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ON CRITERIA OF EXISTENCE FOR NONLINEAR KATUGAMPOLA FRACTIONAL DIFFERENTIAL EQUATIONS WITH p–LAPLACIAN OPERATOR

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Abstract. This paper is devoted to establishing vital criteria of existence and uniqueness for a class of nonlinear Katugampola fractional differential equations (KFDEs) with p-Laplacian operator subjecting to mixed boundary conditions. The reasoning is inspired by diverse classical fixed point theory, such as the Guo-Krasnosel'skii type fixed point principle and Banach contraction theorem. Additionally, several expressive examples are afforded to show the effectiveness of our theoretical results.

1. Introduction

Fractional calculus generalizes the order of derivative and integral from positive integers to real numbers, or even to complex numbers. In the last few decades, it is found that a series of natural phenomena can be modelled robustly in terms of fractional calculus, see [9, 14, 24]. As a result, fractional calculus gained a rapid development recently, both in the aspect of mathematics and many disciplines of applied sciences, being nowadays recognized as an excellent tool for describing complex systems and practical matters, especially involving long range memory effects and non-locality, such as viscoelastic theory, fluid dynamics, biology, image processing, one may refer [5, 16, 17, 18, 19, 20, 21, 22, 26].

In effect, the p-Laplacian operator arises in mathematical modeling, such as in non-Newtonian fluid flow, turbulent filtration in porous media, rheology, glaciology. Problems involving the p-Laplacian have been investigated extensively in the literature during the last several decades, see [23, 25]. Amongst them, there do exist some impressive description on applications of the p-Laplacian operator to fractional differential equations (FDEs), one may refer to [4, 8, 10, 13, 15, 27, 28, 29, 30], and the

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references cited therein. However, according to the survey of the authors, there are no papers dedicate to the investigation of existence and uniqueness of solutions to nonlinear Katugampola fractional differential equations (KFDEs) with p-Laplacian operator, we therefore supply a gap in the literature. Accordingly, we consider

$$\begin{cases} {}^{\rho}\mathscr{D}_{0^+}^{\beta}\left(\phi_p\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi(t)\right)\right)+\gamma f(t,\varphi(t))=0,\ 0<t<T \\ \varphi(0)=0,\ \varphi(T)=0 \\ \phi_p\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi\right)(0)=0,\ \phi_p\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi\right)(T)=0 \end{cases}, \tag{1}$$

where $\gamma\in\mathbb{R}$, and ${}^{\rho}\mathscr{D}_{0^+}^{\alpha}$ for $\rho>0$, presents Katugampola derivative with order $1<\alpha,\beta\leqslant 2$. $\phi_p(x)=|x|^{p-2}x$, the p-Laplacian operator $(p>1)$, $(\phi_p)^{-1}=\phi_r$, $1/p+1/r=1$. $f:[0,T]\times[0,\infty)\to[0,\infty)$ is a continuous function with finite positive constant T.

2. Preliminaries

As be introduced in [14], let us denote $X_c^p[0,T]$, $(c\in\mathbb{R},\ 1\leqslant p\leqslant\infty)$ which means the space of Lebesgue measurable functions φ on $[0,T]$ for which $\|\varphi\|_{X_c^p}<\infty$, is defined by

$$\|\varphi\|_{X_c^p}=\left(\int_0^T|s^c\varphi(s)|^p\frac{ds}{s}\right)^{\frac{1}{p}}<\infty,$$

for $1\leqslant p<\infty$, $c\in\mathbb{R}$, and

$$\|\varphi\|_{X_c^\infty}=ess\sup_{0\leqslant t\leqslant T}\left[t^c|\varphi(t)|\right],\ (c\in\mathbb{R}).$$

For $C[0,T]$, it is a Banach space with all continuous real functions from $[0,T]$ into $\mathbb{R}$ endowed with the maximum norm

$$\|\varphi\|=\max_{0\leqslant t\leqslant T}|\varphi(t)|.$$

DEFINITION 2.1. ([11]) The left-sided Katugampola fractional integral with order $\alpha>0$ of $\varphi\in X_c^p[0,T]$ is defined as

$$\left({}^{\rho}\mathscr{I}_{0^+}^{\alpha}\varphi\right)(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{s^{\rho-1}\varphi(s)}{(t^\rho-s^\rho)^{1-\alpha}}ds, \tag{2}$$

where $\rho>0$, $t\in[0,T]$, and $\Gamma(\alpha)=\int_0^{+\infty}e^{-s}s^{\alpha-1}ds$ is the Euler gamma function.

DEFINITION 2.2. ([12]) Let $\alpha,\ \rho\in\mathbb{R}^+$, and $n-1\leqslant\alpha<n\in\mathbb{N}$, then the Katugampola fractional derivative of a function φ is defined for $0\leqslant t\leqslant T<\infty$ as

$$\begin{aligned} {}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n\left({}^{\rho}\mathscr{I}_{0^+}^{n-\alpha}\varphi\right)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_0^t\frac{s^{\rho-1}\varphi(s)}{(t^\rho-s^\rho)^{\alpha-n+1}}ds. \end{aligned} \tag{3}$$

It deserves to be remarked that the Katugampola derivative could be viewed as the generalization of Riemann-Liouville derivative for $(\rho \to 1)$ and Hadamard derivative for $(\rho \to 0^+)$ settings due to the facts reported in [12].

