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EXISTENCE RESULTS FOR NON–INSTANTANEOUS IMPULSIVE FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION WITH INFINITE DELAY

JAYANTA BORAH* AND SWAROOP NANDAN BORA

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Abstract. In this article, we investigate the existence of mild solution of a class of impulsive fractional functional differential equations with infinite delay in a Banach space. By employing fractional calculus and fixed point theorems, the results are obtained under the assumption that the linear part of the equations generates a compact analytic semigroup.

1. Introduction

Study of evolution equations subjected to impulsive action, which starts abruptly and stays active on a finite time interval, has been a subject of interest in the last few years due to its applicability in practical problems. The pioneering work of such a model, known as non-instantaneous impulsive differential equation, is reported in the work of Hernández and O'Regan [8]. Thereafter, several researchers have carried out the research in this field and some interesting results have been obtained [1, 2, 3, 4, 5, 7, 14, 15].

Many physical models pertaining to memory and hereditary properties have been modeled more successfully with the help of fractional calculus compared to classical settings. The investigation of qualitative as well quantitative properties of the solutions of fractional differential equations is an active subject of research. For the theory and recent development on fractional differential equations, one can refer to some monographs [10, 16, 21].

Integro evolution equations in abstract spaces generalize many partial differential equations and integro differential equations appearing in scientific and engineering problems. So it is meaningful to discuss the nature of solution of this type equations in qualitative ways.

Chang [3] discussed the controllability of a first order impulsive functional differential systems with infinite delay in Banach spaces. Mophou and N'Guérékata [12] established the existence of mild solution of some fractional differential equations with

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* Corresponding author.

nonlocal conditions. Hu et al. [9] established the existence of mild solution of a class of Riemann-Liouville fractional evolution equation with nonlocal conditions and infinite delay in a Banach space, in which the linear part is the infinitesimal generator of a compact analytic semigroup. Mahmudov and Zorlu [11] investigated the approximate controllability of fractional evolution equations with compact analytic semigroup. The controllability of a impulsive neutral functional integro differential equation in a Banach space was established in [17] . Motivated by above consideration, here we investigate the existence of mild solution of a class of fractional evolution equation with infinite delay of the following form:

$$ {}^{C}D_t^{\alpha}x(t) = -Ax(t)+f(t,x_t,\int_0^t B(t,s,x_s)ds), \ \ t \in J_i=(s_i,t_{i+1}], \ \ i=0,1,\dots,N, \quad (1)$$

$$ x(t) = g_i(t,x_t), \ \ t \in I_i=(t_i,s_i], \ \ i=1,2,\dots, \ \ N, \quad (2)$$

$$ x(t) = \phi(t), \ \ t \in (-\infty,0], \quad (3)$$

where $x(.)$ takes values in a Banach space $(X,\|.\|)$; $0<\alpha<1$; $-A:D(A)\subset X \longrightarrow X$ the infinitesimal generator of a compact analytic semigroup $Q(t)$ of uniformly bounded linear operators on X; $0=s_0<t_1\leqslant s_1\leqslant t_2\ldots\leqslant t_N\leqslant s_N<t_{N+1}=T$ is a partition of the interval $J=[0,T]$; the functions $g_i\in C((t_i,s_i]\times\mathscr{P}_0,X_\beta)$ for each $i=1,2,\ldots.N$ and $f:[0,T]\times\mathscr{P}_0\times X_\beta\rightarrow X_\beta$ suitable functions. Here $\mathscr{P}_0$ is a phase space and X_β is a fractional power space defined in the next section. For a function x defined on $(-\infty,T]$ and any $t\in J$, we denote $x_t(.)$ to represent the portion of the function from $-\infty$ to present time t, that is,

$$ x_t(\theta)=x(t+\theta), \ \ \theta\in(-\infty,0]. $$

In section 2 we recall some definitions and preliminaries which are required to develop the article. In section 3, we give sufficient conditions for the existence of mild solution of the system (1)–(3). At the end an example is presented to validate the results obtained.

2. Preliminaries

The definition and the theorems related to analytic semigroup theory is taken from te book by Pazy [13].

Let $0\in\rho(A)$. Then for any $0<\beta<1$, we can define $A^{-\beta}$ as a closed linear operator on its domain $D(A^{-\beta})$ as follows:

$$ A^{-\beta}=\frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}Q(t)dt. \quad (4)$$

The operator defined by (4) is a bounded linear operator and each $A^{-\beta}$ is an injective continuous endomorphism of X. So it is possible to define A^β, for $0<\beta<1$, as a closed linear operator on its domain $D(A^\beta)$. The subspace $D(A^\beta)$ is dense in X, and the expression

$$ \|u\|_\beta=\|A^\beta u\| $$

defines a norm on $D(A^{\beta})$ which makes it a Banach space. We denote $X_{\beta} = D(A^{\beta})$. For $0 < \gamma < \beta \leqslant 1$, $X_{\beta} \hookrightarrow X_{\gamma}$ and the embedding is compact whenever the resolvent operator of A is compact.

LEMMA 1. *A^{β} and $Q(t)$ have the following properties:*

- *There exists a constant $M > 1$ such that $\|Q(t)\| \leqslant M$.*
- *$Q(t) : X \to X_{\beta}$ for each $t > 0$ and $\beta \geqslant 0$.*
- *$A^{\beta}Q(t)x = Q(t)A^{\beta}x$ for each $x \in X_{\beta}$ and $t \geqslant 0$.*
- *For every $t > 0$, $A^{\beta}Q(t)$ is bounded in X and there exists $M_{\beta} > 0$ such that*

$$\|A^{\beta}Q(t)\| \leqslant M_{\beta}t^{-\beta}.$$

- *$A^{-\beta}$ is a bounded linear operator in X with $D(A^{\beta}) = Im(A^{-\beta})$.*

LEMMA 2. [19] *The restriction $Q_{\beta}(t)$ of $Q(t)$ to X_{β} is exactly the part of $Q(t)$ to X_{β}. Also $\{Q(t)\}_{t\geqslant 0}$ is a family of strongly continuous semigroup on X_{β} and $\|Q_{\beta}(t)\| \leqslant \|Q(t)\| \leqslant M$ for all $t \geqslant 0$.*

Now we introduce some definitions and fundamental results of fractional calculus from the work by Wang and Zhou [21].

Let $J = [a,b]$, $-\infty < a < b < \infty$ be a finite interval on the real axis $\mathbb{R}$.

DEFINITION 1. The Riemann-Liouville fractional integral ${}_aD_t^{-\alpha}f(t)$ of order $\alpha > 0$ is defined by

$$I_a^{\alpha}f(t) = {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,$$

provided the right-hand side is point-wise defined on $[a,b]$.

DEFINITION 2. The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C_{\alpha}^n$, $n \in \mathbb{N}$ is defined as

$${}_a^CD_t^{\alpha}f(t) = {}_aD_t^{-(n-\alpha)}D^nf(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^n(s)ds, \ \ t > a, \ \ n = [\alpha]+1.$$

If there is no confusion about the base point of both the operators defined above, we simply remove it.

Finding the solution of a differential equation with infinite delay at any time not only requires the knowledge of state at current time but also that of the state in past. Thus the choice of the phase space is one of the most important characteristics in the solution of such equations. In view of Hale and Kato [6], it is a usual practice to take the phasespace as a semi-normed space satisfying some axioms. Here we define the phase space in the following ways.

