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ON NONLINEAR VOLTERRA–FREDHOLM TYPE DISCRETE FRACTIONAL SUM INEQUALITIES

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Abstract. In the current paper, we establish some new forms of the discrete fractional sum inequalities of the Volterra-Fredholm type. These inequalities can serve as a powerful tool in the analysis of boundedness and uniqueness of the solutions of certain Volterra-Fredholm fractional sum-difference equations and their variants. Some applications are also given to convey the importance of our results.

1. Introduction

It is widely familiar that in the theory of difference, differential, integral and fractional differential equations, Gronwall-Bellman type inequalities, and their various extensions and generalizations have attained a huge scale celebration. In recent years, several mathematicians have taken appreciable efforts to produce the theory of fractional calculus for the real functions, which have diverse applications in many physical processes. Today Riemann-Liouville and Caputo derivatives are used commonly, though this theory has passed through various stages of a variety of definitions of fractional derivatives (for example, see [28]).

It is noteworthy that an immense development in the theory of difference, integral, and differential equations have taken place as a consequence of the number of inequalities established to solve and analyze these mentioned classes of equations (for example, see [1, 5, 6, 7, 8, 9, 10, 11, 16]). Similarly, the theory of discrete fractional calculus is heading towards the peak of development in the past few decades (for example, see [2, 3, 4, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and references cited therein).

Recently, F. M. Atici (see [12, 13, 14, 15]) developed various theoretical advancements of the fractional difference, before which several definitions were initiated by Diaz and Olser [26], Gray and Zhang [27], Miller and Ross [24]. These advancements have given rise to various interesting discrete fractional sum inequalities applicable to several problems in the discrete fractional calculus.

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In this paper, we extend and generalize some of the discrete fractional sum inequalities of the Volterra-Fredholm type developed by Liu and Meng [2]. We also provide some applications to show the importance of our results.

In subsequent discussion, we consider $N_\omega = \{\omega, \omega+1, \omega+2, \ldots\}$ and for $m, M \in N_\omega$, $I_m = [m, M] \cap N_\omega$. We denote the collection of all continuously differentiable functions from X to Y by $C(X,Y)$ and $\mathscr{F}_+(\mathscr{U})$ stands for the set of all nonnegative real valued functions on $\mathscr{U}$. In usual notations, we let, $\mathbb{R}_+ = [0, \infty)$ and for $t < s$, $\sum_{n=s}^{t} a(n) = 0$ and $\prod_{n=s}^{t} a(n) = 1$.

DEFINITION 1. (F. M. Atici [12]) Let a be any real number, ω be any positive real number, and $\sigma(s) = s+1$. Let $f : N_a \to \mathbb{R}$, then ω-th fractional sum (ω-sum) of f is defined by

$$\Delta_a^{-\omega} f(t) = \frac{1}{\Gamma(\omega)} \sum_{s=a}^{t-\omega} (t - \sigma(s))^{(\omega-1)} f(s),$$

where $\Delta_a^{-\omega} f$ is a map which assigns to each function defined on N_a, a function defined on $N_{a+\omega}$, and the falling factorial $t^{(\omega)}$ is defined as $t^{(\omega)} = \frac{\Gamma(t+1)}{\Gamma(t-\omega+1)}$.

DEFINITION 2. (F. M. Atici [12]) The μ-th fractional difference is defined as

$$\Delta^\mu x(t) = \Delta^{m-v} x(t) = \Delta^m \left(\Delta^{-v} x(t) \right),$$

where $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, and $-v = \mu - m$.

THEOREM 1. (Haidong Liu and Fanwei Meng [2]) *Assume that* $0 < \alpha \leqslant 1$ *is a constant,* $u : N_{\alpha-1} \to \mathbb{R}_+$, $f, g : N_0 \to \mathbb{R}_+$ *are functions,* $c \geqslant 0$ *is a constant, and* $p > q > 0$ *are constants. Suppose that* u *satisfies*

$$\begin{aligned} u^p(n) \leqslant & c + \Delta_0^{-\alpha} [f(n) u^q(n+\alpha-1)] \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T-\alpha} (T-s-1)^{(\alpha-1)} g(s) u^p(s+\alpha-1), \quad n \in I_{\alpha-1}. \end{aligned}$$

If

$$\lambda = 2^{\frac{q}{p-q}} \sum_{s=0}^{T-\alpha} G(s,T) < 1,$$

then

$$u(n) \leqslant \left[A^{\frac{p-q}{p}}(T) + \frac{p-q}{p} \sum_{s=\alpha}^{n} f(s-\alpha) \right]^{\frac{1}{p-q}}, \quad n \in I_{\alpha-1},$$

where

$$A(T)=\frac{1}{1-\lambda}\left\{k+2^{\frac{q}{p-q}}\sum_{s=0}^{T-\alpha}G(s,T)\left[\frac{p-q}{p}\sum_{\tau=\alpha}^{s+\alpha-1}f(\tau-\alpha)\right]^{\frac{p}{p-q}}\right\},$$

$$G(s,n)=\frac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}g(s).$$

LEMMA 1. *If $a\geqslant 0, b\geqslant 0$ and $\nu\geqslant 1$, then*

$$(a+b)^{\nu}\leqslant 2^{\nu-1}(a^{\nu}+b^{\nu}).$$

2. Main results

THEOREM 2. *Suppose $0<\omega\leqslant 1$, $c\in\mathbb{R}_+$ are constants and let $x\in\mathscr{F}_+(N_{\omega-1})$, $f_1,f_2\in\mathscr{F}_+(N_0)$ be any functions. Let $M\in N_{\omega-1}$ be a constant. Let $h\in C(\mathbb{R}_+,\mathbb{R}_+)$ be continuous nondecreasing function such that $h(m)>0$ for $m>0$. If, for $p>q>0$,*

$$x^p(t)\leqslant c+h\Bigg(\frac{1}{\Gamma(\omega)}\sum_{n=0}^{t-\omega}(t-n-1)^{(\omega-1)}f_1(s)x^q(n+\omega-1)+\frac{1}{\Gamma(\omega)}\sum_{n=0}^{M-\omega}(M-n-1)^{(\omega-1)}f_2(s)x^q(n+\omega-1)\Bigg),\ t\in I_{\omega-1},\quad (1)$$

