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A UNIQUENESS CRITERION FOR NONTRIVIAL SOLUTIONS OF THE NONLINEAR HIGHER–ORDER ∇–DIFFERENCE SYSTEMS OF FRACTIONAL–ORDER

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Dedicated to the my first and best teacher Prof. Samad Alidoost

(*Communicated by F. Atici*)

Abstract. The main aim of this article is to establish a uniqueness criterion for coupled systems of the nonlinear higher-order ∇-difference boundary value problems. To this end, the coincidence degree theory has chosen to make a solvability space for the existence of at least one solution to the under investigation fractional-order system. Next, we create some conditions that enable us to prove the existence of the exactly one solution of the under study fractional-order system. At the end, a numerical example is given to illustrate the applicability of the obtained theoretical criterion.

1. Introduction and preliminaries

The theory of fractional calculus basically acts on the integral/differential operators as $D_t^\alpha \equiv d^\alpha/dt^\alpha$ having the sliding arbitrary order $\alpha \in \mathbb{R}$, that generalize the integer order integration/differentiation. Nowadays the knowledge of fractional mathematics appears in many specialized sciences such as viscoelastic materials, porous and fractured media, bioengineering, electrochemical processes and so many other real life phenomena, see [21], [24]–[26], [31]–[34] for details.

On the other hand, the literature is witnessed the boom of the abstract developments of the theory of fractional calculus inspiring by the traditional foundations of the fractional calculus such as [21], [24], [25], [26]. Mostly, these developments are appeared regarding to some particular properties of the fractional order operators such as those fractional-order operators that are constructed on the shoulders of the Riemann-Liouville fractional integration and differentiation operators acting on the appropriate

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functions and given by:

$$\mathscr{I}^{\rho}_{a^+(b_-)}f(t)=\begin{cases}\mathscr{I}^{\rho}_{a^+}f(t)=\dfrac{1}{\Gamma(\rho)}\displaystyle\int_a^t(t-s)^{\rho-1}f(s)ds; & \rho>0,\\ \mathscr{I}^{\rho}_{b_-}f(t)=\dfrac{1}{\Gamma(\rho)}\displaystyle\int_t^b(s-t)^{\rho-1}f(s)ds; & \rho>0,\\ f(t) ; & \rho=0,\end{cases}$$

and

$$\mathscr{D}^{\rho}_{a^+(b_-)}f(t)=\begin{cases}\mathscr{D}^{\rho}_{a^+}f(t)=\dfrac{1}{\Gamma(n-\rho)}\left(\dfrac{d^n}{dt^n}\right)\displaystyle\int_a^t(t-s)^{n-\rho-1}f(s)ds; & \rho>0,\\ \mathscr{D}^{\rho}_{b_-}f(t)=\dfrac{(-1)^n}{\Gamma(n-\rho)}\left(\dfrac{d^n}{dt^n}\right)\displaystyle\int_t^b(s-t)^{n-\rho-1}f(s)ds; & \rho>0,\\ f(t) ; & \rho=0,\end{cases}$$

respectively, such that $\alpha>0$ and $n=[\alpha]+1$. Some of these fractional-order operators are the Hadamard, Erdeli-Kober, Hilfer and Caputo fractional operators (see [21] for more details), or, concentrating on the lack of the Leibniz-rule in the Riemann-Liouville based fractional-order operators (see [30]), one can mentioned the recently introduced fractional-order operators that generalize the limit approach of classic differentiation and admitting the fractional version of the Leibniz-rule, namely the fractional conformable integration and differentiation operators that were introduced by [19], generalized by [1] and are defined as:

$$I^{\nu}f(t)=\begin{cases}I^{\nu}_a f(t)=\dfrac{1}{n!}\displaystyle\int_a^t(t-s)^n(s-a)^{\nu-n-1}f(s)ds,\\ {}_bI^{\nu}f(t)=\dfrac{1}{n!}\displaystyle\int_t^b(s-t)^n(b-s)^{\nu-n-1}f(s)ds,\end{cases}$$

for $f\in L^1(a,b)$ and

$$T^{\nu}f(t)=\begin{cases}T^{\nu}_a f(t)=\lim\limits_{\varepsilon\to 0}\dfrac{f^{([\nu]-1)}(t+\varepsilon(t-a)^{[\nu]-\nu})-f^{([\nu]-1)}(t)}{\varepsilon},\\ {}_bT^{\nu}f(t)=(-1)^{n+1}\lim\limits_{\varepsilon\to 0}\dfrac{f^{([\nu]-1)}(t+\varepsilon(b-t)^{[\nu]-\nu})-f^{[\nu]-1}(t)}{\varepsilon},\end{cases}$$

respectively, where $n<\nu\leqslant n+1$, $n\in\mathbb{N}\cup\{0\}$ and f is n-differentiable function.

This variety in the theory naturally makes variety in the applications. In this way, almost all of the applications are constructed on the solutions of the considered fractional-order problems, that is why the concept of solvability of the fractional-order problems is of the greatest importance. Here we suggest a collection of the solvability tools for the fractional-order differential/difference equations, [3]–[11], [14]–[18], [20], [22], [23], [27]–[29], [31]–[35] and related cited bibliography therein. But the main purpose of this paper is devoted to the completely different fractional-order operators. Indeed, we are dealt with the study about fractional difference operators that are divided into the fractional Δ-differences and fractional ∇-differences. As a useful collection of the references about discrete fractional calculus, we suggest here the sample-wise references [2]–[11], [13], [14], [20], [22], [23], [27], [29] and the cited bibliography to more consultation on topic.

Between the aforementioned research works, we are interested in the investigation about the existence and uniqueness criteria for the discrete version of the following system of the higher-order fractional boundary value problems

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t, v(t), v'(t), v''(t), \dots, v^{(N-1)}(t)), \\ D^{\beta}_{0^+}v(t) = f(t, u(t), u'(t), u''(t), \dots, u^{(N-1)}(t)), \end{cases} \tag{1.1}$$

subject to the coupled two-point boundary conditions

$$\begin{cases} u^{(k)}(0) = 0,\ u^{(N-1)}(0) = u^{(N-1)}(1), \\ v^{(k)}(0) = 0,\ v^{(N-1)}(0) = v^{(N-1)}(1), \end{cases} \tag{1.2}$$

where $N-1 < \alpha, \beta \leqslant N$, $N \in \mathbb{N}_2$, $0 \leqslant k \leqslant N-2$ and $0 < t < 1$. In this work, f and g are two given continuous functions. In fact, this fractional-order system is the main problem in the research work [17], where the authors in the light of the coincidence degree theory, make some sufficient conditions to reach at least one solution for the fractional-order system (1.1)–(1.2), and then, presenting some additional ones, they attempt to find a unique solution for this system. So, we consider the following nonlinear higher-order fractional ∇-difference system

$$\begin{cases} \nabla^{\alpha}_{a^+}y(t) = f(t, z, \nabla^{\alpha-(n-1)}_{a^+}z, \nabla^{\alpha-(n-2)}_{a^+}z, \dots, \nabla^{\alpha-1}_{a^+}z), \\ \nabla^{\alpha}_{a^+}z(t) = g(t, y, \nabla^{\alpha-(n-1)}_{a^+}y, \nabla^{\alpha-(n-2)}_{a^+}y, \dots, \nabla^{\alpha-1}_{a^+}y), \end{cases} \tag{1.3}$$

subject to the two-point boundary conditions

$$\begin{cases} \nabla^{\alpha-n}_{a^+}y(a+1) = \dots = \nabla^{\alpha-2}_{a^+}y(a+1) = 0, \quad \nabla^{\alpha-1}_{a^+}y(a+1) = \nabla^{\alpha-1}_{a^+}y(b), \\ \nabla^{\alpha-n}_{a^+}z(a+1) = \dots = \nabla^{\alpha-2}_{a^+}z(a+1) = 0, \quad \nabla^{\alpha-1}_{a^+}z(a+1) = \nabla^{\alpha-1}_{a^+}z(b), \end{cases} \tag{1.4}$$

where $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}_2$, and $t \in \mathbb{N}^b_a$, $a, b \in \mathbb{Z}_0$, $b \geqslant a+3$. $\nabla^{\alpha}_{a^+}$ stands for the fractional ∇-difference operator of order $\alpha > 0$. Throughout this paper we will assume that $f, g : \mathbb{N}^b_{a+1} \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