DEFINITION 3. $\mathscr{P}_0$ is a linear space of functions from $(-\infty,0]$ to X_β endowed with a semi-norm $\|.\|_{\mathscr{P}_0}$, which satisfies the following axioms:

(A1): If $x:(-\infty,T]\to X$, $T>0$ is such that $x_0\in\mathscr{P}_0$ and for every $t\in[0,T)$, the following conditions hold:

- $x_t\in\mathscr{P}_0$,
- $\|x(t)\|_\beta \leqslant C\|x_t\|_{\mathscr{P}_0}$,
- $\|x_t\|_{\mathscr{P}_0}\leqslant C_1(t)\sup_{t\in[0,T]}\|x(t)\|_\beta + C_2\|x_0\|_{\mathscr{P}_0}$, where $C>0$ is a constant, $C_1,C_2:[0,\infty)\to[0,\infty)$, C_1 is continuous and C_2 is locally bounded and C_1,C_2 are independent of $x(.)$.

(A2): For a function $x(.)$ in $(\mathbf{A1})$, x_t is a $\mathscr{P}_0$-valued function for $t\in[0,T)$.

(A3): The space $\mathscr{P}_0$ is complete.

LEMMA 3. *Let $f_1,f_2:J\to\mathbb{R}$ be positive real continuous functions. Assume that there exist a constant $c>0$ and a continuous nondecreasing function $h:\mathbb{R}\to(0,\infty)$ such that*

$$h(t)\leqslant c+\int_a^t f_1(s)h(f_2(s))ds,\ \ \forall t\in J.$$

Then

$$f_2(t)\leqslant H^{-1}\Big(\int_a^t f_1(s)ds\Big),\ \ \forall t\in J$$

provided

$$\int_c^\infty \frac{dy}{h(y)} > \int_a^b f_1(s)ds.$$

Here H^{-1} refers to the inverse of the function $H(y)=\displaystyle\int_c^y \frac{dy}{h(y)}$ for $y\geqslant c$.

LEMMA 4. (Burton-Kirk's fixed point theorem) *Let X be a Banach space and F_1, F_2 be two operators satisfying*

(a) F_1 is a contraction and

(b) F_2 is completely continuous.

Then, either the operator equation $x=F_1(x)+F_2(x)$ possesses a solution, or the set $\mathscr{E}=\{x\in X:\lambda F_1(\frac{x}{\lambda})+\lambda F_2(x)=x,$ for some $0<\lambda<1\}$ is unbounded.

By $PC(J,X_\beta)$ we denote the Banach space of piecewise continuous functions from J into X_β with the norm

$$\|x\|_{PC(X^\beta)}=\sup_{t\in J}\|A^\beta x(t)\|.$$

To deal with the impulsive as well delay conditions, we consider the space $\mathscr{P}_T=\{x:(-\infty,T]\to X_\beta$ to be such that $x_k\in C(J_k,X_\beta)$, for $k=0,1,2,\dots,N$; $x(t_k{}^+),x(t_k{}^-)$ exist; $x(t_k{}^-)=x(t_k)$, $k=0,1,2,\dots,N$; $x_0=\phi\in\mathscr{P}_0$ and $\sup_{t\in[0,T]}\|x(t)\|_\beta<\infty\}$ endowed with the norm

$$\|x\|_{\mathscr{P}_T}=\|x\|_{PC(X^\beta)}+\|\phi\|_{\mathscr{P}_0}.$$

If $x \in \mathscr{P}_T$, then for any $i = 0,1,2,\ldots N$, the function $\tilde{x}_i \in C([t_i,t_{i+1}],X_\beta)$ is constructed as follows:

$$\tilde{x}_i(t) = \begin{cases} x(t), & \text{for } t \in (t_i,t_{i+1}], \\ x(t_i^+), & \text{for } t = t_i. \end{cases}$$

For $\mathscr{B} \subset PC(J,X_\beta)$, we denote $\tilde{\mathscr{B}}_i = \{\tilde{x}_i : x \in \mathscr{B}\}$.

LEMMA 5. [8] *A set $\mathscr{B} \subset PC(J,X_\beta)$ is relatively compact in $PC(J,X_\beta)$ if and only if each set $\tilde{\mathscr{B}}_i$ is relatively compact in $C([t_i,t_{i+1}],X_\beta)$.*

DEFINITION 4. [22] Consider the fractional evolution equation

$$^{C}D_t^{\alpha}x(t) = -Ax(t) + f(t,x_t), \quad t \in J, \quad 0 < \alpha < 1, \tag{5}$$

$$x(t) = \phi \in \mathscr{P}_0. \tag{6}$$

For $f : J \times \mathscr{P}_0 \to X$ and A generating an analytic semigroup $\{T(t)\}_{t \geqslant 0}$, a continuous function $x : J \to X_\beta$ satisfying the integral equation $x(t) = \mathscr{T}(t)\phi(0) + \int_0^t \mathscr{S}(t-s)f(s,x_s)ds$ is called a mild solution of (5)–(6), where

$$\mathscr{T}(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \mathscr{S}(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}(\theta)^{-1-\frac{1}{\alpha}}\bar{\omega}_\alpha(\theta)^{-\frac{1}{\alpha}} \geqslant 0,$$

$$\bar{\omega}_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty),$$

with ξ_α as the probability density function defined on $(0,\infty)$, that is,

$$\xi_\alpha(\theta) \geqslant 0, \quad \theta \in (0,\infty) \text{ and } \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

LEMMA 6. [19] *The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

1. *For fixed $t \geqslant 0$ and any $x \in X_\beta$, we have*

$$\|\mathscr{T}(t)x\|_\beta \leqslant M\|x\|_\beta, \|\mathscr{S}(t)x\|_\beta \leqslant \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|_\beta.$$

2. *$\mathscr{T}_\beta(t)$ and $\mathscr{S}_\beta(t)$, $t > 0$ are uniformly continuous, where*

$$\mathscr{T}_\beta(t) = \int_0^\infty \xi_\alpha(\theta)Q_\beta(t^\alpha\theta)d\theta \text{ and } \mathscr{S}_\beta(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)Q_\beta(t^\alpha\theta)d\theta.$$

3. Existence of PC-mild solution

In this section we first formulate the definition of PC-mild solution of our problem and then prove the existence of solutions with infinite delay.

Motivated from the definition 4 and the work in [8], we define the mild solution as follows:

DEFINITION 5. A function $x \in \mathscr{P}_T$ is said to be PC-mild solution of the problem (1)–(3) if

$$x(t) = \begin{cases} \phi(t), \ t \in (-\infty, 0], \\ \mathscr{T}(t)x_0 + \int_0^t \mathscr{S}(t-s) f(s, x_s, \int_0^s B(s,\tau,x_\tau)d\tau)ds, \ t \in J_1, \\ g_i(t, x_t), \ t \in t \in I_i, \ 1 \leqslant i \leqslant N, \\ \mathscr{T}(t-s_i)g_i(s_i, x_{s_i}) + \int_{s_i}^t \mathscr{S}(t-s) f(s, x_s, \int_0^s B(s,\tau,x_\tau)d\tau)ds, \ t \in J_i, \ 1 \leqslant i \leqslant N. \end{cases}$$

We introduce the following hypotheses:

(H1) $f : J \times \mathscr{P}_0 \times X_\beta \to X_\beta$ is continuous and there exist $\alpha_1 \in (0, \alpha)$ and $\Omega \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t,\psi_1,u) - f(t,\psi_2,v)\|_\beta \leqslant \Omega(t)[\|\psi_1 - \psi_2\|_{\mathscr{P}_0} + \|u - v\|_\beta],$$

for all $\psi_1, \psi_2 \in \mathscr{P}_0$, $u, v \in X_\beta$, $t \in J_i$ and $i = 0, 1, \ldots, N$.