then

$$x(t)\leqslant\left\{c+h\left(\Phi^{-1}\left[\Phi\left(\mathscr{S}^{-1}\left[\sum_{n=\omega}^{M}f(n-\omega)\right]\right)+\sum_{n=\omega}^{t}f(n-\omega)\right]\right)\right\}^{\frac{1}{p}},\ t\in I_{\omega-1},\quad (2)$$

where $f\in\mathscr{F}_+(N_0)$ is such that it is greater than or equal to both f_1 and f_2, $\Phi(r)=\int_{r_0}^{r}\frac{ds}{(c+h(s))^{\frac{q}{p}}}$, $r\geqslant r_0>0$, $\mathscr{S}(r)=\Phi(2r)-\Phi(r)$ is strictly increasing function and Φ^{-1}, $\mathscr{S}^{-1}$ are inverse functions of $\Phi,\mathscr{S}$ respectively.

Proof. Consider the case, $c>0$. Making use of suppositions on f_1, f_2 in (1), we get

$$\begin{aligned}x^p(t)&\leqslant c+h\Bigg(\frac{1}{\Gamma(\omega)}\sum_{n=0}^{t-\omega}(t-n-1)^{(\omega-1)}f(s)x^q(n+\omega-1)+\frac{1}{\Gamma(\omega)}\sum_{n=0}^{M-\omega}(M-n-1)^{(\omega-1)}f(s)x^q(n+\omega-1)\Bigg)\\&=c+h\Bigg(\sum_{n=0}^{t-\omega}J(n,t)x^q(n+\omega-1)+\sum_{n=0}^{M-\omega}J(n,M)x^q(n+\omega-1)\Bigg),\ t\in I_{\omega-1},\quad (3)\end{aligned}$$

where

$$J(n,t) = \frac{1}{\Gamma(\omega)}(t-n-1)^{(\omega-1)}f(s).$$

Let

$$y(t) = \sum_{n=0}^{t-\omega} J(n,t)x^q(n+\omega-1) + \sum_{n=0}^{M-\omega} J(n,M)x^q(n+\omega-1),\ t \in I_{\omega-1}. \tag{4}$$

It is easy to observe that $y(t) \geqslant 0$ is nondecreasing. Further

$$y(\omega-1) = \sum_{n=0}^{M-\omega} J(n,M)x^q(n+\omega-1), \tag{5}$$

and

$$x^p(t) \leqslant c + h(y(t)),\ t \in I_{\omega-1}. \tag{6}$$

We can easily see that $J(n,t)$ is decreasing function in t for fixed n in N_0, hence for $t \in I_\omega$,

$$\begin{aligned}
\Delta y(t-1) &= y(t) - y(t-1) \\
&= J(t-\omega,t)x^q(t-1) + \sum_{n=0}^{t-\omega-1}(J(n,t)-J(n,t-1))x^q(n+\omega-1) \\
&\leqslant J(t-\omega,t)x^q(t-1) \\
&\leqslant J(t-\omega,t)\Big(c+h(y(t-1))\Big)^{\frac{q}{p}} \\
&= f(t-\omega)\Big(c+h(y(t-1))\Big)^{\frac{q}{p}}.
\end{aligned} \tag{7}$$

Further, since $h, y(t)$ are nondecreasing in nature and $\frac{q}{p} > 1$, we have

$$\Big(c+h(y(t-1))\Big)^{\frac{q}{p}} \geqslant \Big(c+h(y(\omega-1))\Big)^{\frac{q}{p}} > 0,\ \ t \in I_\omega. \tag{8}$$

Thus

$$\frac{\Delta y(t-1)}{\Big(c+h(y(t-1))\Big)^{\frac{q}{p}}} \leqslant f(t-\omega). \tag{9}$$

Now, from the definition of Φ, we get

$$\Phi(y(t)) - \Phi(y(t-1)) = \int_{y(t-1)}^{y(t)} \frac{ds}{(c+h(s))^{\frac{q}{p}}} \leqslant \frac{\Delta y(t-1)}{\Big(c+h(y(t-1))\Big)^{\frac{q}{p}}} \leqslant f(t-\omega). \tag{10}$$

Set $t=n$ in (10) and sum it over n from ω to $t-1$ to get

$$\Phi(y(t-1)) \leqslant \Phi(y(\omega-1)) + \sum_{n=\omega}^{t-1} f(n-\omega),\ t \in I_{\omega}. \tag{11}$$

As Φ is an increasing function, from (11), we obtain

$$y(t-1) \leqslant \Phi^{-1}\left[\Phi(y(\omega-1)) + \sum_{n=\omega}^{t-1} f(n-\omega)\right],\ t \in I_{\omega}. \tag{12}$$

The inequality (12) can be restructured as

$$y(t) \leqslant \Phi^{-1}\left[\Phi(y(\omega-1)) + \sum_{n=\omega}^{t} f(n-\omega)\right],\ t \in I_{\omega-1}. \tag{13}$$

From (5) and the definition of $y(t)$, we conclude that

$$2y(\omega-1) = y(M) \leqslant \Phi^{-1}\left[\Phi(y(\omega-1)) + \sum_{n=\omega}^{M} f(n-\omega)\right]. \tag{14}$$

This gives us

$$\Phi\Big(2y(\omega-1)\Big) - \Phi(y(\omega-1)) \leqslant \sum_{n=\omega}^{M} f(n-\omega). \tag{15}$$

But since $\mathscr{S}(t) = \Phi(2t) - \Phi(t)$ is a strictly increasing function, we have

$$y(\omega-1) \leqslant \mathscr{S}^{-1}\left[\sum_{n=\omega}^{M} f(n-\omega)\right]. \tag{16}$$

Using (16) in (13), we arrive to the conclusion that

$$y(t) \leqslant \Phi^{-1}\left[\Phi\left(\mathscr{S}^{-1}\left[\sum_{n=\omega}^{M} f(n-\omega)\right]\right) + \sum_{n=\omega}^{t} f(n-\omega)\right],\ t \in I_{\omega-1}. \tag{17}$$