At the end of this section, we state the organization of the rest of the paper. In section 2, some definitions and technical lemmas regarding the ∇-fractional calculus are given. Besides, we will have a brief tour on the coincidence degree theory as the main solvability key to the fractional-order ∇-difference system (1.3)–(1.4). Section 3, as the main body of this paper includes the existence and uniqueness criteria for the fractional-order ∇-difference system (1.3)–(1.4). In order to justify that the presented existence and uniqueness criteria are implementable in practice, a numerical example in section 4 is given. Finally, we have the section 5 as the conclusion of the paper, where the managed investigation on the fractional-order ∇-difference system (1.3)–(1.4) will be summarized.

2. Basic requirements

This segment of the paper, is actually the setting garage of the solvability machine of the fractional-order ∇-difference system (1.3)–(1.4). In order to activate this mechanism, we start with the fractional rising functions that construct the kernels of the fractional ∇-difference operators.

DEFINITION 2.1. [2], [[13], Chap. 3] For a natural number m, the rising function of t is defined by

$$t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \quad t^{\overline{0}} = 1.$$

Its generalization for any real number α that is called fractional rising function is given by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{\dots,-2,-1,0\}, \quad 0^{\overline{\alpha}} = 0. \tag{2.1}$$

Note that $\nabla\left(t^{\overline{\alpha}}\right) = \alpha t^{\overline{\alpha-1}}$.

NOTATION. For each $a,b \in \mathbb{R}$,

$$\mathbb{N}_a = \{a,a+1,a+2,\dots\}, \ {}_b\mathbb{N} = \{b,b-1,b-2,\dots\}, \ \mathbb{N}_a^b = \{a,a+1,\dots,b-1,b\}. \tag{2.2}$$

Having the above background in hand, now we are enable to define the fractional ∇-difference operators as follows.

DEFINITION 2.2. [2], [[13], Chap. 3] Fractional left and right sided ∇-sums of order $\alpha > 0$ are defined as

$$\nabla^{-\alpha} f(t) = \begin{cases} \nabla_{a^+}^{-\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{s=a}^{t} (t-\delta(s))^{\overline{\alpha-1}} f(s), \\ \nabla_{b_-}^{-\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{s=t}^{b} (s-\delta(t)^{\overline{\alpha-1}} f(s), \end{cases} \tag{2.3}$$

where $\delta(s) = s-1$.

REMARK 2.3. [2], [[13], Chap. 3] Fractional left and right sided ∇-sums of order $\alpha > 0$, defined by (2.3) have the following properties:

(i) $\nabla_{a^+}^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$.

(ii) $\nabla_{b_-}^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

DEFINITION 2.4. [2], [[13], Chap. 3] Fractional left and right sided ∇-differences of order $0 \leqslant n-1 < \alpha \leqslant n$ for $n \in \mathbb{N}$ are given by

$$\nabla^{\alpha} f(t) = \begin{cases} \nabla^{\alpha}_{a^+} f(t) = \dfrac{1}{\Gamma(n-\alpha)} \nabla^n_t \left(\displaystyle\sum_{s=a}^{t} (t-\delta(s))^{\overline{n-\alpha-1}} f(s) \right), \\ \nabla^{\alpha}_{b_-} f(t) = \dfrac{(-1)^n}{\Gamma(n-\alpha)} \Delta^n_t \left(\displaystyle\sum_{s=t}^{b} (s-\delta(t)^{\overline{n-\alpha-1}} f(s) \right), \end{cases} \tag{2.4}$$

such that $\alpha > 0$, $n = [\alpha]+1$ and Δ_t denotes the forward difference operator with respect to the variable t.

REMARK 2.5. [2], [[13], Chap. 3] Fractional left and right sided ∇-differences of order $\alpha > 0$, defined by (2.4) have the following properties:

(i) $\nabla^{\alpha}_{a^+}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n}$,

(ii) $\nabla^{\alpha}_{b_-}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-n}\mathbb{N}$,

where $n = [\alpha]+1$.

Here is worthy time to state some of the most important properties of the fractional ∇-difference operators that will be the road lights to establish the main results.

LEMMA 2.6. [2], [[13], Chap. 3] *Assume that f is a real-valued function and $\mu > 0$, $0 \leqslant n-1 < \nu \leqslant n$. Then*

(Q_1) $\nabla^{-\mu}_{a^+} \nabla^{-\nu}_{a^+} f(t) = \nabla^{-(\mu+\nu)}_{a^+} f(t) = \nabla^{-\nu}_{a^+} \nabla^{-\mu}_{a^+} f(t)$,

(Q_2) $\nabla^{-\nu}_{a^+}\ \nabla^{\nu}_{a^+} f(t) = f(t) + c_1(t-a)^{\overline{\nu-1}} + c_2(t-a)^{\overline{\nu-2}} + \ldots + c_n(t-a)^{\overline{\nu-n}}$, $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$.

(Q_3) $\nabla^{\nu}_{a^+}\ \nabla^{-\nu}_{a^+} f(t) = f(t)$.

(Q_4) $\nabla^{\nu}_{a^+} (t-a)^{\overline{\mu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)} (t-a)^{\overline{\mu-\nu}}$, $\mu-\nu+1 \notin (-\mathbb{Z}_0)$.

