty a_{ij}x_j$ such that for any $x \in l^\infty(\mathbb{N})$,

$$\sum_{j=1}^\infty |a_{ij}| < \infty \quad \forall\ i \in \mathbb{N} \tag{81}$$

and

$$\sup_{i\in\mathbb{N}} \left\{ \sum_{j=1}^\infty |a_{ij}| \right\} < \infty, \tag{82}$$

f is a nonlinear function defined on $[0,1]\times l^{\infty}(\mathbb{N})$ by

$$f(t,x)=\frac{\sin(\|x\|_{\infty})}{3e^{t^2}}. \tag{83}$$

From (81) and (82), A is a bounded linear operator on $l^{\infty}(\mathbb{N})$ and its norm is given by:

$$\|A\|_{\infty}=\sup_{i\in\mathbb{N}}\left\{\sum_{j=1}^{\infty}|a_{ij}|\right\}.$$

Let us define T by:

$$T(t)=e^{tA}=\sum_{k=0}^{\infty}\frac{(tA)^k}{k!}\quad \forall\, t\geqslant 0 \tag{84}$$

such that

$$\|T(t)\|_{\infty}\leqslant\frac{5}{2}e^{-t}\quad \forall\ t\geqslant 0. \tag{85}$$

Then, A is clearly a generator of the C_0-semigroup $\{T(t)\}_{t\geqslant 0}$.

Furthermore, $0\leqslant t\leqslant 1\Longrightarrow 1\leqslant e^{t^2}\leqslant e$ and we have:

$$\begin{aligned}\|f(t,x)\|_{\infty} &\leqslant \frac{1}{3}|\sin(\|x\|_{\infty})|\\ &=\frac{1}{3}|\sin(\|x\|_{\infty})-\sin(0)|\end{aligned}$$

By Mean Value Theorem, there exists $m\in(0,\|x\|_{\infty})$ such that:

$$|\sin(\|x\|_{\infty})-\sin(0)|\leqslant|\cos(m)|\|x\|_{\infty}.$$

So,

$$\|f(t,x)\|_{\infty}\ \leqslant\ \frac{1}{3}\|x\|_{\infty}. \tag{86}$$

In addition, we have for all $x,y\in l^{\infty}(\mathbb{N})$,

$$\|f(t,x)-f(t,y)\|_{\infty}\ \leqslant\ \frac{1}{3}|\sin(\|x\|_{\infty})-\sin(\|y\|_{\infty})|.$$

By Mean Value Theorem, there exists $r\in(\|x\|_{\infty},\|y\|_{\infty})$ such that

$$|\sin(\|x\|_{\infty})-\sin(\|y\|_{\infty})|\leqslant|\cos(r)|\,|\|x\|_{\infty}-\|y\|_{\infty}|$$

which by reverse triangular inequality gives:

$$\|f(t,x)-f(t,y)\|_{\infty}\ \leqslant\ \frac{1}{3}\|x-y\|_{\infty}. \tag{87}$$

From (86) and (87) we deduce that the hypothesis H_1 and H_2 are satisfied with $L=\dfrac{1}{3}$.

We have also the following:

$$0 \leqslant t \leqslant 1 \Longrightarrow -1 \leqslant -t \leqslant 0 \Longrightarrow e^{-1} \leqslant e^{-t} \leqslant 1.$$

So, from (85) we deduce

$$\|T(t)\|_\infty \leqslant \frac{5}{2} = M \tag{88}$$

and therefore,

$$\Omega_{\frac{9}{10}} = \frac{5}{3 \times 2 \times \frac{9}{10}\Gamma(\frac{9}{10})} \approx 0.8665 < 1.$$

We then conclude by Theorem 3.1 that the initial value problem (79) has a unique mild solution on $[0,1]$.

Furthermore, from (85) we have $\tilde{\mu} = 1$ and $\tilde{\mathscr{M}} = \frac{5}{2}$. So,

$$1 - \frac{5}{3 \times 2 \times \Gamma(\frac{9}{10})} \approx 0.2205 > 0.$$

Hence, according to Theorem 4.1, the solution to problem (79) is exponentially stable.

7. Conclusion

We used the Banach contraction principle to prove the existence and uniqueness of mild solution (which we constructed based on the Laplace transform) to the semilinear fractional differential equation (1) in an infinite Banach space with Caputo derivative order $\alpha \in (0,1]$. By means of the Gronwall lemma, we have also proven under some conditions which we clearly specified that the latter system is exponentially stable, and also exponentially stabilizable when the control acts linearly on the system. Finally, we have provided an example to illustrate our approach.

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