A simple substitution of the bound obtained in (17) in inequality (6) gives us the required result in (2). The case of $c=0$ can be handled in a parallel way by taking $c+\varepsilon,\ \ \varepsilon>0$, followed by limit as $\varepsilon \to 0$. This completes the proof of our theorem. □

THEOREM 3. *Suppose that* ω, x, p, M *and* q *are as defined in Theorem 2 and assume that* $1 \leqslant \tilde{c} \in \mathbb{R}_+, f, \tilde{f}, g, \tilde{g} \in \mathscr{F}_+(N_0)$. *If*

$$\begin{aligned} x^p(t) \leqslant \tilde{c} &+ \frac{1}{\Gamma(\omega)} \sum_{n=0}^{t-\omega} (t-n-1)^{(\omega-1)}\Big[f(n)x^q(n+\omega-1) + \tilde{f}(n)\Big] \\ &+ \frac{1}{\Gamma(\omega)} \sum_{n=0}^{M-\omega} (M-n-1)^{(\omega-1)}\Big[g(n)x^p(n+\omega-1) + \tilde{g}(n)\Big],\ t \in I_{\omega-1}, \end{aligned} \tag{18}$$

then

$$x(t) \leqslant \left\{ \mathscr{H}^{\frac{p-q}{p}}(M) + \frac{p-q}{p} \sum_{n=\omega}^{t} \Big(f(n-\omega) + \tilde{f}(n-\omega) \Big) \right\}^{\frac{1}{p-q}}, \ t \in I_{\omega-1}, \tag{19}$$

where

$$\mathscr{H}(M) = \frac{1}{1-\mu} \Bigg\{ \tilde{c} + \sum_{n=0}^{M-\omega} J(n,M) \Bigg[g(s) 2^{\frac{q}{p-q}} \Bigg\{ \frac{p-q}{p} \sum_{m=\omega}^{n+\omega-1} \Big(f(m-\omega) + \tilde{f}(m-\omega) \Big) \Bigg\}^{\frac{p}{p-q}} + \tilde{g}(s) \Bigg] \Bigg\},$$

and

$$\mu = \sum_{n=0}^{M-\omega} J(n,M) g(s) 2^{\frac{q}{p-q}} < 1.$$

Proof. If we let

$$\hat{J}(n,t) = \frac{1}{\Gamma(\omega)} (t-n-1)^{(\omega-1)}, \tag{20}$$

then inequality (18) can be rewritten as

$$x^p(t) \leqslant \tilde{c} + \sum_{n=0}^{t-\omega} \hat{J}(n,t) \Big[f(n) x^q(n+\omega-1) + \tilde{f}(n) \Big] + \sum_{n=0}^{M-\omega} \hat{J}(n,M) \Big[g(n) x^p(n+\omega-1) + \tilde{g}(n) \Big], \ t \in I_{\omega-1}. \tag{21}$$

Define a function $y(t)$ as

$$y(t) = \tilde{c} + \sum_{n=0}^{t-\omega} \hat{J}(n,t) \Big[f(n) x^q(n+\omega-1) + \tilde{f}(n) \Big] + \sum_{n=0}^{M-\omega} \hat{J}(n,M) \Big[g(n) x^p(n+\omega-1) + \tilde{g}(n) \Big], \ t \in I_{\omega-1}. \tag{22}$$

It is obvious that $y(t) \geqslant 1$ and it is a nondecreasing function. Thus from (18), we have

$$x^p(t) \leqslant y(t), \ t \in I_{\omega-1}, \tag{23}$$

and

$$y(\omega-1) = \tilde{c} + \sum_{n=0}^{M-\omega} \hat{J}(n,M) \Big[g(s) y(n+\omega-1) + \tilde{g}(s) \Big]. \tag{24}$$

Using the definition of $t^{(\omega)}$, we can easily conclude that $\hat{J}(n,t)$ is a decreasing function in t for each n in N_0. Hence, by simple calculations along with the fact $1 \leqslant y(t)$, we have for $t \in I_\omega$,

$$\begin{aligned}
\Delta y(t-1) &= y(t) - y(t-1) \\
&= \hat{J}(t-\omega,t)\Big[f(t-\omega)x^q(t-1) + \tilde{f}(t-\omega)\Big] \\
&\quad + \sum_{n=0}^{t-\omega-1}\Big\{\hat{J}(n,t) - \hat{J}(n,t-1)\Big\}\Big[f(n)x^q(n+\omega-1) + \tilde{f}(n)\Big] \\
&\leqslant \hat{J}(t-\omega,t)\Big[f(t-\omega)x^q(t-1) + \tilde{f}(t-\omega)\Big] \\
&\leqslant \hat{J}(t-\omega,t)\Big[f(t-\omega)y^{\frac{q}{p}}(t-1) + \tilde{f}(t-\omega)\Big] \\
&= f(t-\omega)y^{\frac{q}{p}}(t-1) + \tilde{f}(t-\omega) \\
&\leqslant \Big[f(t-\omega) + \tilde{f}(t-\omega)\Big]y^{\frac{q}{p}}(t-1).
\end{aligned} \tag{25}$$

Further, it is easy to notice that $y^{\frac{q}{p}}(t-1) \geqslant y^{\frac{q}{p}}(\omega-1) > 0$, so we get

$$\frac{\Delta y(t-1)}{y^{\frac{q}{p}}(t-1)} \leqslant f(t-\omega) + \tilde{f}(t-\omega). \tag{26}$$

An application of mean value theorem provides us

$$\begin{aligned}
\frac{p}{p-q}\Delta(y^{\frac{p-q}{p}}(t-1)) &= \frac{p}{p-q}\Big(y^{\frac{p-q}{p}}(t) - y^{\frac{p-q}{p}}(t-1)\Big) \\
&= \frac{\Delta y(t-1)}{\eta^{\frac{q}{p}}},
\end{aligned} \tag{27}$$

for some $\eta \in [y(t-1), y(t)]$. Thus from (25) and (27), we have

$$\frac{p}{p-q}\Delta(y^{\frac{p-q}{p}}(t-1)) \leqslant \frac{\Delta y(t-1)}{y^{\frac{q}{p}}(t-1)} \leqslant f(t-\omega) + \tilde{f}(t-\omega), \quad t \in I_\omega. \tag{28}$$