Since our main aim is to sake unique nontrivial solution for the fractional-order ∇-difference system (1.3)–(1.4), then, we continue as follows. As we know

$$\nabla^{\alpha}_{a^+} u(t) = 0 \quad \text{iff} \quad u(t) = c_1(t-a)^{\overline{\nu-1}} + c_2(t-a)^{\overline{\nu-2}} + \ldots + c_n(t-a)^{\overline{\nu-n}},\ c_i \in \mathbb{R},\ i \in \mathbb{N}^n_1. \tag{2.5}$$

Therefore, thanks to the properties (Q_2) and (Q_4) in Lemma 2.6 we have

$$\begin{aligned} \nabla^{\alpha-j}_{a^+} u(t) &= \nabla^{\alpha-j}_{a^+} \left(\sum_{i=1}^{n} c_i (t-a)^{\overline{\alpha-i}} \right) = \sum_{i=1}^{n} c_i \nabla^{\alpha-j}_{a^+} (t-a)^{\overline{\alpha-i}} \\ &= \sum_{i=1}^{n} c_i \frac{\Gamma(\alpha-i+1)}{\Gamma(j-i+1)} (t-a)^{\overline{j-i}},\ j \in \mathbb{N}^n_2. \end{aligned} \tag{2.6}$$

Using $1^{\overline{\beta}} = \beta+1$ and considering the substitution $t = a+1$, yields the following algebraic system:

$$\begin{cases} c_1\Gamma(\alpha)+c_2\Gamma(\alpha-1)=0, \\ c_1\Gamma(\alpha)+c_2\Gamma(\alpha-1)+c_3\Gamma(\alpha-2)=0, \\ c_1\Gamma(\alpha)+c_2\Gamma(\alpha-1)+c_3\Gamma(\alpha-2)+c_4\Gamma(\alpha-3)=0, \\ \vdots \\ c_1\Gamma(\alpha)+c_2\Gamma(\alpha-1)+c_3\Gamma(\alpha-2)+\ldots+c_n\Gamma(\alpha-n+1)=0. \end{cases} \tag{2.7}$$

Some manipulations on the algebraic system (2.7), we get that

$$\alpha c_1+c_2=0, \quad c_i=0,\ i=3,4,\ldots,n.$$

So the homogeneous fractional ∇-difference equation $\nabla^{\alpha}_{a^+}u(t)=0$ has a nontrivial solution. Relying on this fact, now we are in such a position that can establish a solvability framework to reach a uniqueness criterion for the nontrivial solutions of the fractional-order ∇-difference system (1.3)–(1.4). Prior to this analysis, it will be better understanding of the existence path provided that we have a quick overview on the coincidence degree theory as follows [see [12]; Chapters IV,V, for more details].

DEFINITION 2.7. Assume that $\mathfrak{B}$ and $\mathfrak{D}$ be real normed spaces. A linear mapping $L : dom\ L \subset \mathfrak{B} \to \mathfrak{D}$ is called a Fredholm mapping provided that the following conditions hold:

(i) $\ker L$ has a finite dimension,

(ii) $\mathrm{Im}L$ is closed and has a finite codimension.

Let L is a Fredholm mapping. Then its *index* is given by

$$Ind\ L = dim\ kerL - codim\ ImL.$$

Assume that L is a Fredholm mapping with index zero and there exist continuous projectors $P : \mathfrak{B} \to \mathfrak{B}$ and $Q : \mathfrak{D} \to \mathfrak{D}$ such that

$$ImP = \ker L, \quad \ker Q = ImL, \quad \mathfrak{B} = \ker L \oplus \ker P, \quad \mathfrak{D} = ImL \oplus ImQ.$$

It follows that the mapping

$$L|_{domL\cap\ker P} : domL\cap\ker P \to ImL$$

is invertible. Let us denote the inverse by $K_P : ImL \to domL\cap\ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \to domL\cap\ker P$ is defined by $K_{P,Q} = K_P(I-Q)$.

If L is a Fredholm mapping of index zero, then for every isomorphism $J : ImQ \to \ker L$, the mapping $JQ+K_{P,Q} : Z \to domL$ is an isomorphism and, for every $u \in domL$,

$$(JQ+K_{P,Q})^{-1}u = (L+J^{-1}P)u.$$

DEFINITION 2.8. Let $L : domL \subset \mathfrak{B} \to \mathfrak{D}$ be a Fredholm mapping, E be a metric space, and $N : E \to \mathfrak{D}$ be a mapping. N is to be called L-compact on E provided that, $QN : E \to \mathfrak{D}$ is continuous and $K_{P,Q} : E \to \mathfrak{B}$ is compact on E. In addition, we say that, N is L-completely continuous if it is L-compact on every bounded $E \subset \mathfrak{B}$.

THEOREM 2.9. *Let $\Omega \subset \mathfrak{B}$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

- (i) *$Lu \neq \lambda Nu$ for every $(u,\lambda) \in ((domL \setminus \ker L) \cap \partial\Omega) \times (0,1)$;*
- (ii) *$Nu \notin ImL$ for every $u \in \ker L \cap \partial\Omega$;*
- (iii) *$\deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$ with $Q : \mathfrak{D} \to \mathfrak{D}$ a continuous projector such that $\ker Q = ImL$ and $J : ImQ \to \ker L$ is an isomorphism.*

Then the abstract operator equation $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$.

3. Main results

As stated above, in order to find at least one solution for the fractional-order ∇-difference system (1.3)–(1.4), at first we have to establish an appropriate functional space that gives us enough ability to control the growth of the nonlinearities f and g, introduced in the right-hand sides of the governing equations (1.3). So, we begin as follows.

$$(E, \|.\|_E), \quad E = C\left(\mathbb{N}_{a+1}^b, \mathbb{R}\right), \quad \|f\|_E = \max_{t \in \mathbb{N}_{a+1}^b} |f(t)|, \tag{3.1a}$$

$$(X, \|.\|_X), \quad X = \left\{ u \,\middle|\, u \in E,\ \nabla_{a^+}^{\alpha-i} u \in F_i,\ F_i = C\big(\mathbb{N}_{a+n-i+1}^{b+n-i}, \mathbb{R}\big),\ i = 1,2,\ldots,n-1 \right\}, \tag{3.1b}$$

$$\|u\|_X = \max\left\{ \|u\|_E, \left\|\nabla_{a^+}^{\alpha-i} u\right\|_{F_i},\ i = 1,2,\ldots,n-1 \right\}. \tag{3.1c}$$

Based on the recent Banach spaces, the desired Banach spaces that will be applied in this paper are introduced as follows:

$$(\mathfrak{B}, \|.\|_{\mathfrak{B}}), \quad \begin{cases} \mathfrak{B} = X \times X, \\ \|(u,v)\|_{\mathfrak{B}} = \max\{\|u\|_X, \|v\|_X\}, \end{cases} \tag{3.2a}$$

$$(\mathfrak{D}, \|.\|_{\mathfrak{D}}), \quad \begin{cases} \mathfrak{D} = E \times E, \\ \|(u,v)\|_{\mathfrak{D}} = \max\{\|u\|_E, \|v\|_E\}. \end{cases} \tag{3.2b}$$

Now it is time to provide the elements of the coincidence degree theory. To this aim, we begin with defining the operators $L_i : \mathrm{dom}L_i \cap X \to E$, $i = 1,2$ as follows:

$$L_1 y = \nabla_{a^+}^{\alpha} y,\ \mathrm{dom}L_1 = \left\{ y \in X \,\middle|\, \nabla_{a^+}^{\alpha-i} y(a+1) = 0,\ \nabla_{a^+}^{\alpha-1} y(a+1) = \nabla_{a^+}^{\alpha-1} y(b),\ i \in \mathbb{N}_1^n \right\}, \tag{3.3a}$$

$$L_2 z = \nabla_{a^+}^{\alpha} z,\ \mathrm{dom}L_2 = \left\{ z \in X \,\middle|\, \nabla_{a^+}^{\alpha-i} z(a+1) = 0,\ \nabla_{a^+}^{\alpha-1} z(a+1) = \nabla_{a^+}^{\alpha-1} z(b),\ i \in \mathbb{N}_1^n \right\}. \tag{3.3b}$$