(H2) $B : D := \{(t,s) \in J \times J : s \leqslant t\} \times \mathscr{P}_0 \to X_\beta$ is continuous and there exists constant M_B such that for all $(t,s) \in D, x, y \in \mathscr{P}_0$,

$$\| \int_0^t [B(t,s,\psi_1) - B(t,s,\psi_2)]ds\|_\beta \leqslant M_B \|\psi_1 - \psi_2\|_{\mathscr{P}_0}.$$

(H3) There exist constants $L_{g_i} > 0$, for all $\psi_1, \psi_2 \in \mathscr{P}_0$, $t \in J_i$ and $i = 1, \ldots, N$ such that

$$\|g_i(t,\psi_1) - g_i(t,\psi_2)\|_\beta \leqslant L_{g_i} \|\psi_1 - \psi_2\|_{\mathscr{P}_0}$$

and $g_i \in C(J_i \times \mathscr{P}_0, X_\beta)$, for all $i = 1, 2, \ldots, N$.

THEOREM 1. *Assume that the hypotheses* $(H1)$–$(H3)$ *hold and* $\phi \in X_\beta$. *Then the system of equations* (1)–(2) *has a unique PC-mild solution* $x \in \mathscr{P}_T$, *provided*

$$\Theta = \max \left\{ \frac{M}{\Gamma(\alpha)} \tilde{C}_1 \frac{{t_1}^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|\Omega\|_{L^{\frac{1}{\alpha_1}} J_1}, M\Big[\tilde{C}_1 L_{g_i} + \frac{\tilde{C}_1(1+M_B)}{\Gamma(\alpha)} \frac{(t_{i+1} - s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|\Omega\|_{L^{\frac{1}{\alpha_1}} J_i}\Big], \ i = 1, \ldots, N \right\} < 1. \quad (7)$$

Proof. We define the operator

$$\mathscr{F} : \mathscr{P}_T \longrightarrow \mathscr{P}_T \quad as$$

$$\mathscr{F}(x)t = \begin{cases} \phi(t), \ t \in (-\infty, 0], \\ \mathscr{T}(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, x_s, \int_0^s B(s,\tau,x_\tau)d\tau)ds, \ t \in [0,t_1], \\ g_i(t,x_t), \ t \in (t_i, s_i], \\ \mathscr{T}(t-s_i)g_i(s_i,x_{s_i}) + \int_{s_i}^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, x_s, \int_0^s B(s,\tau,x_\tau)d\tau)ds, \\ \qquad t \in J_i, \ 1 \leqslant i \leqslant N. \end{cases}$$

Consider the extension Φ of $\phi \in \mathscr{P}_0$ as

$$\Phi(t) = \begin{cases} \phi(t), \ t \in (-\infty, 0], \\ \mathscr{T}(t)\phi(0), \ t \in [0,t_1], \\ 0, \ t \in (t_1, T]. \end{cases}$$

Then $\Phi \in \mathscr{P}_T$.

Let $x(t) = z(t) + \Phi(t)$, $-\infty < t \leqslant T$. If x satisfies the integral equation (3), then $z_0 = 0$, $x_t = z_t + \Phi_t$, for every $t \in J$ and the function $z(t)$ satisfies

$$z(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, z_s + \Phi_s, \int_0^s B(s.\tau, z_\tau + \Phi_\tau)d\tau)ds, \ t \in J_1, \\ g_i(t, z_t + \Phi_t), \ t \in (t_i, s_i], \\ \int_{s_i}^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, z_s + \Phi_s, \int_0^s B(s,\tau, z_\tau + \Phi_\tau)d\tau)ds, \ t \in J_i, \ 1 \leqslant i \leqslant N. \end{cases}$$

Let

$$\acute{\mathscr{P}}_T = \{z \in \mathscr{P}_T : z_0 = 0\}.$$

For any $z \in \acute{\mathscr{P}}_T$, we have

$$\|z\|_{\acute{\mathscr{P}}_T} = \sup_{t \in J} \|z(t)\|_\beta.$$

Thus $(\acute{\mathscr{P}}_T, \|.\|_{\acute{\mathscr{P}}_T})$ is a Banach space. We define the operator $\tilde{\mathscr{F}} : \acute{\mathscr{P}}_T \to \acute{\mathscr{P}}_T$ by

$$\tilde{\mathscr{F}}(z)t = \begin{cases} 0, \ t \in (-\infty, 0], \\ \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, z_s + \Phi_s, \int_0^s B(s,\tau, z_\tau + \Phi_\tau)d\tau)ds, \ t \in J_1, \\ g_i(t,x_t), \ t \in (t_i, s_i], \\ \mathscr{T}(t-s_i)g_i(s_i, z_{s_i} + \Phi_{s_i}) + \int_{s_i}^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, z_s + \Phi_s, \\ \qquad \int_0^s B(s,\tau, z_\tau + \Phi_\tau)d\tau)ds, \ t \in J_i, \ 1 \leqslant i \leqslant N. \end{cases}$$

It is clear that the operator $\tilde{\mathscr{F}}$ determines the fixed point of the operator $\mathscr{F}$. We show that $\tilde{\mathscr{F}}$ is a contraction mapping.

For $x,y \in \acute{\mathscr{P}}_T$ and $t \in [0,t_1]$, we get

$$\begin{aligned}
\|\tilde{\mathscr{F}}u(t) - \tilde{\mathscr{F}}v(t)\|_\beta &= \|\int_0^t \mathscr{S}(t-s)[f(s,u_s+\Phi_s,\int_0^s B(s,\tau,u_\tau+\Phi_\tau)d\tau) \\
&\quad - f(s,v_s+\Phi_s,\int_0^s B(s,\tau,v_\tau+\Phi_\tau)d\tau)]ds\|_\beta \\
&\leqslant \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Omega(s)[\|u_s-v_s\|_{\mathscr{P}_0} \\
&\quad + \|\int_0^s B(s,\tau,u_\tau+\Phi_\tau)d\tau - \int_0^s B(s,\tau,v_\tau+\Phi_\tau)d\tau\|_\beta]ds \\
&\leqslant \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Omega(s)(1+M_B)\|u_s-v_s\|_{\mathscr{P}_0}ds \\
&\leqslant \frac{M}{\Gamma(\alpha)}(1+M_B)\tilde{C_1}\Big(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}ds}\Big)^{1-\alpha_1} ds \\
&\qquad \times \|\Omega\|_{L^{\frac{1}{\alpha_1}}J_1} \sup_{t\in J_1}\|u(t)-v(t)\|_\beta \\
&\leqslant \frac{M}{\Gamma(\alpha)}\tilde{C_1}\frac{{t_1}^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}\|\Omega\|_{L^{\frac{1}{\alpha_1}}J_1}\|u-v\|_{\acute{\mathscr{P}}_T}.
\end{aligned}$$

For $t \in J_i$,

$$\begin{aligned}
&\|\tilde{\mathscr{F}}u(t) - \tilde{\mathscr{F}}v(t)\|_\beta \\
&\leqslant \|\mathscr{T}(t-s_i)g_i(s_i,u(s_i)+\Phi(s_i)) - \mathscr{T}(t-s_i)g_i(s_i,v(s_i)+\Phi_{s_i})\|_\beta \\
&\quad + \|\int_{s_i}^t \mathscr{S}(t-s)\big(f(s,u_s+\Phi_s,\int_0^s B(s,\tau,u_\tau+\Phi_\tau)d\tau) \\
&\quad - f(s,v_s+\Phi_s,\int_0^s B(s,\tau,u_\tau+\Phi_\tau)d\tau)\big)ds\|_\beta \\
&\leqslant L_{g_i}M\|u_{s_i}-v_{s_i}\|_{\mathscr{P}_0} + \frac{M}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}\Omega(s)(1+M_B)\|u_s-v_s\|_{\mathscr{P}_0}ds \\
&\leqslant L_{g_i}M\sup_{t\in J_i}\|u(t)-v(t)\|_\beta + \tilde{C_1}\frac{M}{\Gamma(\alpha)}(1+M_B)\frac{(t_{i+1}-s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}\| \\
&\qquad \times \Omega\|_{L^{\frac{1}{\alpha_1}}J_i}\sup_{t\in(s_i,t_{i+1}]}\|u(t)-v(t)\|_\beta \\
&\leqslant M\big[\tilde{C_1}L_{g_i} + \frac{\tilde{C_1}(1+M_B)}{\Gamma(\alpha)}\frac{(t_{i+1}-s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}}\|\Omega\|_{L^{\frac{1}{\alpha_1}}J_i}\big]\|u-v\|_{\acute{\mathscr{P}}_T}.
\end{aligned}$$