In the sequel, some fixed-point theorems/principles are addressed which are vital in the acquirement of the main results.

LEMMA 2.1. (Guo-Krasnosel'skii [1], [7]) *Assume that* $\mathscr{Q}$ *be a cone in Banach space* E, *and* Ω_1, Ω_2 *are open subsets of* E *with* $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. *Let* $F : \mathscr{Q} \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right) \to \mathscr{Q}$ *is a completely continuous operator satisfying*

(i) $\|Fz\| \leqslant \|z\|$, *for* $z \in \mathscr{Q} \cap \partial\Omega_1$ *and* $\|Fz\| \geqslant \|z\|$, *for* $z \in \mathscr{Q} \cap \partial\Omega_2$, *or*

(ii) $\|Fz\| \geqslant \|z\|$, *for* $z \in \mathscr{Q} \cap \partial\Omega_1$ *and* $\|Fz\| \leqslant \|z\|$, *for* $z \in \mathscr{Q} \cap \partial\Omega_2$.

Then F *admits a fixed point in* $\mathscr{Q} \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right)$.

LEMMA 2.2. (Leray-Schauder type nonlinear alternative [6]) *Suppose that* $\mathscr{Q}$ *is a convex and closed subset of Banach space* X, $\Omega \subset \mathscr{Q}$ *is open and* $0 \in \Omega$. *Assume that* $F : \bar{\Omega} \to \mathscr{Q}$ *be continuous and compact, so,*

(i) F *owns a fixed point in* $\bar{\Omega}$; *or*

(ii) *there exists a point* $\varphi \in \partial\Omega$ *and* $\varphi = \lambda F(\varphi)$ *with* $\lambda \in (0, 1)$.

LEMMA 2.3. (Banach [6]) *Suppose that* E *be a Banach space,* $\mathscr{Q} \subset E$ *a non-empty closed subset. If* $F : \mathscr{Q} \to \mathscr{Q}$ *is a contraction mapping, then* F *has a unique fixed point in* $\mathscr{Q}$.

LEMMA 2.4. ([13]) *Let* φ_p *be a* p*-Laplacian operator.*

(i) *For* $1 < p \leqslant 2$, $z_1 z_2 > 0$, *and* $|z_1|, |z_2| \geqslant m > 0$, *then*

$$\left|\varphi_p(z_1) - \varphi_p(z_2)\right| \leqslant (p-1)m^{p-2}|z_1 - z_2|.$$

(ii) *For* $p > 2$, $|z_1|, |z_2| \leqslant M$, *then*

$$\left|\varphi_p(z_1) - \varphi_p(z_2)\right| \leqslant (p-1)M^{p-2}|z_1 - z_2|.$$

3. Main results

In the sequel, we always choose $T \leqslant (pc)^{\frac{1}{pc}}$, where $p \geqslant 1$, $c > 0$ for the sake of Remark of [3]. If the above conditions for such constants satisfied, then one has $C[0, T] \hookrightarrow X_c^p[0, T]$, and $\|\varphi\|_{X_c^p} \leqslant \|\varphi\|$.

Now, we present several vital lemmas which play a key role in the proofs of the main results.

LEMMA 3.1. *Assume* $\alpha, \rho \in \mathbb{R}^+$, *be such that* $1 < \alpha, \beta \leqslant 2$. *If* ${}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi \in C[0, T]$, *and* $z(t)$ *admits a continuous function, then*

$$\begin{cases} {}^{\rho}\mathscr{D}_{0^+}^{\beta}\left(\phi_p\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi(t)\right)\right) + \gamma z(t) = 0, \ 0 < t < T \\ \varphi(0) = 0, \ \varphi(T) = 0 \\ \phi_p\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi\right)(0) = 0, \ \phi_p\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha}\varphi\right)(T) = 0 \end{cases}, \tag{4}$$

is equivalent to the integral equation

$$\varphi(t) = \int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma\int_0^T G_\beta(s,\tau)\,z(\tau)\,d\tau\right)ds, \tag{5}$$

where

$$G_\alpha(t,s) = \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\begin{cases}\left(\frac{t}{T}\right)^{\rho(\alpha-1)}(T^\rho - s^\rho)^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant T,\\ \left(\frac{t}{T}\right)^{\rho(\alpha-1)}(T^\rho - s^\rho)^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant T,\end{cases} \tag{6}$$

and

$$G_\beta(s,\tau) = \frac{\rho^{1-\beta}\tau^{\rho-1}}{\Gamma(\beta)}\begin{cases}\left(\frac{s}{T}\right)^{\rho(\beta-1)}(T^\rho - \tau^\rho)^{\beta-1} - (s^\rho - \tau^\rho)^{\beta-1}, & 0 \leqslant \tau \leqslant s \leqslant T,\\ \left(\frac{s}{T}\right)^{\rho(\beta-1)}(T^\rho - \tau^\rho)^{\beta-1}, & 0 \leqslant s \leqslant \tau \leqslant T.\end{cases} \tag{7}$$

Proof. From Lemma 3.1 of [3], the solution to (4) could be read as the following equivalent form

$$\phi_p\left({}^\rho\mathscr{D}_{0^+}^\alpha\varphi(t)\right) = -\gamma\,{}^\rho\mathscr{I}_{0^+}^\beta z(t) - C_1 t^{\rho(\beta-1)} - C_2 t^{\rho(\beta-2)},$$

where C_1 and C_2 are real numbers associated with some initial conditions.