If we let $t = n$ in (28) and sum it from $n = \omega$ to $n = t-1$, then

$$y^{\frac{p-q}{p}}(t-1) \leqslant y^{\frac{p-q}{p}}(\omega-1) + \frac{p-q}{p}\sum_{n=\omega}^{t-1}\Big(f(n-\omega) + \tilde{f}(n-\omega)\Big), \quad t \in I_\omega. \tag{29}$$

The restatement of inequality (29) can be done as

$$y(t) \leqslant \left\{y^{\frac{p-q}{p}}(\omega-1) + \frac{p-q}{p}\sum_{n=\omega}^{t}\Big(f(n-\omega) + \tilde{f}(n-\omega)\Big)\right\}^{\frac{p}{p-q}}, \quad t \in I_{\omega-1}. \tag{30}$$

Making use of (30) in (24), followed by applying lemma 1, the bound on $y(t-\omega)$ can be expressed as

$$
\begin{aligned}
y(\omega-1) &\leqslant \tilde{c}+\sum_{n=0}^{M-\omega}\hat{J}(n,M)\Bigg[g(s)\Bigg\{y^{\frac{p-q}{p}}(\omega-1)+\frac{p-q}{p}\\
&\qquad \sum_{m=\omega}^{n+\omega-1}\Big(f(m-\omega)+\tilde{f}(m-\omega)\Big)\Bigg\}^{\frac{p}{p-q}}+\tilde{g}(s)\Bigg]\\
&\leqslant \tilde{c}+\sum_{n=0}^{M-\omega}\hat{J}(n,M)\Bigg[g(s)2^{\frac{q}{p-q}}\Bigg\{y(\omega-1)+\Bigg[\frac{p-q}{p}\\
&\qquad \sum_{m=\omega}^{n+\omega-1}\Big(f(m-\omega)+\tilde{f}(m-\omega)\Big)\Bigg]^{\frac{p}{p-q}}\Bigg\}+\tilde{g}(s)\Bigg]. \qquad (31)
\end{aligned}
$$

After simplification, (31) gives us that

$$
\begin{aligned}
y(\omega-1) \leqslant \frac{1}{1-\mu}\Bigg\{\tilde{c}+\sum_{n=0}^{M-\omega}\hat{J}(n,M)\Bigg[g(s)2^{\frac{q}{p-q}}\Bigg\{\frac{p-q}{p}\\
\sum_{m=\omega}^{n+\omega-1}\Big(f(m-\omega)+\tilde{f}(m-\omega)\Big)\Bigg\}^{\frac{p}{p-q}}+\tilde{g}(s)\Bigg]\Bigg\}, \qquad (32)
\end{aligned}
$$

provided $\mu=\sum_{n=0}^{M-\omega}\hat{J}(n,M)g(s)2^{\frac{q}{p-q}}<1$. Denote the right side of (32) by $\mathscr{H}(M)$, then (30) can be viewed as

$$
y(t)\leqslant\Bigg\{\mathscr{H}^{\frac{p-q}{p}}(M)+\frac{p-q}{p}\sum_{n=\omega}^{t}\Big(f(n-\omega)+\tilde{f}(n-\omega)\Big)\Bigg\}^{\frac{p}{p-q}}, \quad t\in I_{\omega-1}. \qquad (33)
$$

Substituting (33) in (23), we obtain the desired inequality in (19). This concludes the proof of our theorem. □

THEOREM 4. *Suppose that ω,x,c,p,M and q are as defined in Theorem 2 and assume that $f_i,g_i\in\mathscr{F}_+(N_0)$, $1\leqslant i\leqslant k$. If*

$$
\begin{aligned}
x^p(t)\leqslant c+\sum_{i=1}^{k}\sum_{n=0}^{t-\omega}\frac{(t-n-1)^{(\omega-1)}}{\Gamma(\omega)}f_i(n)x^q(n+\omega-1)\\
+\sum_{i=1}^{k}\sum_{n=0}^{M-\omega}\frac{(M-n-1)^{(\omega-1)}}{\Gamma(\omega)}g_i(n)x^p(n+\omega-1),\ t\in I_{\omega-1}, \qquad (34)
\end{aligned}
$$

then

$$
x(t)\leqslant\Bigg\{\overline{\mathscr{H}}^{\frac{p-q}{p}}(M)+\frac{p-q}{p}\sum_{n=\omega}^{t}\sum_{i=1}^{k}f_i(n-\omega)\Bigg\}^{\frac{1}{p-q}}, \quad t\in I_{\omega-1}, \qquad (35)
$$

where

$$\overline{\mathscr{H}}(M)=\frac{1}{1-\overline{\mu}}\left\{c+\sum_{i=1}^{k}\sum_{n=0}^{M-\omega}\tilde{J}_i(n,M)2^{\frac{q}{p-q}}\left[\frac{p-q}{p}\sum_{m=\omega}^{n+\omega-1}\left(\sum_{i=1}^{k}f_i(m-\omega)\right)\right]^{\frac{p}{p-q}}\right\} \quad \textit{and}$$

$$\overline{\mu}=\sum_{i=1}^{k}\sum_{n=0}^{M-\omega}\tilde{J}_i(n,M)2^{\frac{q}{p-q}}<1.$$

Proof. Let us assume that $c>0$. If we consider

$$J_i(n,t)=\frac{1}{\Gamma(\omega)}(t-n-1)^{(\omega-1)}f_i(n) \text{ and } \tilde{J}_i(n,M)=\frac{1}{\Gamma(\omega)}(M-n-1)^{(\omega-1)}g_i(n), \tag{36}$$

then, we can restate the inequality (34) as

$$x^p(t)\leqslant c+\sum_{i=1}^{k}\sum_{n=0}^{t-\omega}J_i(n,t)x^q(n+\omega-1) \\ +\sum_{i=1}^{k}\sum_{n=0}^{M-\omega}\tilde{J}_i(n,M)x^p(n+\omega-1),\ t\in I_{\omega-1}. \tag{37}$$