Thereafter, we can define the operator $L : \mathrm{dom}L \cap \mathfrak{B} \to \mathfrak{D}$ as

$$L(y,z) = (L_1 y, L_2 z), \tag{3.4}$$

where

$$\mathrm{dom}L = \{(y,z) \in \mathfrak{B} |\ y \in \mathrm{dom}L_1,\ z \in \mathrm{dom}L_2\}. \tag{3.5}$$

Next, we define the operator $N : \mathfrak{B} \to \mathfrak{D}$ as

$$N(y,z) = (N_1 z, N_2 y), \tag{3.6}$$

where $N_\gamma : X \to E$ for $\gamma \in \{1,2\}$, are defined as below

$$N_1 z = f(t, z, \nabla_{a^+}^{\alpha-(n-1)} z, \nabla_{a^+}^{\alpha-(n-2)} z, \ldots, \nabla_{a^+}^{\alpha-1} z), \tag{3.7a}$$

$$N_2 y = g(t, y, \nabla_{a^+}^{\alpha-(n-1)} y, \nabla_{a^+}^{\alpha-(n-2)} y, \ldots, \nabla_{a^+}^{\alpha-1} y). \tag{3.7b}$$

Turning to the coincidence degree theory discussed in previous section and comparing the main problem (1.3)–(1.4) with (3.3a)–(3.7b), we find that $L(y,z) = N(y,z)$. Continuing our research path, we have to prove that the operator L defined by (3.3a)–(3.4) is a Fredholm operator having index zero.

LEMMA 3.1. *The operator $L : domL \cap \mathfrak{B} \to \mathfrak{D}$ defined by* (3.3a)–(3.4) *is a Fredholm operator with index zero.*

Proof. Applying the property (Q_2) in Lemma 2.6, it follows that $\ker L = (t-a)^{\overline{\alpha-1}}(c_1,d_1)$, $c_1, d_1 \in \mathbb{R}$. Thus, $\ker L \cong \mathbb{R}^2$. Assuming $(u,v) \in \mathrm{Im}L$, there exists $(y,z) \in \mathrm{dom}L$, such that $L(y,z) = (u,v)$. Thereby

$$\begin{aligned} y(t) &= \nabla_{a^+}^{-\alpha} u(t) + c_1 (t-a)^{\overline{\alpha-1}} + c_2 (t-a)^{\overline{\alpha-2}} + \ldots + c_n (t-a)^{\overline{\alpha-n}}, \\ z(t) &= \nabla_{a^+}^{-\alpha} v(t) + d_1 (t-a)^{\overline{\alpha-1}} + d_2 (t-a)^{\overline{\alpha-2}} + \ldots + d_n (t-a)^{\overline{\alpha-n}}. \end{aligned}$$

Definition of the operator L in (3.3a)–(3.4), consequences that $c_i = d_i = 0$, $i = 2,3,\ldots,n$. So, we conclude that

$$\begin{aligned} y(t) &= \nabla_{a^+}^{-\alpha} u(t) + c_1 (t-a)^{\overline{\alpha-1}}, \\ z(t) &= \nabla_{a^+}^{-\alpha} v(t) + d_1 (t-a)^{\overline{\alpha-1}}. \end{aligned}$$

Hence

$$\begin{aligned} \nabla_{a^+}^{\alpha-1} y(t) &= \nabla_{a^+}^{\alpha-1} \left(\nabla_{a^+}^{-\alpha} u\right)(t) + c_1 \Gamma(\alpha), \\ \nabla_{a^+}^{\alpha-1} z(t) &= \nabla_{a^+}^{\alpha-1} \left(\nabla_{a^+}^{-\alpha} v\right)(t) + d_1 \Gamma(\alpha). \end{aligned} \tag{3.8}$$

Here we impose the boundary conditions

$$\nabla_{a^+}^{\alpha-1} y(a+1) = \nabla_{a^+}^{\alpha-1} y(b),$$
$$\nabla_{a^+}^{\alpha-1} z(a+1) = \nabla_{a^+}^{\alpha-1} z(b),$$

on (3.8) to arrive at

$$\sum_{s=a+2}^{b} u(s) = 0, \qquad \sum_{s=a+2}^{b} v(s) = 0.$$

Assume given (u,v) fulfils the recent equalities. Taking $y(t) = \nabla_{a^+}^{-\alpha} u(t)$ and $z(t) = \nabla_{a^+}^{-\alpha} v(t)$, one can directly conclude that $(y,z) \in \mathrm{dom}L$. Consequently, we have

$$\mathrm{Im}L = \left\{ (u,v) \middle| \sum_{s=a+2}^{b} u(s) = 0, \quad \sum_{s=a+2}^{b} v(s) = 0 \right\}. \tag{3.9}$$

Now we define the operators $Q_\gamma : E_\alpha \to E_\alpha$, $\gamma \in \{1,2\}$ as

$$Q_1 u(t) = \frac{1}{b-a-1} \sum_{s=a+2}^{b} u(s), \quad Q_2 v(t) = \frac{1}{b-a-1} \sum_{s=a+2}^{b} v(s). \tag{3.10}$$

Obviously $Q(u,v) = (Q_1 u, Q_2 v) \cong \mathbb{R}^2$. Thereby, it is easy to check that for $u,v \in E$, upcoming properties hold:

$$Q_1^2 u(t) = Q_1 u(t), \quad Q_2^2 v(t) = Q_2 v(t),$$

that is $Q^2(u,v) = Q(u,v)$. Using the identity $(u,v) = (u,v) - Q(u,v) + Q(u,v)$, one can derive $\mathfrak{D} = \mathrm{Im}L + \mathrm{Im}Q$. In addition, since $\mathrm{Im}L \cap \mathrm{Im}Q = \{(0,0)\}$, then, we deduce that $\mathfrak{D} = \mathrm{Im}L \oplus \mathrm{Im}Q$. Finally, by Definition 2.7, we have:

$$\mathrm{Ind}L = \dim\ker L - \mathrm{codim}\ \mathrm{Im}L = \dim\ker L - [\dim \mathfrak{D} - \dim\ \mathrm{Im}L] = 2 - [4-2] = 0.$$

Therefore, the operator L defined above is a Fredholm operator having index zero. □

In this position, let us consider the operators $P_\gamma : X \to X$, $\gamma \in \{1,2\}$ as

$$P_1 u(t) = \frac{\nabla_{a^+}^{\alpha-1} u(a)}{\Gamma(\alpha)} (t-a)^{\overline{\alpha-1}}, \quad P_2 v(t) = \frac{\nabla_{a^+}^{\alpha-1} v(a)}{\Gamma(\alpha)} (t-a)^{\overline{\alpha-1}}. \tag{3.11}$$

Using the property (Q_4) in Lemma 2.6, it is immediate that $P_1^2 u = P_1 u$ and $P_2^2 v = P_2 v$. Thus, if we define $P : \mathfrak{B} \to \mathfrak{B}$ as $P(u,v) = (P_1 u, P_2 v)$, then, we have

$$\ker P = \left\{ (u,v) \middle| \nabla_{a^+}^{\alpha-1} u(a) = 0,\ \nabla_{a^+}^{\alpha-1} v(a) = 0 \right\}.$$

If we take the setting $(u,v) = (u,v) - P(u,v) + P(u,v)$, so, it is easy to check that $\mathfrak{B} = \ker P + \ker L$, and as a result of $\ker P \cap \ker L = \{(0,0)\}$, we deduce that $\mathfrak{B} = \ker P \oplus \ker L$.