In terms of $\phi_p\left({}^\rho\mathscr{D}_{0^+}^\alpha\varphi\right)(0) = 0$, $\phi_p\left({}^\rho\mathscr{D}_{0^+}^\alpha\varphi\right)(T) = 0$, it yields to $C_2 = 0$ and

$$C_1 = -\frac{\gamma}{T^{\rho(\beta-1)}}\left({}^\rho\mathscr{I}_{0^+}^\beta z\right)(T).$$

Hence,

$$\begin{aligned}\phi_p\left({}^\rho\mathscr{D}_{0^+}^\alpha\varphi(t)\right) &= -\gamma\,{}^\rho\mathscr{I}_{0^+}^\beta z(t) + \gamma\left(\frac{t}{T}\right)^{\rho(\beta-1)}\left({}^\rho\mathscr{I}_{0^+}^\beta z\right)(T)\\ &= \gamma\int_0^T G_\beta(t,\tau)\,z(\tau)\,d\tau,\end{aligned}$$

which implies

$${}^\rho\mathscr{D}_{0^+}^\alpha\varphi(t) = \phi_r\left(\gamma\int_0^T G_\beta(t,\tau)\,z(\tau)\,d\tau\right),$$

where $\phi_r = (\phi_p)^{-1}$, $\frac{1}{p} + \frac{1}{r} = 1$. Accordingly, boundary value problem (BVP) (4) is equivalent to the following problem

$$\begin{cases}{}^\rho\mathscr{D}_{0^+}^\alpha\varphi(t) = \phi_r\left(\gamma\int_0^T G_\beta(t,\tau)\,z(\tau)\,d\tau\right), & 0 < t < T,\\ \varphi(0) = 0,\ \varphi(T) = 0.\end{cases} \tag{8}$$

It immediately follows from Lemma 3.2 of [3] and (8) that BVP (4) admits a unique solution described by (5). □

In the sequel, we demonstrate some properties about the Green's function $G_\alpha(t,s)$ given by (6).

LEMMA 3.2. *Let* $1<\alpha\leqslant 2$ *and* $0<\rho\leqslant 1$, *the function* G_α *defined by* (6) *satisfies the following items*

(i) $G_\alpha(t,s)>0$, *for* $t,s\in(0,T)$.

(ii) $\max\limits_{0\leqslant t\leqslant T} G_\alpha(t,s)=G_\alpha(s,s)$, *for each* $s\in[0,T]$.

(iii) *For any* $t\in\left[\frac{T}{4},\frac{3T}{4}\right]$, *there exists a positive function* $b\in C(0,T)$ *such that*

$$G_\alpha(t,s)\geqslant b(s)\max_{0\leqslant t\leqslant T} G_\alpha(t,s)=b(s)G_\alpha(s,s),\ \text{for any } 0<s<T. \tag{9}$$

Proof. Items (i) and (ii) are showed in [3].

(iii) Likewise, we could prove it as the proof of Lemma 2.4 in [2]. For given $s\in(0,T)$, $G_\alpha(t,s)$ is increasing with respect to t for $t\leqslant s$ and decreasing with respect to t for $s\leqslant t$. Consequently, setting

$$g_1(t,s)=\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\left(\frac{t}{T}\right)^{\rho(\alpha-1)}(T^\rho-s^\rho)^{\alpha-1}-(t^\rho-s^\rho)^{\alpha-1}\right],$$

$$g_2(t,s)=\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left(\frac{t}{T}\right)^{\rho(\alpha-1)}(T^\rho-s^\rho)^{\alpha-1},$$

one has

$$\begin{aligned}\min_{\frac{T}{4}\leqslant t\leqslant\frac{3T}{4}} G_\alpha(t,s)&=\begin{cases} g_1\left(\frac{3T}{4},s\right), & s\in\left(0,\frac{T}{4}\right],\\ \min\left\{g_1\left(\frac{3T}{4},s\right),g_2\left(\frac{T}{4},s\right)\right\}, & s\in\left[\frac{T}{4},\frac{3T}{4}\right],\\ g_2\left(\frac{T}{4},s\right), & s\in\left[\frac{3T}{4},T\right),\end{cases}\\ &=\begin{cases} g_1\left(\frac{3T}{4},s\right), & s\in(0,\bar{r}],\\ g_2\left(\frac{T}{4},s\right), & s\in[\bar{r},T),\end{cases}\\ &=\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\begin{cases}\left[\left(\frac{3}{4}\right)^{\rho(\alpha-1)}(T^\rho-s^\rho)^{\alpha-1}-\left(\left(\frac{3T}{4}\right)^\rho-s^\rho\right)^{\alpha-1}\right], & s\in(0,\bar{r}],\\ \left(\frac{1}{4}\right)^{\rho(\alpha-1)}(T^\rho-s^\rho)^{\alpha-1}, & s\in[\bar{r},T),\end{cases}\end{aligned}$$

where $\frac{T}{4}<\bar{r}<\frac{3T}{4}$ is the unique solution of the equation $g_1\left(\frac{3T}{4},s\right)=g_2\left(\frac{T}{4},s\right)$.