For $t \in (t_i,s_i]$,

$$\|\tilde{\mathscr{F}}u(t) - \tilde{\mathscr{F}}v(t)\| \leqslant \tilde{C_1}L_{g_i}\|u-v\|_{\acute{\mathscr{P}}_T}.$$

Thus,

$$\|\tilde{\mathscr{F}}u - \tilde{\mathscr{F}}v\|_{\acute{\mathscr{P}}_T} \leqslant \Theta\|u-v\|_{\acute{\mathscr{P}}_T}.$$

Therefore, $\tilde{\mathscr{F}}$ is a contraction on $\acute{\mathscr{P}}_T$ and there exists a unique PC-mild solution of (1)–(3). □

In the previous theorem, we established the existence and uniqueness of PC-mild solution by using Lipschitz conditions on both source as well as impulse functions. In our next result, we relax the Lipschitz condition on the source function and use Burton-Kirk's fixed point theorem to obtain the existence of PC-mild solution.

Consider the ball $B_r = \{z \in \acute{\mathscr{P}}_T : \|z\|_{\acute{\mathscr{P}}_T} \leqslant r\}$.

Clearly B_r is a closed, bounded convex set in $\acute{\mathscr{P}}_T$.

In the sequel we make the following assumptions:

(H4) For each $t \in J$, the function $f(t,.,.) : \mathscr{P}_0 \times X_\beta \to X_\beta$ is continuous and for each $\psi \in \mathscr{P}_0$, $u \in X_\beta$, the function $f(.,\psi,u) : J \to X_\beta$ is strongly measurable.

(H5) There exist a function $m \in L(J,\mathbb{R}^+)$ and a non decreasing function $W_f \in C([0,\infty),\mathbb{R}^+)$ such that

$$ {}_{s_i}D_t^{-\alpha} m \in C(J_i,\mathbb{R}^+), \quad \lim_{t\to s_i^+} {}_{s_i}D^{-\alpha} m(t) = 0, \quad i = 0,1,2,\dots N, $$

and

$$ \|f(t,x_t,Bx)\|_\beta \leqslant m(t) W_f(\|x_t\|_{\mathscr{P}_0}), \quad x \in \mathscr{P} \text{ and almost all } t \in J. $$

THEOREM 2. *Assume that assumptions $(H3)$–$(H5)$ hold. If $\|g_i(t,0)\|_\beta$, $i = 1,2,\dots N$ are bounded, $L_1 = \max_{1\leqslant i\leqslant N} \tilde{C}_1 M L_{g_i} < 1$ and the following condition holds:*

$$ K_1 \int_{s_i}^{t_{i+1}} (t-s)^{\alpha-1} m(s)ds < \int_{K_0}^{\infty} \frac{ds}{W_f(s)}, \quad i = 0,1,\dots N, $$

where

$$ \text{for } t \in [s_i,t_{i+1}], \ K_0 = \frac{\tilde{C}_1 M \|g_i(t,0)\|_\beta + \tilde{C}_1 \|\phi(0)\|_\beta + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}}{1-\tilde{C}_1 M L_{g_i}}, $$

$$ K_1 = \frac{\tilde{C}_1 M}{\Gamma(\alpha)(1-\tilde{C}_1 M L_{g_i})} \quad \text{and} $$

$$ \text{for } t \in [0,t_1], \quad K_0 = M\tilde{C}_1 \|\phi(0)\|_\beta + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}, K_1 = \frac{M}{\Gamma(\alpha)} \tilde{C}_1, $$

then the problem possesses at least one mild solution on $(-\infty,T]$.

Proof. To apply the fixed point theorem, let us split our operator $\tilde{\mathscr{F}} : B_r \to B_r$, introduced in the previous theorem, into two parts given by

$$ \tilde{\mathscr{F}} = \sum_{i=0}^{N} \tilde{\mathscr{F}}_i^1 + \sum_{i=0}^{N} \tilde{\mathscr{F}}_i^2, $$

where $\tilde{\mathscr{F}}_i^j : \acute{\mathscr{P}}_T \longrightarrow \acute{\mathscr{P}}_T$, $i = 0,\dots,N;\ j = 1,2$ are given by

$$ \tilde{\mathscr{F}}_i^1 x(t) = \{ 0, g_i(t,z_t+\Phi_t), \mathscr{T}(t-s_i) g_i(s_i, z_{s_i}+\Phi_{s_i}), $$

$$ \tilde{\mathscr{F}}_i^2 x(t) = \begin{cases} \int_{s_i}^t \mathscr{S}(t-s) f(s, z_s+\Phi_s, \int_0^s B(s,\tau,z_\tau+\Phi_\tau)d\tau)) ds, & t \in J_i, \\ 0, & t \notin J_i. \end{cases} $$

The proof is split into several steps as follows:

Step I. To show that $\tilde{\mathscr{F}}^2$ maps a bounded set into a bounded set in $\acute{\mathscr{P}}_T$.

Note that for $z \in B_r$,

$$\|z_s + \Phi_s\|_{\mathscr{P}_0} \leqslant C_1 r + C_1 M\|\phi(0)\|_\beta + C_2\|\phi\|_{\mathscr{P}_0} =: r^*.$$

Let $u \in B_r$. For $i \geqslant 1$ and $t \in J_i$, $i = 0,1,\ldots N$, we get

$$\begin{aligned}
\|\tilde{\mathscr{T}} u(t)\|_\beta &= \int_{s_i}^t \mathscr{S}(t-s) f(s, z_s + \Phi_s, \int_0^s B(s,\tau,z_\tau + \Phi_\tau)d\tau))ds\|_\beta \\
&\leqslant \frac{M}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} m(s) W_f(\|z_s + \Phi_s\|_{\mathscr{P}_0})ds \\
&\leqslant \frac{M}{\Gamma(\alpha)} W_f(r^*) \sup_{t\in J_i} \int_{s_i}^t (t-s)^{\alpha-1} m(s)ds \\
&\leqslant \frac{M}{\Gamma(\alpha)} W_f(r^*) \sup_{t\in J_i} \int_{s_i}^t (t-s)^{\alpha-1} m(s)ds \\
&=: l,
\end{aligned}$$

where

$$l = \max\left\{ \frac{M}{\Gamma(\alpha)} W_f(r^*) \sup_{t\in J_i} \int_{s_i}^t (t-s)^{\alpha-1} m(s)ds, \ \ 0 \leqslant i \leqslant N \right\}.$$

Then for each $z \in B_r$, we have $\|\tilde{\mathscr{F}}^2(u)\|_{\acute{\mathscr{P}}} \leqslant l$.

For convenience, from next step onwards, we take $B(z+\Phi)(s) = \int_0^s B(s,\tau,z_\tau + \Phi_\tau)d\tau$.

Step II. To show that $[\widetilde{\mathscr{F}_i^2} x : x \in B_r]_i$, $i = 0,1,\ldots,N$, is an equicontinuous family of functions on $C([t_i, t_{i+1}], X_\beta)$.