Next, we define the operator $K_P : \mathrm{Im}\, L \to \mathrm{dom}\, L \cap \ker P$ as:

$$K_P(u,v) = \left(\nabla_{a^+}^{-\alpha} u, \nabla_{a^+}^{-\alpha} v\right). \tag{3.12}$$

Thus, for each $(u,v) \in ImL$, we have

$$LK_P(u,v) = L\left(\nabla_{a^+}^{-\alpha} u, \nabla_{a^+}^{-\alpha} v\right) = \left(\nabla_{a^+}^{\alpha} \nabla_{a^+}^{-\alpha} u, \nabla_{a^+}^{\alpha} \nabla_{a^+}^{-\alpha} v\right) = (u,v). \tag{3.13}$$

On the other hand, since for every $(u,v) \in \mathrm{dom}L \cap \ker P$, we have $\nabla_{a^+}^{\alpha-1} u(a) = 0$ and $\nabla_{a^+}^{\alpha-1} v(a) = 0$, hence, in the identities

$$u(t) = \nabla_{a^+}^{-\alpha} \nabla_{a^+}^{\alpha} u(t) + c_1(t-a)^{\overline{\alpha-1}} + c_2(t-a)^{\overline{\alpha-2}} + \ldots + c_n(t-a)^{\overline{\alpha-n}},$$
$$v(t) = \nabla_{a^+}^{-\alpha} \nabla_{a^+}^{\alpha} v(t) + d_1(t-a)^{\overline{\alpha-1}} + d_2(t-a)^{\overline{\alpha-2}} + \ldots + d_n(t-a)^{\overline{\alpha-n}},$$

all of the coefficients $c_i, d_i = 0$ for $i = 1,2,\ldots,n$. Equivalently, we have

$$K_P L(u,v) = \left(\nabla_{a^+}^{-\alpha} \nabla_{a^+}^{\alpha} u, \nabla_{a^+}^{-\alpha} \nabla_{a^+}^{\alpha} v\right) = (u,v). \tag{3.14}$$

Therefore, using (3.13) and (3.14), it has proven that the following identity is satisfied: $K_P = (L_{\mathrm{dom}L \cap \ker P})^{-1}$.

LEMMA 3.2. *Assume that* Ω *be an open and bounded subset of* $\mathfrak{B}$ *such that* $domL \cap \overline{\Omega} \neq \varnothing$. *Then, the operator* N *defined by* (3.6)–(3.7b) *is* L*-compact.*

Proof. Continuity of $f, g : \mathbb{N}_{a+1}^{b} \times \mathbb{R}^n \to \mathbb{R}$ implies that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. So, for applying the $Arzela-Ascoli$ theorem, it is sufficient that $K_P(I-Q)N(\overline{\Omega}) \subset \mathfrak{B}$ be equicontinuous. This is immediate by the discrete nature of fractional ∇-difference operators. □

In this position we present an assessment for an upper bound of fractional rising function $(t-a)^{\overline{\alpha-1}}$, that will play crucial role in the simplification of the related computations.

LEMMA 3.3. *Let* $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}_1$. *Then*

$$\left\|(t-a)^{\overline{\alpha-1}}\right\|_X \leqslant \max\left\{(b-a)^{\overline{\alpha-1}}, \Gamma(\alpha)(b-a+1)^{\overline{n-2}}\right\}. \tag{3.15}$$

Proof. Considering (3.1c) we have

$$\left\|(t-a)^{\overline{\alpha-1}}\right\|_X = \max\left\{\left\|(t-a)^{\overline{\alpha-1}}\right\|_E, \left\|\nabla_{a^+}^{\alpha-i}(t-a)^{\overline{\alpha-1}}\right\|_{F_i}; \; i = 1,2,3,\ldots,n-1\right\}.$$

On the other hand, by the property (Q_4) in Lemma 2.6, we have

$$\nabla_{a^+}^{\alpha-i}(t-a)^{\overline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma(i)}(t-a)^{\overline{i-1}}, \quad i = 1,2,3,\ldots,n-1.$$

Therefore, we come to the conclusion that

$$\left\|(t-a)^{\overline{\alpha-1}}\right\|_X = \max\left\{\left\|(t-a)^{\overline{\alpha-1}}\right\|_E, \frac{\Gamma(\alpha)}{\Gamma(i)}\left\|(t-a)^{\overline{i-1}}\right\|_{F_i}, i=1,2,3,\dots,n-1\right\},$$
$$\leqslant \max\left\{\left\|(t-a)^{\overline{\alpha-1}}\right\|_E, \Gamma(\alpha)\left\|(t-a)^{\overline{i-1}}\right\|_{F_i}, i=1,2,3,\dots,n-1\right\}.$$

Since

$$\Delta(t-a)^{\overline{i-1}} = (t-a)^{\overline{i-1}}\left(\frac{i-1}{t-a}\right) \geqslant 0 \quad t\in\mathbb{N}_{a+n-i+1}^{b+n-i},\ i=1,2,\dots,n-1,$$
$$\Delta_i(b+n-i-a)^{\overline{i-1}} = \frac{\Gamma(b+n-a-1)}{\Gamma(b+n-i-a)}(b+n-i-a-2)\geqslant 0, \quad i=1,2,\dots,n-1,$$

thus, it follows that

$$\max_{t\in\mathbb{N}_{a+n-i+1}^{b+n-i}, i=1,2,\dots,n-1}(t-a)^{\overline{i-1}} = (b-a+1)^{\overline{n-2}}. \tag{3.16}$$

Similarly we have

$$\max_{t\in\mathbb{N}_{a+1}^{b}}(t-a)^{\overline{\alpha-1}} = (b-a)^{\overline{\alpha-1}}. \tag{3.17}$$

Therefore, it has proven that

$$\left\|(t-a)^{\overline{\alpha-1}}\right\|_X \leqslant \max\left\{(b-a)^{\overline{\alpha-1}}, \Gamma(\alpha)(b-a+1)^{\overline{n-2}}\right\}.$$

The proof is completed. □

REMARK 3.4. By similar arguments as represented in Lemma 3.3, one can conclude

$$\left\|(t-a-1)^{\overline{\alpha-1}}\right\|_X \leqslant \max\left\{(b-a-1)^{\overline{\alpha-1}}, \Gamma(\alpha)(b-a)^{\overline{n-2}}\right\}. \tag{3.18}$$

REMARK 3.5. For each $(u,v)\in\mathfrak{B}$, Lemma 3.3 implies that

$$\begin{aligned}\|P(u,v)\|_{\mathfrak{B}} &= \|(P_1(u),P_2(v))\|_{\mathfrak{B}} = \max\left\{\|P_1(u)\|_X, \|P_2(v)\|_X\right\}\\ &= \max\left\{\frac{|\nabla_{a^+}^{\alpha-1}u(a)|}{\Gamma(\alpha)}\|(t-a)^{\overline{\alpha-1}}\|_X, \frac{|\nabla_{a^+}^{\alpha-1}v(a)|}{\Gamma(\alpha)}\|(t-a)^{\overline{\alpha-1}}\|_X\right\}\\ &\leqslant \Lambda_1\max\left\{|\nabla_{a^+}^{\alpha-1}u(a)|, |\nabla_{a^+}^{\alpha-1}v(a)|\right\},\end{aligned} \tag{3.19}$$

where

$$\Lambda_1 = \frac{\max\left\{(b-a)^{\overline{\alpha-1}}, \Gamma(\alpha)(b-a+1)^{\overline{n-2}}\right\}}{\Gamma(\alpha)}. \tag{3.20}$$