Secondly, with the use (ii),

$$\max_{0\leqslant t\leqslant T} G_\alpha(t,s)=G_\alpha(s,s)=\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left(\frac{s}{T}\right)^{\rho(\alpha-1)}(T^\rho-s^\rho)^{\alpha-1}.$$

Thus, setting

$$b(s)=\begin{cases}\dfrac{\left(\frac{3}{4}\right)^{\rho(\alpha-1)}(T^\rho-s^\rho)^{\alpha-1}-\left(\left(\frac{3T}{4}\right)^\rho-s^\rho\right)^{\alpha-1}}{\left(\frac{s}{T}\right)^{\rho(\alpha-1)}(T^\rho-s^\rho)^{\alpha-1}}, & s\in(0,\bar{r}],\\ \left(\frac{T}{4s}\right)^{\rho(\alpha-1)}, & s\in[\bar{r},T),\end{cases}$$

the proof is complete. $\square$

REMARK 1. Note that the same properties (i), (ii) and (iii) are valid for G_β with replacement β by α. Denote by

$$\omega_\alpha = \int_0^T G_\alpha(s,s)\,ds, \qquad \omega_\beta = \int_0^T G_\beta(\tau,\tau)\,d\tau.$$
$$\bar{\omega}_\alpha = \int_{\frac{T}{4}}^{\frac{3T}{4}} b(s)\,G_\alpha(s,s)\,ds, \quad \bar{\omega}_\beta = \int_{\frac{T}{4}}^{\frac{3T}{4}} b(\tau)\,G_\beta(\tau,\tau)\,d\tau.$$

Assume that $E = C([0,T],\mathbb{R})$ with maximum norm $\|\varphi\| = \max_{t\in[0,T]} |\varphi(t)|$ be Banach space. Define a mapping $F: E \to E$ by

$$F\varphi(t) = \int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma \int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right) ds, \tag{10}$$

where $G_\alpha(t,s)$, $G_\beta(s,\tau)$ are defined by (6) and (7), respectively.

By Lemma 3.1, φ is a solution to (1) which indicates that there exist a $\varphi \in E$ such that $F\varphi = \varphi$. That being the case, we first consider the existence of fixed points of the operator F.

3.1. Existence of solution

Define a cone set $\mathcal{Q}$ on E as

$$\mathcal{Q} = \{\varphi \in E \mid \varphi(t) \geqslant 0,\ \forall t \in [0,T]\}. \tag{11}$$

LEMMA 3.3. *If $f(t,\varphi)$ is a continuous function on $[0,T]\times[0,\infty)$, then $F: \mathcal{Q} \to \mathcal{Q}$ is a completely continuous operator.*

Proof. In the light of continuity of $G_\alpha(t,s)$, $G_\beta(s,\tau)$, and $f(t,\varphi)$, $F: \mathcal{Q} \to \mathcal{Q}$ is continuous. Let $\Omega \subset \mathcal{Q}$ be bounded, i.e., $\exists M_0 > 0$ such that $\|\varphi\| \leqslant M_0$, for all $\varphi \in \Omega$.

Let $L = \max_{0\leqslant t\leqslant T,\ 0\leqslant\varphi\leqslant M_0} |f(t,\varphi)| + 1$, we have

$$\begin{aligned}
|F\varphi(t)| &= \left|\int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma \int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right) ds\right| \\
&\leqslant \int_0^T G_\alpha(s,s)\,\phi_r\left(|\gamma| L \int_0^T G_\beta(\tau,\tau)\,d\tau\right) ds \\
&\leqslant \omega_\alpha \phi_r\left(|\gamma| L \omega_\beta\right).
\end{aligned} \tag{12}$$

Hence, $F(\Omega)$ is bounded.

Likewise, motivated by Lemma 3.6 in [3], for each $u \in \Omega$, $t_1, t_2 \in [0, T]$, $t_1 < t_2$, we find that

$$\begin{aligned} |F\varphi(t_2) - F\varphi(t_1)| &= \left| \begin{array}{l} \int_0^T G_\alpha(t_2,s)\,\phi_r\left(\gamma \int_0^T G_\beta(s,\tau) f(\tau,\varphi(\tau))\,d\tau\right) \\ -\int_0^T G_\alpha(t_1,s)\,\phi_r\left(\gamma \int_0^T G_\beta(s,\tau) f(\tau,\varphi(\tau))\,d\tau\right) \end{array} \right| \\ &\leqslant \phi_r\left(|\gamma| L\omega_\beta\right) \int_0^T \left[G_\alpha(t_2,s) - G_\alpha(t_1,s)\right] ds \\ &\leqslant \frac{\phi_r\left(|\gamma| L\omega_\beta\right) T^\rho}{\rho^\alpha \Gamma(\alpha)} \left(t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}\right), \end{aligned}$$

the second side of this inequality tends to zero if $t_2 \to t_1$. Hence, $F(\Omega)$ is equicontinuous. So, by Arzela-Ascoli theorem, $F : \mathscr{Q} \to \mathscr{Q}$ is completely continuous. $\square$