Let us define $y(t)$ by the right side of the inequality (37). It can be easily seen that $y(t)$ is a nonnegative and nondecreasing function. This implies

$$x^p(t)\leqslant y(t),\ t\in I_{\omega-1}, \tag{38}$$

and further

$$y(\omega-1)=c+\sum_{i=1}^{k}\sum_{n=0}^{M-\omega}\tilde{J}_i(n,M)y(n+\omega-1). \tag{39}$$

It is obvious to see that for each $1\leqslant i\leqslant n,\ J_i(n,t)$ is a decreasing function in t for each n in N_0, hence for $t\in I_\omega$, we have

$$\begin{aligned}\Delta y(t-1)&=y(t)-y(t-1)\\ &=\sum_{i=1}^{k}J_i(t-\omega,t)x^q(t-1)+\sum_{i=1}^{k}\sum_{n=0}^{t-\omega-1}\Big\{J_i(n,t)-J_i(n,t-1)\Big\}x^q(n+\omega-1)\\ &\leqslant\left(\sum_{i=1}^{k}f_i(t-\omega)\right)y^{\frac{q}{p}}(t-1).\end{aligned} \tag{40}$$

Hereafter following the parallel steps as in the proof of Theorem 3 below the inequality (25), we obtain the required result. This finishes the proof of our theorem. □

REMARK 1. If we assume the value of $k=1$, then the inequality mentioned in Theorem 1 can be achieved as a special case of this inequality.

THEOREM 5. *Suppose that* ω, x, M *are as defined in Theorem 2. Let us assume that* $1 \leqslant \tilde{c} \in \mathbb{R}_+, f_1, f_2, g_1, g_2 \in \mathscr{F}_+(N_0)$ *and* $\psi, \varphi_1, \varphi_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be such that* ψ *is strictly increasing and* φ_1, φ_2 *are nondecreasing with* $\varphi_1(x) > 0$ *and* $\varphi_2(x) > 0$ *for* $x > 0$ *with* $\varphi(1) = 1 = \psi(1)$. *If*

$$\psi(x(t)) \leqslant \tilde{c} + h\Bigg(\frac{1}{\Gamma(\omega)} \sum_{n=0}^{t-\omega} (t-n-1)^{(\omega-1)} \Big[f_1(n)\varphi_1(x(n+\omega-1)) + g_1(n)\Big] + \frac{1}{\Gamma(\omega)} \sum_{n=0}^{M-\omega} (M-n-1)^{(\omega-1)} \Big[f_2(n)\varphi_2(x(n+\omega-1)) + g_2(n)\Big]\Bigg), \quad (41)$$

where $t \in I_{\omega-1}$, *then*

$$x(t) \leqslant \psi^{-1}\Bigg(\tilde{c} + h\Bigg\{\Phi^{-1}\Bigg[\Phi\Bigg(\mathscr{S}^{-1}\Bigg[\sum_{n=\omega}^{M}\Big[f(n-\omega) + g(n-\omega)\Big]\Bigg]\Bigg) + \sum_{n=\omega}^{t}\Big[f(n-\omega) + g(n-\omega)\Big]\Bigg]\Bigg\}\Bigg), \quad (42)$$

for $t \in I_{\omega-1}$, *where* $f, g \in \mathscr{F}_+(N_0), \varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are such that both* f_1, f_2, *both* g_1, g_2 *and both* φ_1, φ_2 *are less than or equal to* f, g *and* φ *respectively,* $\Phi(r) = \int_{r_0}^{r} \frac{ds}{\varphi(\psi^{-1}[\tilde{c}+h(s)])}$, $r \geqslant r_0 > 0$, $\mathscr{S}(r) = \Phi(2r) - \Phi(r)$ *is strictly increasing function and* $\Phi^{-1}, \mathscr{S}^{-1}$ *are inverse functions of* $\Phi, \mathscr{S}$ *respectively.*

Proof. Making use of suppositions on $f_1, f_2, g_1, g_2, \varphi_1, \varphi_2$ and (41), we get

$$\psi(x(t)) \leqslant \tilde{c} + h\Bigg(\sum_{n=0}^{t-\omega} \hat{J}(n,t)\Big[f(n)\varphi(x(n+\omega-1)) + g(n)\Big] + \sum_{n=0}^{M-\omega} \hat{J}(n,M)\Big[f(n)\varphi(x(n+\omega-1)) + g(n)\Big]\Bigg), \quad (43)$$

where $\hat{J}(n,t)$ is as defined in (20). Let us assume that

$$y(t) = \sum_{n=0}^{t-\omega} \hat{J}(n,t)\Big[f(n)\varphi(x(n+\omega-1)) + g(n)\Big] + \sum_{n=0}^{M-\omega} \hat{J}(n,M)\Big[f(n)\varphi(x(n+\omega-1)) + g(n)\Big], \quad (44)$$

where $t \in I_{\omega-1}$. It is simple to check that $y(t)$ is nonnegative and a nondecreasing function. Thus inequality (43) takes the form

$$x(t) \leqslant \psi^{-1}\Big(\tilde{c} + h(y(t))\Big), \quad t \in I_{\omega-1}, \quad (45)$$

and

$$y(\omega-1)=\sum_{n=0}^{M-\omega}\hat{J}(n,M)\Big[f(n)\varphi(x(n+\omega-1))+g(n)\Big]. \tag{46}$$

Using the definition of $t^{(\omega)}$, we can easily conclude that $\hat{J}(n,t)$ is a decreasing function in t for each n in N_0. Hence, the monotonicity of φ and ψ along with the fact that $\varphi(1)=1=\psi(1)$ gives us for $t\in I_\omega$,

$$\begin{aligned}\Delta y(t-1)&=y(t)-y(t-1)\\&=\hat{J}(t-\omega,t)\Big[f(t-\omega)\varphi(x(t-1))+g(t-\omega)\Big]\\&\quad+\sum_{n=0}^{t-\omega-1}\Big\{\hat{J}(n,t)-\hat{J}(n,t-1)\Big\}\Big[f(n)\varphi(x(n+\omega-1))+g(n)\Big]\\&\leqslant\hat{J}(t-\omega,t)\Big[f(t-\omega)\varphi(x(t-1))+g(t-\omega)\Big]\\&\leqslant\Big[f(t-\omega)\varphi\Big(\psi^{-1}\Big[\tilde{c}+h(y(t-1))\Big]\Big)+g(t-\omega)\Big]\\&\leqslant\Big[f(t-\omega)+g(t-\omega)\Big]\varphi\Big(\psi^{-1}\Big[\tilde{c}+h(y(t-1))\Big]\Big).\end{aligned} \tag{47}$$