A direct computation shows that

$$\Delta_s(t-s+1)^{\overline{\alpha-1}} \leqslant 0, \quad t \geqslant s,\ s=a+1,\dots,b.$$

Therefore, the operator K_P defined by (3.12) for $(u,v) \in \mathrm{Im}L$, satisfies the following inequality:

$$\begin{aligned} \|K_P(u,v)\|_{\mathfrak{B}} &= \left\| \left(\nabla_{a^+}^{-\alpha}u, \nabla_{a^+}^{-\alpha}v\right) \right\|_{\mathfrak{B}} = \max\left\{ \left\|\nabla_{a^+}^{-\alpha}u\right\|_X, \left\|\nabla_{a^+}^{-\alpha}v\right\|_X \right\} \\ &\leqslant (b-a+1)\Lambda_2 \max\left\{ \|u\|_E, \|v\|_E \right\}, \end{aligned} \tag{3.21}$$

where

$$\Lambda_2 = \frac{\max\left\{ (b-a-1)^{\overline{\alpha-1}}, \Gamma(\alpha)(b-a)^{\overline{n-2}} \right\}}{\Gamma(\alpha)}. \tag{3.22}$$

In order to find at least one solution for the fractional-order ∇-difference system (1.3)–(1.4), we present the following hypotheses.

(C_1) Continuous functions f,g have the following properties

$$f: \mathbb{N}_{a+1}^b \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}, \text{ or } \quad f: \mathbb{N}_{a+1}^b \times \mathbb{R}^n \to \mathbb{R}^- \cup \{0\}, \tag{3.23a}$$

and

$$g: \mathbb{N}_{a+1}^b \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}, \text{ or } \quad g: \mathbb{N}_{a+1}^b \times \mathbb{R}^n \to \mathbb{R}^- \cup \{0\}. \tag{3.23b}$$

(C_2) There exist positive real constants b_k, c_k, d_1, d_2 for $k=1,2,\dots,n$ and real constants $\theta_k, \lambda_k \in [0,1]$ with $k=1,2,\dots,n$ such that for all $(x_1,x_2,\dots,x_n) \in \mathbb{R}^n$,

$$|f(t,x_1,x_2,\dots,x_n)| \leqslant d_1 + \sum_{k=1}^n b_k|x_k|^{\theta_k}, \quad t \in \mathbb{N}_{a+1}^b, \tag{3.24a}$$

and

$$|g(t,x_1,x_2,\dots,x_n)| \leqslant d_2 + \sum_{k=1}^n c_k|x_k|^{\lambda_k}, \quad t \in \mathbb{N}_{a+1}^b. \tag{3.24b}$$

(C_3) There exists a positive real constant B such that for any $w_i, z_i \in \mathbb{R}$, $i=1,2,\dots,n$, if $|w_n| > B$ or $|z_n| > B$, one has either

$$z_n \cdot f(t,w_1,w_2,\dots,w_n) > 0, \text{ or } \quad z_n \cdot f(t,w_1,w_2,\dots,w_n) < 0, \quad t \in \mathbb{N}_{a+1}^b, \tag{3.25a}$$

or

$$w_n \cdot g(t,z_1,z_2,\dots,z_n) > 0, \text{ or } \quad w_n \cdot g(t,z_1,z_2,\dots,z_n) < 0, \quad t \in \mathbb{N}_{a+1}^b. \tag{3.25b}$$

(C_4)

$$\sum_{i=1}^{n}|x_i| < \frac{1}{(b-a+1)(\Lambda_1+\Lambda_2)}, \quad x=b,c, \tag{3.26a}$$

$$\Lambda_1\sum_{i=1}^{n}|c_i|+\Lambda_2\sum_{i=1}^{n}|b_i| < \frac{1}{b-a+1}, \tag{3.26b}$$

$$\Lambda_1\sum_{i=1}^{n}|b_i|+\Lambda_2\sum_{i=1}^{n}|c_i| < \frac{1}{b-a+1}. \tag{3.26c}$$

Let us once again turn to the coincidence degree theory. Then, it can be observed that the existence of at least one solution for coupled system (1.3)–(1.4), depends on the boundedness of the following sets:

$$\Omega_1 = \left\{(u,v)\in domL\setminus\ker L \,\middle|\, L(u,v)=\lambda N(u,v),\ \lambda\in[0,1]\right\}, \tag{3.27a}$$

$$\Omega_2 = \left\{(u,v)\in\ker L \,\middle|\, N(u,v)\in \operatorname{Im} L\right\}, \tag{3.27b}$$

$$\Omega_3 = \left\{(u,v)\in\ker L \,\middle|\, \lambda(u,v)+(1-\lambda)QN(u,v)=(0,0),\ \lambda\in[0,1]\right\}, \tag{3.27c}$$

$$\Omega_4 = \left\{(u,v)\in\ker L \,\middle|\, -\lambda(u,v)+(1-\lambda)QN(u,v)=(0,0),\ \lambda\in[0,1]\right\}. \tag{3.27d}$$

So, we must prove that the quadruplets $\Omega_j,\ j=1,2,3,4$ are bounded.

LEMMA 3.6. *Ω_1 defined by (3.27a) is bounded.*

Proof. We point out this fact that for each $(u,v)\in\Omega_1$, necessarily $\lambda\neq 0$. Therefore, the abstract operator equation

$$L(u,v)=\lambda N(u,v)\in \operatorname{Im}L=\ker Q, \tag{3.28}$$

ensures that

$$\frac{\lambda}{b-a-1}\sum_{s=a+2}^{b} f\left(s,v,\nabla_{a^+}^{\alpha-(n-1)}v,\nabla_{a^+}^{\alpha-(n-2)}v,\ldots,\nabla_{a^+}^{\alpha-1}v\right)=0,$$
$$\frac{\lambda}{b-a-1}\sum_{s=a+2}^{b} g\left(s,u,\beta_{a^+}^{\alpha-(n-1)}u,\nabla_{a^+}^{\alpha-(n-2)}u,\ldots,\nabla_{a^+}^{\alpha-1}u\right)=0.$$

Accordingly, the hypothesis (C_1) implies that there exist $t_0,t_1\in\mathbb{N}_{a+1}^{b}$ such that

$$f\left(t_1,v(t_1),\nabla_{a^+}^{\alpha-(n-1)}v(t_1),\nabla_{a^+}^{\alpha-(n-2)}v(t_1),\ldots,\nabla_{a^+}^{\alpha-1}v(t_1)\right)=0,$$
$$g\left(t_0,u(t_0),\nabla_{a^+}^{\alpha-(n-1)}u(t_0),\nabla_{a^+}^{\alpha-(n-2)}u(t_0),\ldots,\nabla_{a^+}^{\alpha-1}u(t_0)\right)=0.$$

The recent equalities in combination with the hypothesis (C_3), lead us to the following inequalities:

$$|\nabla_{a^+}^{\alpha-1}u(t_0)| \leqslant B, \quad |\nabla_{a^+}^{\alpha-1}v(t_1)| \leqslant B.$$