Denote

$$M = \frac{\phi_p\left(\frac{1}{\omega_\alpha}\right)}{\gamma\omega_\beta}, \quad N = \frac{\phi_p\left(\frac{1}{\bar{\omega}_\alpha}\right)}{\gamma\bar{\omega}_\beta}.$$

THEOREM 3.1. *For two positive constants* $r_2 > r_1 > 0$ *such that*
$(H1)$ $f(t,\varphi) \leqslant M\phi_p(r_2)$, *for* $(t,\varphi) \in [0,T] \times [0,r_2]$.
$(H2)$ $f(t,\varphi) \geqslant N\phi_p(r_1)$, *for* $(t,\varphi) \in [0,T] \times [0,r_1]$.
The BVP (1) has at least one positive solution φ *such that* $r_1 \leqslant \|\varphi\| \leqslant r_2$.

Proof. From Lemma 3.3, F is a completely continuous operator and BVP (1) admits one solution $\varphi = \varphi(t)$ if and only if φ solves the operator equation $\varphi = F\varphi$. It could be achieved from Lemma 2.1, and the main process can be divided into following two steps:

Step 1. Let $\Omega_2 = \{\varphi \in \mathscr{Q} \mid \|\varphi\| < r_2\}$. Let $\varphi \in \mathscr{Q} \cap \partial\Omega_2$. Then, from assumption $(H1)$ and (ii) in Lemma 3.2 for $t \in [0,T]$, we have

$$\begin{aligned} \|F\varphi\| &= \max_{0<t<T} \int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma \int_0^T G_\beta(s,\tau) f(\tau,\varphi(\tau))\,d\tau\right) ds \\ &\leqslant \int_0^T G_\alpha(s,s)\,\phi_r\left(\gamma \int_0^T G_\beta(\tau,\tau) M\phi_p(r_2)\,d\tau\right) ds \\ &\leqslant \int_0^T G_\alpha(s,s)\,\phi_r\left(\gamma \int_0^T G_\beta(\tau,\tau) \frac{\phi_p\left(\frac{1}{\omega_\alpha}\right)}{\gamma\omega_\beta} \phi_p(r_2)\,d\tau\right) ds \\ &\leqslant \frac{r_2}{\omega_\alpha} \int_0^T G_\alpha(s,s)\,ds \\ &\leqslant r_2 = \|\varphi\|. \end{aligned}$$

which implies that

$$\|F\varphi\| \leqslant \|\varphi\|, \text{ for all } \varphi \in \mathscr{Q} \cap \partial\Omega_2.$$

Step 2. Let $\Omega_1 = \{\varphi \in \mathcal{Q} \mid \|\varphi\| < r_1\}$, and $\varphi \in \mathcal{Q} \cap \partial\Omega_1$. Then, from $(H2)$ and (iii) in Lemma 3.2 for $t \in \left[\frac{T}{4}, \frac{3T}{4}\right]$, we have

$$\begin{aligned}
F\varphi(t) &= \int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right)ds\\
&\geqslant \int_0^T b(s)\,G_\alpha(s,s)\,\phi_r\left(\gamma\int_0^T b(\tau)\,G_\beta(\tau,\tau)\,N\phi_p(r_1)\,d\tau\right)ds\\
&\geqslant \int_{\frac{T}{4}}^{\frac{3T}{4}} b(s)\,G_\alpha(s,s)\,\phi_r\left(\gamma\int_{\frac{T}{4}}^{\frac{3T}{4}} b(\tau)\,G_\beta(\tau,\tau)\,\frac{\phi_p\left(\frac{1}{\bar{\omega}_\alpha}\right)}{\gamma\bar{\omega}_\beta}\phi_p(r_1)\,d\tau\right)ds\\
&\geqslant \frac{r_1}{\bar{\omega}_\alpha}\int_{\frac{T}{4}}^{\frac{3T}{4}} b(s)\,G_\alpha(s,s)\,ds\\
&\geqslant r_1 = \|\varphi\|.
\end{aligned}$$

So

$$\|F\varphi\| \geqslant \|\varphi\|, \quad \text{for all } \ \varphi \in \mathcal{Q}\cap\partial\Omega_1.$$

Therefore, by (ii) in Lemma 2.1, F has at least one fixed point in $\mathcal{Q}\cap\left(\bar{\Omega}_2\setminus\Omega_1\right)$. Hence, there exists one solution at least to (1) with $r_1 \leqslant \|\varphi\| \leqslant r_2$. Thus the proof is finish. □

THEOREM 3.2. *Let $f(t,\varphi)$ is continuous on $[0,T]\times[0,\infty)$. If there exists a constant $k>0$ such that*

$$\frac{\omega_\alpha\phi_r\left(|\gamma|\,L\omega_\beta\right)}{k} < 1, \tag{13}$$

where $L = \max\limits_{0\leqslant t\leqslant T,\ 0\leqslant\varphi\leqslant k}|f(t,\varphi)|+1$, then the fractional BVP (1) admits at least one solution.