Let, $l_1, l_2 \in (s_i, t_{i+1}]$, $s_i < l_1 < l_2$ and $x \in B_r$. Then

$$\begin{aligned}
&\|(\widetilde{\mathscr{F}_i^2} x)(l_2) - (\widetilde{\mathscr{F}_i^2} x)(l_1)\|_\beta \\
&= \|\int_{s_i}^{l_2} (l_2 - s)^{\alpha-1} \mathscr{S}(l_2 - s) f(s, z_s + \Phi_s, B(z+\Phi)(s))ds \\
&\quad - \int_{s_i}^{l_1} (l_1 - s)^{\alpha-1} \mathscr{S}(l_1 - s) f(s, x(s), f(s, z_s + \Phi_s, B(z+\Phi)(s))ds\| \\
&= \|\int_{l_1}^{l_2} (l_2 - s)^{\alpha-1} \mathscr{S}(l_2 - s) f(s, z_s + \Phi_s, B(z+\Phi)(s))ds\| \\
&\quad + \|\int_{s_i}^{l_1} (l_2 - s)^{\alpha-1} \mathscr{S}(l_2 - s) f(s, z_s + \Phi_s, B(z+\Phi)(s))ds \\
&\quad - \int_{s_i}^{l_1} (l_1 - s)^{\alpha-1} \mathscr{S}(l_2 - s) f(s, z_s + \Phi_s, B(z+\Phi)(s))ds\| \\
&\quad + \|\int_{s_i}^{l_1} (l_1 - s)^{\alpha-1} [(\mathscr{S}(l_2 - s) - \mathscr{S}(l_1 - s))] f(s, z_s + \Phi_s, B(z+\Phi)(s))ds\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{M\alpha}{\Gamma(\alpha+1)}\int_{l_1}^{l_2}(l_2-s)^{\alpha-1}m(s)W_f(\|z_s+\Phi_s\|_{\mathscr{P}_0})ds\\
&\quad+\frac{M\alpha}{\Gamma(\alpha+1)}\int_{s_i}^{l_1}[(l_1-s)^{\alpha-1}-(l_2-s)^{\alpha-1}]m(s)W_f(\|z_s+\Phi_s\|_{\mathscr{P}_0})ds\\
&\quad+\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|m(s)W_f(\|z_s+\Phi_s\|_{\mathscr{P}_0})ds\\
&\leqslant \frac{M\alpha}{\Gamma(\alpha+1)}\int_{l_1}^{l_2}(l_2-s)^{\alpha-1}m(s)W_f(r^*)ds\\
&\quad+\frac{M\alpha}{\Gamma(\alpha+1)}\int_{s_i}^{l_1}[(l_1-s)^{\alpha-1}-(l_2-s)^{\alpha-1}]m(s)W_f(r^*)ds\\
&\quad+\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|m(s)W_f(r^*)ds\\
&\leqslant W_f(r^*)\frac{M\alpha}{\Gamma(\alpha+1)}\|\int_{s_i}^{l_2}(l_2-s)^{\alpha-1}m(s)ds-\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}m(s)ds\|\\
&\quad+2W_f(r^*)\frac{M\alpha}{\Gamma(\alpha+1)}\int_{s_i}^{l_1}[(l_1-s)^{\alpha-1}-(l_2-s)^{\alpha-1}]m(s)ds\\
&\quad+W_f(r^*)\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|m(s)ds\\
&=: I_1+I_2+I_3.
\end{aligned}
$$

Since ${}_{s_i}D_t^{-\alpha}m\in C(J_i,\mathbb{R}^+)$,

$$I_1\to 0 \quad \text{as} \quad l_2\to l_1.$$

For $l_1<l_2$,

$$I_2\leqslant W_f(r^*)\frac{2M\alpha}{\Gamma(\alpha+1)}\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}m(s)ds.$$

As $\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}m(s)ds<\infty$, we have $I_2\to 0$ as $l_2\to l_1$.

For $\varepsilon>0$ small enough,

$$
\begin{aligned}
I_3 &= W_f(r^*)\int_{s_i}^{l_1-\varepsilon}(l_1-s)^{\alpha-1}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|m(s)ds\\
&\quad+W_f(r^*)\int_{l_1-\varepsilon}^{l_1}(l_1-s)^{\alpha-1}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|m(s)ds\\
&\leqslant W_f(r^*)\int_{s_i}^{l_1-\varepsilon}(l_1-s)^{\alpha-1}m(s)ds\sup_{s\in[s_i,l_1-\varepsilon]}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|\\
&\quad+W_f(r^*)\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|m(s)ds\\
&\quad-W_f(r^*)\int_{s_i}^{l_1-\varepsilon}(l_1-s)^{\alpha-1}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|m(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant W_f(r^*)\int_{s_i}^{l_1-\varepsilon}(l_1-s)^{\alpha-1}m(s)ds\sup_{s\in[s_i,l_1-\varepsilon]}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|\\
&\quad+W_f(r^*)\frac{2M\alpha}{\Gamma(\alpha+1)}[\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}m(s)ds-\int_{s_i}^{l_1-\varepsilon}(l_1-s)^{\alpha-1}m(s)ds]\\
&\leqslant W_f(r^*)\int_{s_i}^{l_1-\varepsilon}(l_1-s)^{\alpha-1}m(s)ds\sup_{s\in[s_i,l_1-\varepsilon]}\|\mathscr{S}(l_2-s)-\mathscr{S}(l_1-s)\|\\
&\quad+W_f(r^*)\frac{2M\alpha}{\Gamma(\alpha+1)}[\int_{s_i}^{l_1}(l_1-s)^{\alpha-1}m(s)ds-\int_{s_i}^{l_1-\varepsilon}(l_1-\varepsilon-s)^{\alpha-1}m(s)ds]\\
&\quad+W_f(r^*)\frac{2M\alpha}{\Gamma(\alpha+1)}\int_{s_i}^{l_1-\varepsilon}[(l_1-\varepsilon-s)^{\alpha-1}-(l_1-s)^{\alpha-1}]m(s)ds\\
&=: I_{31}+I_{32}+I_{33}.
\end{aligned}
$$

Since $\{Q(t)\}_t>0$ is a compact operator, it is an equicontinuous family, so $I_{31}\to 0$ as $l_2\to l_1$. $I_{32}\to 0$, $I_{33}\to 0$ as $\varepsilon\to 0$ by virtue of I_2 and I_3. Therefore, $||(\tilde{\mathscr{F}}_i^2x)(l_2)-(\tilde{\mathscr{F}}_i^2x)(l_1)||\to 0$ is independent of $x\in B_r$ since $l_2\to l_1$. By a similar argument, equicontinuity can be verified in $\tau_1<0<\tau_2\leqslant T$, whereas it is trivial for $\tau_1<\tau_2\leqslant 0$.

Thus, $[\tilde{\mathscr{F}}^2B_r]_i$ is equicontinuous and hence $\mathscr{F}_2$ is completely continuous.