By definition of Ω_1 represented by (3.27a), $(u,v) \in \Omega_1$, if and only if $(u,v) \in \operatorname{dom}L \setminus \ker L$. Thus, because of $P^2 = P$, we conclude that $(I-P)(u,v) \in \operatorname{dom}L \cap \ker P$ and $LP(u,v) = (0,0)$. Hence, (3.21) yields:

$$\begin{aligned} \|(I-P)(u,v)\|_{\mathfrak{B}} &= \|K_P L(I-P)(u,v)\|_{\mathfrak{B}} = \|K_P\,(L_1u, L_2v)\,\|_{\mathfrak{B}} \\ &= \left\| \left(\nabla_{a^+}^{-\alpha}L_1u, \nabla_{a^+}^{-\alpha}L_2u\right) \right\|_{\mathfrak{B}} \\ &\leqslant \lambda(b-a+1)\Lambda_2 \max\left\{ \|N_1v\|_E, \|N_2u\|_E \right\}, \\ &\leqslant (b-a+1)\Lambda_2 \max\left\{ \|N_1v\|_E, \|N_2u\|_E \right\}. \end{aligned} \tag{3.29}$$

By recalling the abstract operator equation

$$L(u,v) = \lambda N(u,v), \quad (u,v) \in domL,$$

equivalently we have

$$\begin{cases} L_1u = \lambda N_1v, \\ L_2v = \lambda N_2u. \end{cases} \tag{3.30}$$

So, using the following well known identity

$$\nabla_{a^+}^{-\alpha}\nabla_{a^+}^{\alpha}u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{\alpha-n+k}}}{\Gamma(\alpha-n+k+1)} \nabla_{a^+}^{k}\left[\nabla_{a^+}^{\alpha-n}u(a)\right],\ 0 \leqslant n-1 < \alpha \leqslant n,\ n \in \mathbb{N},$$

one can transform (3.30) to the following fractional ∇-sum coupled system

$$\begin{cases} u(t) = \lambda\nabla_{a^+}^{-\alpha}N_1v + \sum_{k=0}^{n-1} \dfrac{(t-a)^{\overline{\alpha-n+k}}}{\Gamma(\alpha-n+k+1)} \nabla_{a^+}^{k}\left[\nabla_{a^+}^{\alpha-n}u(a)\right], \\ v(t) = \lambda\nabla_{a^+}^{-\alpha}N_2u + \sum_{k=0}^{n-1} \dfrac{(t-a)^{\overline{\alpha-n+k}}}{\Gamma(\alpha-n+k+1)} \nabla_{a^+}^{k}\left[\nabla_{a^+}^{\alpha-n}v(a)\right]. \end{cases}$$

Now, taking $\nabla_{a^+}^{\alpha-1}$ on the both sides in the last coupled system, it follows that

$$\nabla_{a^+}^{\alpha-1}u(t) = \lambda\nabla_{a^+}^{-1}N_1v + \nabla_{a^+}^{\alpha-1}u(a), \tag{3.31a}$$

$$\nabla_{a^+}^{\alpha-1}v(t) = \lambda\nabla_{a^+}^{-1}N_2u + \nabla_{a^+}^{\alpha-1}v(a). \tag{3.31b}$$

Substituting $t = t_0$ in (3.31a) and $t = t_1$ in (3.31b), as a result of $|\nabla_{a^+}^{\alpha-1}u(t_0)| \leqslant B$ and $|\nabla_{a^+}^{\alpha-1}v(t_1)| \leqslant B$, one may derive:

$$\begin{cases} |\nabla_{a^+}^{\alpha-1}u(a)| \leqslant B + \lambda\sum_{s=a}^{t_1} \left| f(s, v(s), \nabla_{a^+}^{\alpha-(n-1)}v(s), \nabla_{a^+}^{\alpha-(n-2)}v(s), \ldots, \nabla_{a^+}^{\alpha-1}v(s)) \right|, \\ |\nabla_{a^+}^{\alpha-1}v(a)| \leqslant B + \lambda\sum_{s=a}^{t_0} \left| g(s, u(s), \nabla_{a^+}^{\alpha-(n-1)}u(s), \nabla_{a^+}^{\alpha-(n-2)}u(s), \ldots, \nabla_{a^+}^{\alpha-1}u(s)) \right|. \end{cases}$$

Finally, applying the hypothesis (C_2) represented by (3.24a) and (3.24b), we achieve the followings

$$\begin{cases} |\nabla_{a^+}^{\alpha-1}u(a)| \leqslant B+(b-a+1)\left(d_1+b_1\|v\|_E^{\theta_1}+\sum_{i=2}^n b_i\left\|\nabla_{a^+}^{\alpha-i+1}v\right\|_{F_{i-1}}^{\theta_i}\right), \\ |\nabla_{a^+}^{\alpha-1}v(a)| \leqslant B+(b-a+1)\left(d_2+c_1\|u\|_E^{\lambda_1}+\sum_{i=2}^n c_i\left\|\nabla_{a^+}^{\alpha-i+1}u\right\|_{F_{i-1}}^{\lambda_i}\right). \end{cases} \tag{3.32}$$

Reminding the Remark 3.5 together with (3.29), we have

$$\begin{aligned} \|(u,v)\|_{\mathfrak{B}} &= \|P(u,v)+(I-P)(u,v)\|_{\mathfrak{B}} \\ &\leqslant \|P(u,v)\|_{\mathfrak{B}}+\|(I-P)(u,v)\|_{\mathfrak{B}} \\ &\leqslant \max\Big\{\Big\{\Lambda_1\left|\nabla_{a^+}^{\alpha-1}u(a)\right|+(b-a+1)\Lambda_2\|N_1v\|_E\Big\}, \\ &\qquad \Big\{\Lambda_1\left|\nabla_{a^+}^{\alpha-1}v(a)\right|+(b-a+1)\Lambda_2\|N_2u\|_E\Big\}, \\ &\qquad \Big\{\Lambda_1\left|\Delta_{a^+}^{\alpha-1}u(a)\right|+(b-a+1)\Lambda_2\|N_2u\|_E\Big\}, \\ &\qquad \Big\{\Lambda_1\left|\nabla_{a^+}^{\alpha-1}v(a)\right|+(b-a+1)\Lambda_2\|N_1v\|_E\Big\}\Big\}. \end{aligned} \tag{3.33}$$

In order to complete the proof, let us divide the remainder of the proof into four cases as follows:

(i) The hypothesis (C_2) and (3.32), imply that

$$\begin{aligned} \|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1\left|\nabla_{a^+}^{\alpha-1}u(a)\right|+(b-a+1)\Lambda_2\|N_1v\|_E \\ &\leqslant \Lambda_1B+(b-a+1)(\Lambda_1+\Lambda_2)\left(d_1+b_1\|v\|_E^{\theta_1}+\sum_{i=2}^n b_i\left\|\nabla_{a^+}^{\alpha-i+1}v\right\|_{F_{i-1}}^{\theta_i}\right). \end{aligned}$$

(ii) Once again considering the hypothesis (C_2) together with (3.32), similarly we conclude that

$$\begin{aligned} \|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1\left|\nabla_{a^+}^{\alpha-1}v(a)\right|+(b-a+1)\Lambda_2\|N_2u\|_E \\ &\leqslant \Lambda_1B+(b-a+1)(\Lambda_1+\Lambda_2)\left(d_2+c_1\|u\|_E^{\lambda_1}+\sum_{i=2}^n c_i\left\|\nabla_{a^+}^{\alpha-i+1}u\right\|_{F_{i-1}}^{\lambda_i}\right). \end{aligned}$$