Proof. Let $\Omega = \{\varphi \in \mathcal{Q} \mid \|\varphi\| < k\}$. By virtue of Lemma 3.3, the operator $F : \bar{\Omega} \to \mathcal{Q}$ is completely continuous. Let $\varphi \in \partial\Omega$ such that $\varphi = \lambda F\varphi$, $\lambda \in (0,1)$. From (12), we have

$$\begin{aligned}
\|\varphi\| = \lambda\,\|F\varphi\| &\leqslant \int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right)ds\\
&\leqslant \omega_\alpha\phi_r\left(|\gamma|\,L\omega_\beta\right),
\end{aligned}$$

hence,

$$k \leqslant \omega_\alpha\phi_r\left(|\gamma|\,L\omega_\beta\right).$$

which contradicts (13). By Lemma 2.2, the BVP (1) shares one solution $\varphi\in\Omega$ at least. □

3.2. Uniqueness of solution

In this part, the following condition will be compensated.

$(H3)$ Suppose that $f:[0,T]\times[0,\infty)\to[h,\infty)$ is continuous, and there is a positive constant L_0, satisfying

$$|f(t,\varphi)-f(t,\psi)|\leqslant L_0|\varphi-\psi|,\ \text{for any } t\in[0,T],\ \varphi,\psi\in[0,\infty). \tag{14}$$

THEOREM 3.3. *If $(H3)$ is valid, and $p>2$, then there exists a unique solution of the BVP (1), where $|\gamma|<\frac{1}{(p-1)M^{p-2}L_0\omega_\alpha\omega_\beta}$.*

Proof. Assume $B=\{\varphi\in E\mid\|\varphi\|\leqslant R\}$ where $R\geqslant\omega_\alpha\phi_r\left(|\gamma|L\omega_\beta\right)$, be closed subset. Consider the operator $F:B\to B$, defined by (10) as follows

$$F\varphi(t)=\int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right)ds.$$

For $\varphi\in B,$ we have

$$\begin{aligned}\|F\varphi\|&\leqslant\left|\int_0^T G_\alpha(t,s)\,\phi_r\left(\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right)ds\right|\\&\leqslant\omega_\alpha\phi_r\left(|\gamma|L\omega_\beta\right)\leqslant R,\end{aligned}$$

which proves that $F(B)\subset B$.

Now we shall show that F is a contraction mapping. For $t\in[0,T]$, we get

$$\left|\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right|\leqslant|\gamma|\,\omega_\beta L=M,$$

where $L=\max\limits_{0\leqslant t\leqslant T,\ 0\leqslant\varphi\leqslant R}|f(t,\varphi)|+1.$

By (ii) in Lemma 2.4 and (14), for any $\varphi,\psi\in B$ and $t\in[0,T]$, we have

$$\begin{aligned}&|F\varphi(t)-F\psi(t)|\\&\leqslant\int_0^T G_\alpha(t,s)\left|\phi_r\left(\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right)\right.\\&\quad\left.-\phi_r\left(\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\psi(\tau))\,d\tau\right)\right|ds\\&\leqslant|\gamma|(p-1)M^{p-2}\int_0^T G_\alpha(t,s)\left|\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right.\\&\quad\left.-\int_0^T G_\beta(s,\tau)\,f(\tau,\psi(\tau))\,d\tau\right|ds\\&\leqslant|\gamma|(p-1)M^{p-2}\int_0^T G_\alpha(t,s)\left(\int_0^T G_\beta(s,\tau)\,|f(\tau,\varphi(\tau))-f(\tau,\psi(\tau))|\,d\tau\right)ds\\&\leqslant|\gamma|(p-1)M^{p-2}L_0\int_0^T G_\alpha(s,s)\left(\int_0^T G_\beta(\tau,\tau)\,|\varphi(\tau)-\psi(\tau)|\,d\tau\right)ds\\&\leqslant|\gamma|(p-1)M^{p-2}L_0\omega_\alpha\omega_\beta\,\|\varphi-\psi\|,\end{aligned}$$

then

$$\|F\varphi - F\psi\| \leqslant |\gamma|\,(p-1)\,M^{p-2}L_0\,\omega_\alpha\,\omega_\beta\,\|\varphi - \psi\|\,.$$

From (3.3) and the previous estimate, F is a contraction operator. As a consequence of Lemma 2.3, we conclude that F has a unique fixed point which is the unique solution of the problem (1) in B. □

THEOREM 3.4. *If $(H3)$ is valid, and $1 < p < 2$, then there exists a unique solution of the BVP (1), where $|\gamma| < \frac{1}{(p-1)m^{p-2}L_0\,\omega_\alpha\,\omega_\beta}$.*