Step III. $\tilde{\mathscr{F}}^2:\acute{\mathscr{P}}\to\acute{\mathscr{P}}$ is continuous.
Let $\{z^m\}_{m=1}^{\infty}$ be a sequence in $\acute{\mathscr{P}}$ with $z^m\to z\in\acute{\mathscr{P}}$.
By condition $(H3)$, we have

$$\lim_{m\to\infty}f(t,{z^m}_t+\Phi_t,B(z^m(t)+\Phi(t))\to f(t,z_t+\Phi_t,B(x(t)+\Phi_t).$$

For any $t\in J_i$

$$
\begin{aligned}
&(t-s)^{\alpha-1}\|f(s,z_s^m+\Phi_s,B(z^m(s)+\Phi(s)))-f(s,z_s^m+\Phi_s,B(z(s)+\Phi_s))\|\\
&\qquad\leqslant(t-s)^{\alpha-1}m(s)W_f(r^*).
\end{aligned}
$$

By $(H4)$ the function $s\to(t-s)^{\alpha-1}m(s)$ is integrable for $s\in[s_i,t_{i+1}]$.
Hence by Lebesgue's dominated convergence theorem, we get

$$\int_{s_i}^{t}(t-s)^{\alpha-1}\|f(s,z_s^m+\Phi_s,B(z^m(s)+\Phi(s))-f(s,z_s+\Phi_s,B(z(s)+\Phi_s))\|ds\to 0,\quad \text{as } m\to\infty.$$

Thus for $t\in J_i$,

$$
\begin{aligned}
&\|(\tilde{\mathscr{F}}_i^1z^m)(t)-(\tilde{\mathscr{F}}_i^1z)(t)\|\\
&\leqslant\tilde{M}_TW_f(r^*)\int_{s_i}^{t}(t-s)^{\alpha-1}\|f(s,z_s^m,B(z^m(s)+\Phi(s))))-f(s,z(s),B(z(s)+\Phi(s)))\|ds\\
&\to 0,\ \text{ as } m\to\infty.
\end{aligned}
$$

Therefore, $\tilde{\mathscr{F}}_i^1z^m\to\tilde{\mathscr{F}}_i^1z$ pointwise on J_i as $m\to\infty$.

Thus $\tilde{\mathscr{F}}^2$ is continuous in $\acute{\mathscr{P}}$.

Step IV. Let $t \in (s_i, t_{i+1})$ be fixed and $s(<t) \in (s_i, t_{i+1})$ be such that $\varepsilon \in (s_i, s)$. To prove $\bigcup_{\tau\in[s,t]} \tilde{\mathscr{F}}_i^2 z(\tau)$, $z \in B_r$ is relatively compact in X_β.

For any $\delta > 0$, define the set

$$V_\varepsilon^\delta(t) = \{(\tilde{\mathscr{F}}_i^2)_\varepsilon^\delta z)(t) : z \in B_r\},$$

where

$$((\tilde{\mathscr{F}}_i^2)_\varepsilon^\delta z)(t) = \alpha \int_{s_i}^{t-\varepsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^\alpha \theta) f(s, z_s + \phi_s, B(z+\phi)(s)) ds d\theta$$
$$= \alpha Q(\varepsilon^\delta \theta) \int_{s_i}^{t-\varepsilon} \int_\delta^\infty \theta(\tau-s)^{\alpha-1} \xi_\alpha(\theta) Q((\tau-s)^\alpha \theta - \varepsilon^\alpha \delta)$$
$$f(s, z_s + \phi_s, B(z+\phi)(s)) ds d\theta.$$

Since $Q(\varepsilon^\alpha \delta), (\varepsilon^\alpha \delta) > 0$ is compact, the set $V_\varepsilon^\delta(\tau)$ is relatively compact in X_β.

On the other hand, for any $z \in B_r$,

$$|((\tilde{\mathscr{F}}_i^2 z)(\tau) - ((\tilde{\mathscr{F}}_i^2)_\varepsilon^\delta z)(\tau)\|_\beta$$
$$\leqslant \alpha \| \int_{s_i}^\tau \int_0^\psi \theta(\tau-s)^{\alpha-1} \xi_\alpha(\theta) Q((\tau-s)^\alpha(\theta))$$
$$f(s, z_s + \Phi_s, B(z+\Phi_s)(s) ds d\theta\|_\beta + \alpha \| \int_{\tau-\varepsilon}^\tau \int_\delta^\infty \theta(t-s)^{\alpha-1}$$
$$\xi_\alpha(\theta) Q((t-s)^\alpha(\theta)) f(s, z_s + \Phi_s, B(z+\Phi)(s) ds d\theta\|_\beta$$
$$\leqslant W_f(r^*) \alpha M \int_{s_i}^t (t-s)^{\alpha-1} m(s) ds \int_0^\psi \theta \xi_\alpha(\theta) d\theta$$
$$+ W_f(r^*) \frac{M}{\Gamma(\alpha)} \int_{\tau-\varepsilon}^\tau (\tau-s)^{\alpha-1} m(s) ds \to 0 \quad \text{as} \quad \varepsilon \to 0, \; \delta \to 0.$$

Thus there exist relatively compact sets arbitrary close to the set $\bigcup_{\tau\in[s,t]} \tilde{\mathscr{F}}_i^2 B_r(\tau)$. Therefore the set $\bigcup_{\tau\in[s,t]} \tilde{\mathscr{F}}_i^2 B_r(\tau)$ is relatively compact in X_β.

Step V. To prove that $\tilde{\mathscr{F}}^1$ is a contraction in B_r.

Let $u, v \in \acute{\mathscr{P}}_T$ and $t \in J_i$, $1 \leqslant i \leqslant N$. We have

$$\|(\tilde{\mathscr{F}}_i^1 u)(t) - (\tilde{\mathscr{F}}_i^1 v)(t)\|_\beta \leqslant L_{g_i} \|u_s - v_s\|_{\mathscr{P}_0}$$
$$\leqslant \tilde{C}_1 L_{g_i} \|u - v\|_{\acute{\mathscr{P}}_T}.$$

Also for $u,v \in \acute{\mathscr{P}}_T$ and $t \in J_i,\ i = 1,2,\ldots,N,$ we have

$$\begin{aligned}\|(\tilde{\mathscr{F}}_i^1 u)(t) - (\tilde{\mathscr{F}}_i^1 v)(t)\|_\beta &= \|\mathscr{T}(t-s_i)g_i(s_i,u(s_i)+\Phi(s_i)) \\ &\qquad - \mathscr{T}(t-s_i)g_i(s_i,v(s_i)+\Phi_{s_i})\|_\beta \\ &\leqslant ML_{g_i}\|u_s - v_s\|_{\mathscr{P}_0} \\ &\leqslant M\tilde{C}_1 L_{g_i}\|u-v\|_{\acute{\mathscr{P}}_T}.\end{aligned}$$

Taking the supremum over $t,$ we get

$$\|\tilde{\mathscr{F}}_i^1(u) - \tilde{\mathscr{F}}_i^1(v)\|_{\acute{\mathscr{P}}_T} \leqslant L\|u-v\|_{\acute{\mathscr{P}}_T}.$$

Hence $\tilde{\mathscr{F}}^1$ is a contraction mapping on $\acute{\mathscr{P}}_T$.

Step VI. To establish a priori bounds. Here we show that the set

$$\mathscr{E} = \{z \in \tilde{\mathscr{P}} : z = \lambda\tilde{\mathscr{F}}^1 z + \lambda\tilde{\mathscr{F}}^2(\frac{z}{\lambda}),\ \text{ for some } 0 < \lambda < 1\}$$

is bounded.