It can be simply deduced that

$$\varphi\Big(\psi^{-1}\Big[\tilde{c}+h(y(t-1))\Big]\Big)\geqslant\varphi\Big(\psi^{-1}\Big[\tilde{c}+h(y(\omega-1))\Big]\Big)>0,\ \ t\in I_\omega,$$

which further concludes that

$$\frac{\Delta y(t-1)}{\varphi\Big(\psi^{-1}\Big[\tilde{c}+h(y(t-1))\Big]\Big)}\leqslant\Big[f(t-\omega)+g(t-\omega)\Big]. \tag{48}$$

On the other side

$$\begin{aligned}\Phi(y(t))-\Phi(y(t-1))&=\int_{y(t-1)}^{y(t)}\frac{ds}{\varphi\Big(\psi^{-1}\Big[\tilde{c}+h(s)\Big]\Big)}\\&\leqslant\frac{\Delta y(t-1)}{\varphi\Big(\psi^{-1}\Big[\tilde{c}+h(y(t-1))\Big]\Big)}\\&\leqslant\Big[f(t-\omega)+g(t-\omega)\Big].\end{aligned} \tag{49}$$

Set $t=n$ in (49), and take the summation over the range $n=\omega$ to $n=t-1$ to get

$$\Phi(y(t-1))\leqslant\Phi(y(\omega-1))+\sum_{n=\omega}^{t-1}\Big[f(n-\omega)+g(n-\omega)\Big],\ t\in I_\omega. \tag{50}$$

Using the steps as discussed in the Theorem 2 after (11), we can easily conclude that

$$y(t) \leqslant \Phi^{-1}\left[\Phi\left(\mathscr{S}^{-1}\left[\sum_{n=\omega}^{M}\left[f(n-\omega)+g(n-\omega)\right]\right]\right)+\sum_{n=\omega}^{t}\left[f(n-\omega)+g(n-\omega)\right]\right], \tag{51}$$

for $t \in I_{\omega-1}$. A straightforward substitution of the bound obtained on $y(t)$ in (51) in the inequality (45) gives us the desired inequality in (42). This completes the proof of our theorem. □

COROLLARY 1. *Suppose that $\omega, x, M, \tilde{c}, f_1, f_2, g_1, g_2, f, g$ are as stated in Theorem 5. If*

$$x^p(t) \leqslant \tilde{c}+h\left(\frac{1}{\Gamma(\omega)}\sum_{n=0}^{t-\omega}(t-n-1)^{(\omega-1)}\left[f_1(n)x^q(n+\omega-1)+g_1(n)\right]\right.$$
$$\left.+\frac{1}{\Gamma(\omega)}\sum_{n=0}^{M-\omega}(M-n-1)^{(\omega-1)}\left[f_2(n)x^q(n+\omega-1)+g_2(n)\right]\right), \tag{52}$$

where $p > q > 0$, $t \in I_{\omega-1}$, then

$$x(t) \leqslant \left(\tilde{c}+h\left\{\overline{\Phi}^{-1}\left[\overline{\Phi}\left(\overline{\mathscr{S}}^{-1}\left[\sum_{n=\omega}^{M}\left[f(n-\omega)+g(n-\omega)\right]\right]\right)\right.\right.\right.$$
$$\left.\left.\left.+\sum_{n=\omega}^{t}\left[f(n-\omega)+g(n-\omega)\right]\right]\right\}\right)^{\frac{1}{p}}, \tag{53}$$

for $t \in I_{\omega-1}$, where $\overline{\Phi}(r) = \int_{r_0}^{r}\frac{ds}{[\tilde{c}+h(s)]^{\frac{q}{p}}}$, $r \geqslant r_0 > 0$, $\overline{\mathscr{S}}(r) = \overline{\Phi}(2r) - \overline{\Phi}(r)$ is strictly increasing function and $\overline{\Phi}^{-1}, \overline{\mathscr{S}}^{-1}$ are inverse functions of $\overline{\Phi}, \overline{\mathscr{S}}$ respectively.

Proof. If we substitute $\psi(x) = x^p$ and $\varphi_1(x) = \varphi_2(x) = x^q$, then the required inequality (53) can be simply obatined by using (42). □

THEOREM 6. *Suppose that $\omega, x, M, f_i, g_i, c$ are as defined in Theorem 4 and h, φ, $\varphi_1, \varphi_2, \psi, \Phi, \mathscr{S}$ are as stated in Theorem 5. If $g_i \leqslant f_i$ for each $1 \leqslant i \leqslant k$ and*

$$\psi(x(t)) \leqslant c+h\left(\sum_{i=1}^{k}\sum_{n=0}^{t-\omega}\frac{(t-n-1)^{(\omega-1)}}{\Gamma(\omega)}f_i(n)\varphi_1(x(n+\omega-1))\right.$$
$$\left.+\sum_{i=1}^{k}\sum_{n=0}^{M-\omega}\frac{(M-n-1)^{(\omega-1)}}{\Gamma(\omega)}g_i(n)\varphi_2(x(n+\omega-1))\right), \ t \in I_{\omega-1}, \tag{54}$$

then

$$x(t) \leqslant \psi^{-1}\left(c + h\left\{\Phi^{-1}\left[\Phi\left(\mathscr{S}^{-1}\left[\sum_{i=1}^{k}\sum_{n=\omega}^{M} f_i(n-\omega)\right]\right) + \sum_{i=1}^{k}\sum_{n=\omega}^{t} f_i(n-\omega)\right]\right\}\right), \tag{55}$$

for $t \in I_{\omega-1}$.