Proof. By Lemma 3.2, for $s \in \left[\frac{T}{4}, \frac{3T}{4}\right]$, we get

$$\begin{aligned}\left|\gamma\int_0^T G_\beta(s,\tau)\,f(\tau,\varphi(\tau))\,d\tau\right| &\geqslant |\gamma|\int_{\frac{T}{4}}^{\frac{3T}{4}} b(\tau)\,G_\beta(\tau,\tau)\,f(\tau,\varphi(\tau))\,d\tau \\ &\geqslant |\gamma|\,h\bar{\omega}_\beta = m > 0.\end{aligned}$$

By (i) in Lemma 2.4 and (14), for any $\varphi,\,\psi \in B$ and $t \in [0,\,T]$, we have

$$\|F\varphi - F\psi\| \leqslant |\gamma|\,(p-1)\,m^{p-2}L_0\,\omega_\alpha\,\omega_\beta\,\|\varphi - \psi\|\,.$$

From (3.4) and the previous estimate, F be contraction operator. As a consequence of Lemma 2.3, one could conclude that F admits a unique fixed point that is the unique solution of the problem (1) in B. □

4. Illustrative examples

In this section, several examples are provided to illustrate the results obtained in Theorems 3.1, 3.2, 3.3 and 3.4, respectively.

EXAMPLE 1. Consider the following BVP

$$\begin{cases} {}^1\mathscr{D}_{0^+}^{\frac{3}{2}}\left(\phi_2\left({}^1\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi(t)\right)\right) + \gamma\left(\frac{(\arctan t)^2}{1+\varphi^2} + 100\right) = 0,\ t \in [0,\,1] \\ \varphi(0) = 0,\ \varphi(1) = 0 \\ \phi_2\left({}^1\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi\right)(0) = 0,\ \phi_2\left({}^1\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi\right)(1) = 0 \end{cases}, \tag{15}$$

here, $f(t,\varphi) = \frac{(\arctan t)^2}{1+\varphi^2} + 100$, $\alpha = \frac{5}{4}$, $\beta = \frac{3}{2}$ and $p = 2$. In this case, the function f be jointly continuous for any $t \in [0,\,1]$, and any $\varphi > 0$.

We have

$$\omega_\alpha = \int_0^1 G_\alpha(s,s)\,ds = \frac{1}{\Gamma\left(\frac{5}{4}\right)} \int_0^1 s^{\frac{1}{4}} (1-s)^{\frac{1}{4}}\,ds = 0.68184,$$

$$\omega_\beta = \int_0^1 G_\beta(\tau,\tau)\,d\tau = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 \tau^{\frac{1}{2}} (1-\tau)^{\frac{1}{2}}\,d\tau = 0.44311,$$

$$\bar{\omega}_\alpha = \int_{\frac{1}{4}}^{\frac{3}{4}} b(s)\,G_\alpha(s,s)\,ds = 0.056961,$$

$$\bar{\omega}_\beta = \int_{\frac{1}{4}}^{\frac{3}{4}} b(\tau)\,G_\beta(\tau,\tau)\,d\tau = 0.07318,$$

and

$$M = \frac{\phi_p\left(\frac{1}{\omega_\alpha}\right)}{\gamma\omega_\beta} = \frac{3.3098}{\gamma},\ \ N = \frac{\phi_p\left(\frac{1}{\bar{\omega}_\alpha}\right)}{\gamma\bar{\omega}_\beta} = \frac{239.90}{\gamma}.$$

Choosing $r_1 = \frac{1}{100}$, $r_2 = 1$, for $2.399 \times 10^{-2} \leqslant \gamma \leqslant 3.2894 \times 10^{-2}$, we have

$$f(t,\varphi) = \frac{(\arctan t)^2}{1+\varphi^2} + 100 \leqslant \frac{\pi^2}{16} + 100 \leqslant M\phi_p(r_2),\ \text{for } (t,\ \varphi) \in [0,1] \times [0,1],$$

$$f(t,\varphi) = \frac{(\arctan t)^2}{1+\varphi^2} + 100 \geqslant 100 \geqslant N\phi_p(r_1),\ \text{for } (t,\ \varphi) \in [0,\ 1] \times \left[0,\ \frac{1}{100}\right].$$

Then, the condition $(H1)$, $(H2)$ are satisfied for each $2.399 \times 10^{-2} \leqslant \gamma \leqslant 3.2894 \times 10^{-2}$. Therefore from Theorem 3.1, the problem (15) has at least one solution φ such that $\frac{1}{100} \leqslant \|\varphi\| \leqslant 1$.

EXAMPLE 2. Consider the following BVP

$$\begin{cases} {}^1\mathscr{D}_{0^+}^{\frac{3}{2}}\left(\phi_2\left({}^1\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi(t)\right)\right) + \gamma\frac{\ln(1+\varphi(t))}{1+t^2} = 0,\ t \in [0,1] \\ \varphi(0) = 0,\ \varphi(1) = 0 \\ \phi_2\left({}^1\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi\right)(0) = 0,\ \phi_2\left({}^1\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi\right)(1) = 0 \end{cases}, \tag{16}$$

here, $f(t,\varphi) = \frac{\ln(1+\varphi(t))}{1+t^2}$, $\alpha = \frac{5}{4}$, $\beta = \frac{3}{2}$, $\gamma = \frac{1}{\sqrt{\pi}}$. If choose $k = 1$, one has

$$L = \max_{0\leqslant t\leqslant 1,\ 0\leqslant\varphi\leqslant 1} |f(t,\varphi)| + 1 = 1 + \ln 2,$$

then for $|\gamma| < 1.9548$, we obtain

$$\frac{\omega_\alpha\phi_r\left(|\gamma|L\omega_\beta\right)}{k} = (0.51155)\,|\gamma| < 1.$$

Therefore, (13) is valid. So by Theorem 3.2, such problem (16) owns one solution at least.