For each $t \in [0,t_1]$,

$$z(t) = \int_0^t \mathscr{T}(t-s)(t-s)^{\alpha-1} f(s, z_s + \Phi_s, B(z+\Phi)(s))ds.$$

Hence

$$\|z(t)\|_\beta \leqslant \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s) W_f(\|z_s+\Phi_s\|_{\mathscr{P}})ds. \tag{8}$$

Now

$$\begin{aligned}\|z_s+\Phi_s\|_{\mathscr{P}_0} &\leqslant \|z_s\|_{\mathscr{P}_0} + \|\Phi_s\|_{\mathscr{P}_0} \\ &\leqslant \tilde{C}_1 \sup_{t\in[0,t_1]}\|z(s)\|_\beta + M\tilde{C}_1\|\phi(0)\|_\beta + \tilde{C}_2\|\phi\|_{\mathscr{P}_0} \\ &=: \mu(t).\end{aligned}$$

Hence (8) becomes

$$\|z(t)\|_\beta \leqslant \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} m(s) W_f(\mu(s))ds. \tag{9}$$

Using (9) in the definition of μ, we have

$$\mu(t) \leqslant \frac{M}{\Gamma(\alpha)}\tilde{C}_1\int_0^t (t-s)^{\alpha-1} m(s) W_f(\mu(s))ds + M\tilde{C}_1\|\phi(0)\|_\beta + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}. \tag{10}$$

Thus

$$\mu(t) \leqslant K_0 + K_1\int_0^t (t-s)^{\alpha-1} m(s) W_f(\mu(s))ds, \tag{11}$$

where

$$K_0 = M\tilde{C}_1\|\phi(0)\|_\beta + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}, K_1 = \frac{M}{\Gamma(\alpha)}\tilde{C}_1.$$

If $v(t) = K_0 + K_1 \int_0^t (t-s)^{\alpha-1} m(s) W_f(\mu(s)) ds,$ then

$$\mu(t) \leqslant v(t), \;\; v(0) = K_0 \;\text{ and }\; v'(t) \leqslant K_1 (s-t)^{\alpha-1} m(t) W_f(v(t)).$$

Thus for $t \in [0, t_1],$ we have

$$\int_{v(0)}^{v(t)} \frac{ds}{W_f(s)} \leqslant K_1 \int_0^t (t-s)^{\alpha-1} m(s) ds < \int_{K_0}^{\infty} \frac{ds}{W_f(s)}.$$

By lemma 3, we have

$$v(t) \leqslant H^{-1}(K_1) \int_0^t (t-s)^{\alpha-1} m(s) ds), \;\; t \in [0, t_1],$$

where

$$H(y) = \int_{K_0}^{y} \frac{ds}{W_f(s)}.$$

Hence

$$\|z_s + \Phi_s\|_{\mathscr{P}_0} < M_{t_0}.$$

From (9), we get

$$\|z(t)\|_\beta \leqslant \frac{M}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} m(s) W_f(M_{t_0}) ds.$$

Thus there exists $L_{t_0} > 0$ such that

$$\|z\|_{\mathscr{B}} \leqslant L_{t_0}.$$

For $t \in (t_i, s_i],\; i = 1, 2, \ldots, N,$ we have

$$z(t) = g_i(t, z_t + \phi_t).$$

Hence for each $t \in J_i$

$$\|z(t)\|_\beta \leqslant L_{g_i} \|z_t + \Phi_t\|_{\mathscr{P}_0} + \|g_i(t, 0)\|_\beta. \tag{12}$$

If $\|z_s + \Phi_s\|_{\mathscr{P}_0} \leqslant \mu(t),$ then (12) becomes

$$\|z(t)\|_\beta \leqslant L_{g_i} \mu(t) + \|g_i(t, 0)\|_\beta. \tag{13}$$

From definition of $\mu(t)$, we have

$$\mu(t) \leqslant \tilde{C}_1 (L_{g_i} \mu(t) + \|g_i(t, 0)\|_\beta + M\tilde{C}_1 \|\phi(0)\|_\beta + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}.$$

Hence, (14) gives

$$\mu(t) \leqslant \frac{\tilde{C}_1 (\|g_i(t, 0)\|_\beta + M\|\phi(0)\|_\beta) + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}}{1 - \tilde{C}_1 L_{g_i}} =: M_{t_i}.$$

This shows that there is a constant $M_{t_i} > 0$ such that

$$\mu(t) \leqslant M_{t_i}, t \in (t_i, s_i].$$

Therefore

$$\|z(t)\|_\beta \leqslant L_{g_i} M_{t_i} + \|g_i(t,0)\|_\beta.$$

Thus

$$\|z\|_{\acute{\mathscr{P}}} \leqslant L_{t_i}.$$

Finally for each $t \in J_i$,

$$z(t) = \mathscr{T}(t-s_i)g_i(s_i, z_{s_i} + \Phi_{s_i}) + \int_{s_i}^t (t-s)^{\alpha-1} \mathscr{S} f(s, z_s + \Phi_s + B(z+\Phi)(s))ds).$$

Now

$$\|z(t)\|_\beta \leqslant ML_{g_i} \|z_{s_i} + \Phi_{s_i}\|_{\acute{\mathscr{P}}_0} + M\|g_i(t,0)\|_\beta + \frac{M}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} m(s)$$

$$W_f(\|z_s + \Phi_s\|_{\mathscr{P}_0})ds.$$

Using $\mu(t)$, we get

$$\|z(t)\|_\beta \leqslant ML_{g_i}\mu(t) + M\|g_i(t,0)\|_\beta + \frac{M}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} m(s) W_f(\mu(s))ds. \quad (14)$$

From the definition of μ, we have

$$\mu(t) \leqslant \tilde{C}_1 \big(ML_{g_i}\mu(t) + M\|g_i(t,0)\|_\beta + \frac{M}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} m(s) W_f(\mu(s))ds\big)$$

$$+ \tilde{C}_1\|\phi(0)\|_\beta + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}.$$

Thus, we get

$$\mu(t) \leqslant \tilde{K}_0 + \tilde{K}_1 \int_{s_i}^t (t-s)^{\alpha-1} m(s) W_f(\mu(s))ds,$$

where

$$\tilde{K}_0 = \frac{\tilde{C}_1 M\|g_i(t,0)\|_\beta + \tilde{C}_1\|\phi(0)\|_\beta + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}}{1 - \tilde{C}_1 ML_{g_i}} \text{ and } \tilde{K}_1 = \frac{\tilde{C}_1 M}{\Gamma(\alpha)(1 - \tilde{C}_1 ML_{g_i})}.$$

If we denote the RHS of the above inequality by $v(t)$, we have $v(0) = \tilde{K}_0$, $\mu(t) \leqslant v(t)$, $t \in [s_i, t_{i+1}]$, $i = 1, 2, \dots N$ and $v'(t) \leqslant \tilde{K}_1(s-t)^{\alpha-1} m(t)) W_f(v(t))$. Thus for $t \in [s_i, t_{i+1}]$,

$$\int_{v(i)}^{v(t)} \frac{ds}{W_f(s)} \leqslant \tilde{K}_1 \int_{s_i}^t (t-s)^{\alpha-1} m(s)ds < \int_{\tilde{K}_0}^\infty \frac{ds}{W_f(s)}.$$

By lemma 3, we obtain

$$v(t) \leqslant H^{-1}\big(\tilde{K}_1 \int_{s_i}^{t_{i+1}} (t-s)^{\alpha-1} m(s)ds\big), \;\; s \in [s_i, t_{i+1}],$$

where

$$H(y) = \int_{\tilde{K}_0}^{y} \frac{ds}{W_f(s)}.$$

Therefore,

$$\|z_s + \Phi_s\|_{\mathscr{P}_0} < M_{t_{i+1}}.$$

From (14), we get

$$\|z(t)\|_\beta \leqslant ML_{g_i}M_{t_{i+1}} + M\|g_i(t,0)\|_\beta + \frac{M}{\Gamma(\alpha)}\int_{s_i}^{t}(t-s)^{\alpha-1}m(s)W_f(M_{t_{i+1}})ds =: L_{t_{i+1}}.$$

Thus there exist $L_{t_{i+1}} > 0$ such that

$$\|z\|_{\tilde{\mathscr{P}}} \leqslant L_{t_{i+1}}.$$

This implies that the set $\mathscr{E}$ is bounded. Hence by Burton-Kirk's fixed point theorem the operator $\tilde{\tilde{\mathscr{F}}}$ has a fixed point, since $x(t) = z(t) + \Phi(t)$, $t \in (-\infty, T]$. This establishes that x is a fixed point of the operator $\mathscr{T}$ which is a mild solution of the problem. □