Proof. We can easily finish the proof of this theorem by minutely following the procedures of the proofs discussed in Theorems 5 and 4. We pass over the details here. □

COROLLARY 2. *Suppose that* $\omega, x, M, f_i, g_i, c, h, \varphi, \varphi_1, \varphi_2, \psi$ *are as stated in Theorem* 6. *If*

$$x^p(t) \leqslant c + h\left(\sum_{i=1}^{k}\sum_{n=0}^{t-\omega} \frac{(t-n-1)^{(\omega-1)}}{\Gamma(\omega)} f_i(n)x^q(n+\omega-1) + \sum_{i=1}^{k}\sum_{n=0}^{M-\omega} \frac{(M-n-1)^{(\omega-1)}}{\Gamma(\omega)} g_i(n)x^q(n+\omega-1)\right), \quad t \in I_{\omega-1}, \tag{56}$$

where $p > q > 0$, *then*

$$x(t) \leqslant \left(c + h\left\{\overline{\Phi}^{-1}\left[\overline{\Phi}\left(\overline{\mathscr{S}}^{-1}\left[\sum_{i=1}^{k}\sum_{n=\omega}^{M} f_i(n-\omega)\right]\right) + \sum_{i=1}^{k}\sum_{n=\omega}^{t} f_i(n-\omega)\right]\right\}\right)^{\frac{1}{p}}, \tag{57}$$

for $t \in I_{\omega-1}$, *where* $\overline{\Phi}$ *and* $\overline{\mathscr{S}}$ *are as defined in corollary* 1.

Proof. If we follow the steps taken in the proof of corollary 1, then we can simply achieve the desired inequality in (57). □

3. Applications

EXAMPLE 1. Consider the following Volterra-Fredholm fractional sum-difference equation:

$$x^2(t) = 1 + \frac{1}{\Gamma(0.4)} \sum_{n=0}^{t-0.4} (t-n-1)^{(-0.6)} \left[n\,x(n-0.6) + 2n\right] + \frac{1}{\Gamma(0.4)} \sum_{n=0}^{5} (4.4-n)^{(-0.6)} \left[\frac{1}{n+2}x^2(n-0.6) + 1\right], \; t \in I_{-0.6}. \tag{58}$$

If we consider $\omega = 0.4$, $M = 5.4$, $p = 2$, $q = 1$, $\tilde{c} = 1$, $g(n) = \frac{1}{n+2}$, $\tilde{g}(n) = 1$, $f(n) = n$, and $\tilde{f}(n) = 2n$, then by using the bound from Theorem 3 on equation (58), we have the required bound as

$$x(t) \leqslant \left\{ \sqrt{\mathscr{H}(M)} + \frac{3}{2} \sum_{n=0.4}^{t} (n-0.4) \right\}, \; t \in I_{-0.6}, \tag{59}$$

where $\mathscr{H}(M)$ is as stated in Theorem 3. Further, we have

$$\begin{aligned} \mu &= \sum_{n=0}^{M-\omega} \frac{\Gamma(M-n)}{\Gamma(\omega)\Gamma(M-n-\omega+1)} g(s) 2^{\frac{q}{p-q}} \\ &= \sum_{n=0}^{5} \frac{\Gamma(5.4-n)}{\Gamma(0.4)\Gamma(6-n)} \frac{2}{n+2} = 0.937533 < 1. \end{aligned} \tag{60}$$

Additionally, using (60), we get

$$\begin{aligned} \mathscr{H}(M) = \frac{1}{1-\mu} \Bigg\{ \tilde{c} + \sum_{n=0}^{M-\omega} \frac{\Gamma(M-n)}{\Gamma(\omega)\Gamma(M-n-\omega+1)} \Bigg[g(s) 2^{\frac{q}{p-q}} \Bigg\{ \frac{p-q}{p} \\ \sum_{m=\omega}^{n+\omega-1} \Big(f(m-\omega) + \tilde{f}(m-\omega) \Big) \Bigg\}^{\frac{p}{p-q}} + \tilde{g}(s) \Bigg] \Bigg\} \\ = \frac{1}{0.062467} \left\{ 1 + \sum_{n=0}^{5} \frac{\Gamma(5.4-n)}{\Gamma(0.4)\Gamma(6-n)} \left[\frac{9}{2n+4} \left\{ \sum_{m=0.4}^{n-0.6} (m-0.4) \right\}^2 + 1 \right] \right\} \\ = 1294.57. \end{aligned} \tag{61}$$

Using the value obtained in (61) in (59), we get the explicit bound on solution $x(t)$ of (58) as

$$x(t) \leqslant \left\{ 35.9801 + \frac{3}{2} \sum_{n=0.4}^{t} (n-0.4) \right\}, \; t \in I_{-0.6}. \tag{62}$$

EXAMPLE 2. Consider the following Volterra-Fredholm fractional sum-difference equation:

$$x^3(t) \leqslant \sqrt{F(t)}, \; t \in I_{-0.3}, \tag{63}$$

where

$$\begin{aligned} F(t) = \frac{1}{\Gamma(0.7)} \sum_{n=0}^{t-0.7} (t-n-1)^{(-0.3)} n^2 x^2(n-0.3) \\ + \frac{1}{\Gamma(0.7)} \sum_{n=0}^{6} (5.7-n)^{(-0.3)} n^2 x^2(n-0.3). \end{aligned}$$

If we consider $\omega = 0.7,\ M = 6.7,\ p = 3,\ q = 2,\ c = 0,\ f_1(n) = n^2 = f_2(n),$ and $h(s) = \sqrt{s}$, then we can obtain the bound on the solution of (63) using the Theorem 2, and the desired bound is

$$x(t) \leqslant \left\{ \sqrt{\Phi^{-1}\left[\Phi\left(\mathscr{S}^{-1}\left[\sum_{n=0.7}^{6.7}(n-0.7)^2\right]\right) + \sum_{n=0.7}^{t}(n-0.7)^2\right]} \right\}^{\frac{1}{3}},\ t \in I_{-0.3}, \tag{64}$$

where $\Phi, \mathscr{S}, \Phi^{-1}, \mathscr{S}^{-1}$ are as stated in Theorem 2. By computing them, we get for $r \in \mathbb{R}_+$,

$$\left.\begin{aligned} \Phi(r) &= \int_1^r \frac{ds}{(h(s))^{\frac{q}{p}}} = \int_1^r \frac{ds}{\sqrt[3]{s}} = \frac{3(r^{\frac{2}{3}}-1)}{2} \\ \mathscr{S}(r) &= \Phi(2r) - \Phi(r) = (0.8811)r^{\frac{2}{3}} \\ \Phi^{-1}(r) &= \left(\frac{2r}{3}+1\right)^{\frac{3}{2}} \\ \mathscr{S}^{-1}(r) &= [(1.135)r]^{\frac{3}{2}} \end{aligned}\right\} r > 1.$$