EXAMPLE 3. Consider the following BVP

$$\begin{cases} {}^{1}\mathscr{D}_{0^+}^{\frac{3}{2}}\left(\phi_p\left({}^{1}\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi(t)\right)\right)+\frac{\cos(t)(2+|\varphi(t)|)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)(1+|\varphi(t)|)}=0,\ t\in\left[0,\frac{\pi}{4}\right] \\ \varphi(0)=0,\ \varphi\left(\frac{\pi}{4}\right)=0 \\ \phi_p\left({}^{1}\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi\right)(0)=0,\ \phi_p\left({}^{1}\mathscr{D}_{0^+}^{\frac{5}{4}}\varphi\right)\left(\frac{\pi}{4}\right)=0 \end{cases}, \tag{17}$$

here, $f(t,\varphi)=\frac{\cos(t)(2+|\varphi|)}{\left(\sqrt{2}\cos(t)+\sin(t)\right)(1+|\varphi|)}$, $\alpha=\frac{5}{4}$, $\beta=\frac{3}{2}$, $\gamma=\frac{1}{\sqrt{\pi}}$.

Let $B=\{\varphi\in E \mid \|\varphi\|\leqslant 1\}$, one has

$$L=\max_{0\leqslant t\leqslant\frac{\pi}{4},\ 0\leqslant\varphi\leqslant 1}|f(t,\varphi)|+1=\frac{3}{\sqrt{2}}+1,$$
$$M=|\gamma|\,\omega_\beta L\simeq 0.97344,\ m=|\gamma|\,h\bar{\omega}_\beta\simeq 1.3762\times 10^{-2}.$$

Clearly, $f:\left[0,\frac{\pi}{4}\right]\times[0,\infty)\to\left[\frac{1}{3},\infty\right)$ is continuous.

For any $\varphi,\psi\in\mathbb{R}$ and $t\in\left[0,\frac{\pi}{4}\right]$, due to $\frac{\sqrt{2}}{2}\leqslant\cos(t)\leqslant 1$ and $0\leqslant\sin(t)\leqslant\frac{\sqrt{2}}{2}$, then

$$\begin{aligned}|f(t,\varphi)-f(t,\psi)| &= \left|\frac{\cos(t)(2+|\varphi|)}{\left(\sqrt{2}\cos(t)+\sin(t)\right)(1+|\varphi|)}-\frac{\cos(t)(2+|\psi|)}{\left(\sqrt{2}\cos(t)+\sin(t)\right)(1+|\psi|)}\right| \\ &= \left|\frac{\cos(t)}{\sqrt{2}\cos(t)+\sin(t)}\right|\left|\frac{2+|\varphi|}{1+|\varphi|}-\frac{2+|\psi|}{1+|\psi|}\right| \\ &\leqslant \left||\varphi|-|\psi|\right|\leqslant|\varphi-\psi|.\end{aligned}$$

Hence, the condition (14) is satisfied with $L_0=1$.

(i) If $p=3$, it remains to show that the condition (3.3)

$$|\gamma|=\frac{1}{\sqrt{\pi}}<\frac{1}{(p-1)M^{p-2}L_0\omega_\alpha\omega_\beta}=1.7001,$$

is valid. Therefore, by Theorem 3.3, system (17) takes a unique solution in B.

(ii) If $p=\frac{3}{2}$, it remains to show that the condition (3.4)

$$|\gamma|=\frac{1}{\sqrt{\pi}}<\frac{1}{(p-1)m^{p-2}L_0\omega_\alpha\omega_\beta}=0.77656,$$

is valid. Therefore, by Theorem 3.4, system (17) takes a unique solution in B.

5. Conclusions

In our paper, we study the existence and uniqueness of solution for the nonlinear KFDEs with classic p-Laplacian operator subjecting to mixed boundary conditions. In order to demonstrate the existence and uniqueness of solutions of a class of nonlinear KFDEs involving the Katugampola fractional derivative with p-Laplacian operator subjecting to mixed boundary conditions, we firstly prove the equivalence of a nonlinear KFDE to a Volterra integral equation. Next, the Green functions of the corresponding nonlinear KFDEs are constructed and some properties of such Green functions are also analyzed. Thereupon, by virtue of established properties of the Green functions and suitable fixed point theorems on cones, some existence and uniqueness of solutions including existence of positive solutions, are addressed and illustrated.

The current study be more emphasis on existence and uniqueness of such problem, but it is known that the dynamics including stability and instability of solutions to KFDEs is still in its infancy. Such closed topics are of great interest and will be investigated in the sequel.

Conflict of interest. The authors declare that there are no any associative or commercial interest that represents a conflict of interest in connection with the work submitted.

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