4. An example

Consider the space $X = L^2([0,\pi],\mathbb{R})$ and the following fractional partial differential equation with infinite delay:

$$^{C}D_t^{\alpha}x(t,z) = \frac{\partial^2 x(t,z)}{\partial z^2} + \sigma\Big(t, x_t(.,z), \int_0^t \sigma_1(t,s,x_t(.,z))ds\Big), \quad t \in [s_i, t_{i+1}],\ z \in [0,\pi], \tag{15}$$

$$x(t,z) = G_i(t, x_{(}.,z)),\ i = 1,2,\ldots N, \tag{16}$$

$$x(t,0) = x(t,\pi) = 0,\ t \in [0,T], \tag{17}$$

$$x(t,z) = \phi(t,z),\quad -\infty \leqslant t \leqslant 0,\ 0 \leqslant z \leqslant \pi, \tag{18}$$

where $s_i \in (t_i, t_{i+1}]$ $(i = 1,2,\ldots,N,)$ in the partition $0 = t_0 < t_1 < \ldots < t_{N+1} = T$ of the interval $[0,T]$ with $s_0 = 0$ an x_t indicates the portion of the solution $x(.,.) : (-\infty,T] \times [0,\pi] \to X$, that is, for any $t \geqslant 0$, $x_t(.,.) : (-\infty,0] \times [0,\pi] \to X$ is given by

$$x_t(\theta,z) = x(t+\theta,z),\ \text{ for }\ \theta \in (-\infty,0].$$

Let $X = L^2[0,\pi]$ and define $A : D(A) \subset X \to X$ by $Ax = x''$, on

$$D(A) = \{x \in X : \frac{\partial x}{\partial z}, \frac{\partial^2 x}{\partial z^2} \in X\ \text{ and }\ x(0) = x(\pi) = 0\}.$$

Then A generates a compact analytic semigroup $Q(t)_{t \geqslant 0}$ on X and there exists a constant $M \geqslant 1$ such that $\|Q(t)\| \leqslant M$.

Consider the functions

$$x(t)z = x(t,z), \;\; t \in J, z \in [0,\pi], \tag{19}$$

$$g_i(t,\phi)z = G_i(t,\phi(\theta,z)), \;\; \theta \in (-\infty,0], z \in [0,\pi], \tag{20}$$

$$f(t,\phi,\int_0^t B(t,s,\phi)ds)z = \sigma(t,\phi(\theta,z),\int_0^t \sigma_1(t,s,\phi(\theta,z)ds), \tag{21}$$

$$\theta \in (-\infty,0], \;\; z \in [0,\pi],$$

$$\phi(\theta)(z) = \phi(\theta,z), \;\; \theta \in (-\infty,0], \;\; z \in [0,\pi], \tag{22}$$

with the following assumptions:

(i) For each $i = 0,1,\dots N,$ the function $f : [s_i,t_{i+1}] \times \mathscr{P}_0 \times X \to X$ defined by (22) is continuous and we impose a suitable condition on F to satisfy the hypotheses $(H1)$–$(H2)$.

(ii) For each $i = 1,\dots N,$ the function $g_i : (t_i,s_i] \times \mathscr{P}_0 \to X$ defined by (20) is continuous and we impose a suitable condition on G_i to satisfy the hypothesis $(H3)$.

With the above setting the system of equations (16)–(18) reduces to the system of equations (1)–(3) satisfying the hypotheses of Theorem 1 and hence ensuring a mild solution on $(-\infty,T]$.

5. Conclusion

In this paper, the problem of existence of mild solution of a class of Caputo fractional evolution equation with non instantaneous impulses considered. With the aid of contraction mapping theorem and Burton Kirk's fixed point theorem, we established couple of theorems on the existence of mild solution.

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REFERENCES

[1] A. BOUDAOUI, T. CARABALLO AND A. OUHAB, *Stochastic differential equation with non-instantaneous impulses driven by fractional Brownian motion*, Discrete and Continuous Dynamical Systems-series B. **22 (7)** (2017), 2521–2541.

[2] J. BORAH AND S. N. BORA, *Existence of mild-solution for mixed Volterra-Fredholm integro fractional differential equation with non-instantaneous impulses*, Differential Equations and Dynamical Systems (2018) 1–12.

[3] Y. K. CHANG, A. ANGURAJ AND M. M. ARJUNAN, *Existence results for impulsive neutral functional differential equations with infinite delay*, Nonlinear Analalysis: Hybrid System **2** (2008), 209–218.

[4] P. CHEN, X. ZHANG AND Y. LI, *Existence of mild-solutions to partial differential equations with non-instantaneous impulses*, Electronic Journal of Differential Equations **241** (2016), 1–11.

[5] G. R. GAUTAM AND J. DABAS, *Mild-solutions for class of neutral fractional functional differential equations with not instantaneous impulses*, Applied Mathematics and Computation **259** (2015), 480–489.

[6] J. K. HALE AND J. KATO, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac. **21 (1)** (1978), 11–41.

[7] P. KUMAR, D. N. PANDEY AND D. BAHUGUNA, *On a new class of abstract impulsive functional differential equations of fractional order*, Journal of Nonlinear Science and Applications **7** 2 (2014), 102–114.

[8] E. HERNÁNDEZ AND D. O'REGAN, *On a new class of abstract impulsive differential equations*, Proceeding of American Mathematical Society **141 (5)** (2013), 1641–1649.

[9] L. HU, Y. REN AND R. SAKTHIVEL, *Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delay*, Semigroup Forum **79** 3 (2009), 507–514.

[10] A. A. KILBAS, H. M. SRIVASTAVA AND J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North Holland Mathematics Studies, Elsevier Science Inc., New York, 204 (2006).

[11] N. I. MAHMUDOV AND S. ZORLOU, *On the approximate controllability of fractional evolution equations with compact analytic semigroup*, Journal of Computational and Applied Mathematics **259** (2014) 194–204.

[12] G. M. MOPHOU, G. M. N'GUÉRÉKATA, *Existence of the mild solution for some fractional differential equations with nonlocal conditions*, Semigroup Forum, **79** (2009) 315–322.

[13] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Spinger-Verlag, New York (1983).

[14] M. PIERRI, D. O'REGAN AND V. ROLNIK, *Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses*, Applied Mathematics and Computation **219** 12 (2013), 6743–6749.

[15] M. PIERRI, H. R. HENNRIQUEZ AND A. PROKOPCZYK, *Global solutions of abstract differential equations with non-instantaneous impulse*, Mediterranean Journal of Mathematics **13 (4)** (2016), 1685–1708.

[16] I. PODULBNY, *Fractional Differential Equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, (1998).

[17] Z. TAI AND X. WANG, *Controllability of fractional order impulsive neutral functional infinite delay integro differential systems in Banach space*, Appl. Math. Letters **22** 11 (2009), 1760–1765.

[18] J. WANG AND X. LI, *Periodic boundary value problem for integer/fractional order differential equations with not-instantaneous impulse*, Applied Mathematics and Computing **46** 1–2 (2014), 321–334.

[19] J. WANG AND Y. ZHOU, *A class of fractional evolution equations and optimal controls*, Nonlinear Analysis: Real World Applications **12** 1 (2011), 262–272.

[20] L. ZHANG AND Y. ZHOU, *Fractional Cauchy problems with almost sectorial operators*, Applied Mathematics and Computations **257** (2015), 145–157.

[21] Y. ZHOU, W. JINRONG AND Z. LU, *Basic Theory of Fractional Differential equations*, World Scientific (2016).

[22] Y. ZHOU AND F. JIAO, *Existence of mild solutions for fractional neutral evolution equations*, Computers and Mathematics with Applications, **59**, 3 (2010), 1063–1077.

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Jayanta Borah
Department of Mathematical Sciences
Tezpur University
Tezpur-784028, Assam, India
e-mail: jba@tezu.ernet.in

Swaroop Nandan Bora
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati-781039, Assam, India
e-mail: swaroop@iitg.ac.in