Using these estimates in (64), we obtain

$$x(t) \leqslant \left\{ 103.285 + \frac{1}{3}\sum_{n=0.7}^{t}(n-0.7)^2 \right\}^{\frac{1}{4}},\ t \in I_{-0.3}. \tag{65}$$

4. Conclusions

In this manuscript, we have developed some new Volterra-Fredholm type discrete fractional sum inequalities, which extend and generalize some existing discrete fractional sum inequalities of the Volterra-Fredholm type. These inequalities are developed to solve a more general kind of the Volterra-Fredholm type fractional sum difference equations, where the application of previous existing inequalities is not possible directly. However, in the theory of discrete fractional calculus, we have to deal with more crucial Volterra-Fredholm fractional sum-difference equations that involve critical nonlinear terms. Hence, to overcome such situations, the above inequalities can be further extended and generalized to study certain classes of fractional sum-difference equations.

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REFERENCES

[1] B. G. PACHPATTE, *Inequalities for Finite Difference Equations*, Marcel Dekker, Inc., New York, 2002.

[2] HAIDONG LIU AND FANWEI MENG, *Some new generalized Volterra-Fredholm type discrete fractional sum inequalities and their applications*, Journal of Inequalities and Applications, 2016:213, (2016).

[3] RUI A. C. FERREIRA, *A Discrete fractional Gronwall Inequality*, Proc. of the American Mathematical Society, **140**, 5 (2012), 1605–1612.

[4] BIN ZHENG, *Some New Discrete Fractional Inequalities and their applications in Fractional Difference equations*, Journal of Mathematical Inequalities, **9**, 3 (2015), 823–839.

[5] QING-HUA MA, *Some New nonlinear Volterra-Fredholm-type discrete inequalities and their Applications*, Journal of Computational and Applied Mathematics, **216**, (2008), 451–466.

[6] QING-HUA MA AND WING-SUM CHEUNG, *Some New Nonlinear difference inequalities and their Applications*, Journal of Computational and Applied Mathematics, **202**, (2007), 339–351.

[7] RUN XU AND XIANGTING MA, *Some new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two variables and their applications*, Journal of Inequalities and Applications, 2017:187, (2017).

[8] QING-HUA MA, *Estimates on some power nonlinear Volterra-Fredholm type discrete inequalities and their applications*, Journal of Computational and Applied Mathematics, **233**, (2010), 2170–2180.

[9] L. OU-IANG, *The boundedness of solutions of linear differential equations* $y" + A(t)y' = 0$, Advances in Mathematics, **3**, (1957), 409–415.

[10] E. H. YANG, *On some nonlinear integral and discrete inequalities related to Ou-Iang's inequality*, Acta Mathematica Sinica, **14**, 3 (1998), 353–360.

[11] W. S. CHEUNG AND Q. H. MA, *On certain new Gronwall-Ou-Iang type integral inequalities in two variables and their applications*, Journal of Inequalities and Applications, **2005**, 4 (2005), 347–361.

[12] F. M. ATICI AND S. SENGUL, *Modelling with fractional difference equations*, Journal of Math. Anal. Appl., **369**, (2010), 1–9.

[13] F. M. ATICI AND P. W. ELOE, *A transform method in discrete fractional calculus*, Int. Journal of Diff. eqn., **2**, 2 (2007), 165–176.

[14] F. M. ATICI AND P. W. ELOE, *Initial value problems in discrete fractional calculus*, Proc. of the American Mathematical Society, **137**, 3 (2009), 981–989.

[15] F. M. ATICI AND P. W. ELOE, *Gronwall's inequality on discrete fractional calculus*, Computers and Mathematics with Applications, **64**, 10 (2012), 3193–3200.

[16] WALTER G. KELLEY AND ALLAN C. PETERSON, *Difference Equations- An Introduction with Applications*, Academic Press, New York, (2002).

[17] M. HOLM, *Sum and difference compositions in discrete fractional calculus*, CUBO A Mathematical Journal, **13**, 3 (2011), 153–184.

[18] P. O. MOHAMMED, *A generalized uncertain fractional forward difference equations of Riemann-Liouville type*, Journal of Mathematics Research, **11**, 4 (2019), 43–50.

[19] THABET ABDELJAWAD, *On Riemann and Caputo fractional differences*, Computers and Mathematics with Applications, **62.3**, (2011), 1602–1611.

[20] THABET ABDELJAWAD, *On Delta and Nabla Caputo Fractional Differences and Dual Identities*, Discrete Dynamics in Nature and Society, Hindawi Publishing Corporation, **2013**, Article ID 406910, (2013).

[21] P. O. MOHAMMED, THABET ABDELJAWAD, F. JARAD AND YU-MING CHU, *Existence and Uniqueness of Uncertain Fractional Backward Difference Equations of Riemann-Liouville type*, Mathematical Problems in Engineering, Article ID 6598682, (2020).

[22] M. A. ALQUDAH, P. O. MOHAMMED AND THABET ABDELJAWAD, *Solution of Singular Integral Equations via Riemann-Liouville Fractional Integrals*, Mathematical Problems in Engineering, Article ID 1250970, (2020).

[23] THABET ABDELJAWAD, *Dual identities in fractional difference calculus within Riemann*, Advances in Difference Equations, 2013:36, (2013).

[24] K. S. MILLER AND B. ROSS, *Fractional Difference Calculus*, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and their Applications, Nihon University, Koriyama, Japan, (1988), 139–152.

[25] K. OLDHAM AND J. SPANIER, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover Publications, Inc., Mincola, New York, (2002).
[26] J. B. DIAZ AND T. J. OLSER, *Differences of Fractional order*, Mathematics of Computation, **28**, (1974).
[27] H. GRAY AND N. ZHANG, *On a New Definition of the Fractional Difference*, Mathematics of Computation, **50**, (1988), 513–529.
[28] IGOR PODLUBNY, *Fractional Differential Equations*, Academic Press, New York, (1999).

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