(iii) In third case, we have the following

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1 \left|\nabla_{a^+}^{\alpha-1}u(a)\right| + (b-a+1)\Lambda_2\|N_2u\|_E \\ &\leqslant \Lambda_1\left\{B+(b-a+1)\left(d_1+b_1\|v\|_E^{\theta_1}+\sum_{i=2}^n b_i\left\|\nabla_{a^+}^{\alpha-i+1}v\right\|_{F_{i-1}}^{\theta_i}\right)\right\} \\ &\quad + (b-a+1)\Lambda_2\left(d_2+c_1\|u\|_E^{\lambda_1}+\sum_{i=2}^n c_i\left\|\nabla_{a^+}^{\beta-i+1}u\right\|_{F_{i-1}}^{\lambda_i}\right).\end{aligned}$$

(iv) In the last case, similar to the case (iii), it follows that

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1 \left|\nabla_{a^+}^{\beta-1}v(a)\right| + (b-a+1)\Lambda_2\|N_1v\|_E \\ &\leqslant \Lambda_1\left\{B+(b-a+1)\left(d_2+c_1\|u\|_E^{\lambda_1}+\sum_{i=2}^n c_i\left\|\nabla_{a^+}^{\beta-i+1}u\right\|_{F_{i-1}}^{\lambda_i}\right)\right\} \\ &\quad + (b-a+1)\Lambda_2\left(d_1+b_1\|v\|_E^{\theta_1}+\sum_{i=2}^n b_i\left\|\nabla_{a^+}^{\alpha-i+1}v\right\|_{F_{i-1}}^{\theta_i}\right).\end{aligned}$$

At the last step of the proof applying the inequalities (3.26a)–(3.26c) in the hypothesis (C_4), in the recent four inequalities, we conclude the boundedness of Ω_1 as below:

(i)

$$\|(u,v)\|_{\mathfrak{B}} \leqslant \frac{\Lambda_1 B + |d_1|(b-a+1)(\Lambda_1+\Lambda_2)}{1-(b-a+1)(\Lambda_1+\Lambda_2)\sum_{i=1}^n |b_i|}. \tag{3.34}$$

(ii)

$$\|(u,v)\|_{\mathfrak{B}} \leqslant \frac{\Lambda_1 B + |d_2|(b-a+1)(\Lambda_1+\Lambda_2)}{1-(b-a+1)(\Lambda_1+\Lambda_2)\sum_{i=1}^n |c_i|}. \tag{3.35}$$

(iii)

$$\|(u,v)\|_{\mathfrak{B}} \leqslant \frac{\Lambda_1 B + (b-a+1)(\Lambda_1|d_1|+\Lambda_2|d_2|)}{1-(b-a+1)\left[\Lambda_1\sum_{i=1}^n |b_i| + \Lambda_2\sum_{i=1}^n |c_i|\right]}. \tag{3.36}$$

(iv)

$$\|(u,v)\|_{\mathfrak{B}} \leqslant \frac{\Lambda_1 B + (b-a+1)(\Lambda_1|d_2|+\Lambda_2|d_1|)}{1-(b-a+1)\left[\Lambda_1\sum_{i=1}^n |c_i| + \Lambda_2\sum_{i=1}^n |b_i|\right]}. \tag{3.37}$$

Let us now define

$$\begin{aligned}M = \max\Big\{&\Lambda_1 B + |d_1|(b-a+1)(\Lambda_1+\Lambda_2),\ \Lambda_1 B + |d_2|(b-a+1)(\Lambda_1+\Lambda_2), \\ &\Lambda_1 B + (b-a+1)(\Lambda_1|d_1|+\Lambda_2|d_2|),\ \Lambda_1 B + (b-a+1)(\Lambda_1|d_2|+\Lambda_2|d_1|)\Big\}.\end{aligned} \tag{3.38}$$

Combining the inequalities (3.34)–(3.37) with (3.38), yields $\|(u,v)\|_{\mathfrak{B}} < M$ for each $(u,v) \in \mathfrak{B}$. Thus, Ω_1 defined by (3.27a) is bounded. □

LEMMA 3.7. *Ω_2 defined by (3.27b) is bounded.*

Proof. Suppose $(u,v) \in \Omega_2$. Then, $u = c_1(t-a)^{\overline{\alpha-1}}$, $v = c_2(t-a)^{\overline{\alpha-1}}$, $c_1, c_2 \in \mathbb{R}$. On the other hand, $N(u,v) = (N_1 v, N_2 u) \in \operatorname{Im} L = \ker Q$, implies that

$$\sum_{s=a+2}^{b} f\Big(s, c_2(s-a)^{\overline{\alpha-1}}, c_2 \nabla_{a^+}^{\alpha-(n-1)}(s-a)^{\overline{\alpha-1}}, \\ c_2 \nabla_{a^+}^{\alpha-(n-2)}(s-a)^{\overline{\alpha-1}}, \ldots, c_2 \nabla_{a^+}^{\alpha-1}(s-a)^{\overline{\alpha-1}}\Big) = 0,$$

$$\sum_{s=a+2}^{b} f\Big(s, c_1(s-a)^{\overline{\alpha-1}}, c_1 \nabla_{a^+}^{\alpha-(n-1)}(s-a)^{\overline{\alpha-1}}, \\ c_1 \nabla_{a^+}^{\alpha-(n-2)}(s-a)^{\overline{\alpha-1}}, \ldots, c_1 \nabla_{a^+}^{\alpha-1}(s-a)^{\overline{\alpha-1}}\Big) = 0.$$

So, by means of the hypothesis (C_1), there exist constants $t_0, t_1 \in \mathbb{N}_{a+1}^{b}$ such that

$$f\Big(t_1, c_2(t_1-a)^{\overline{\alpha-1}}, c_2 \nabla_{a^+}^{\alpha-(n-1)}(t_1-a)^{\overline{\alpha-1}}, \\ c_2 \nabla_{a^+}^{\alpha-(n-2)}(t_1-a)^{\overline{\alpha-1}}, \ldots, c_2 \nabla_{a^+}^{\alpha-1}(t_1-a)^{\overline{\alpha-1}}\Big) = 0,$$

$$f\Big(t_0, c_1(t_0-a)^{\overline{\alpha-1}}, c_1 \nabla_{a^+}^{\alpha-(n-1)}(t_0-a)^{\overline{\alpha-1}}, \\ c_1 \nabla_{a^+}^{\alpha-(n-2)}(t_0-a)^{\overline{\alpha-1}}, \ldots, c_1 \nabla_{a^+}^{\alpha-1}(t_0-a)^{\overline{\alpha-1}}\Big) = 0.$$

Thus, relying on the hypothesis (C_3), we arrive at

$$|c_1|, |c_2| \leqslant \frac{B}{\Gamma(\alpha)}. \tag{3.39}$$

The inequalities (3.39) guarantee the boundedness of Ω_2. □

LEMMA 3.8. *Ω_3 defined by (3.27c) is bounded.*

Proof. Let $(u,v) \in \Omega_3$. Hence, $(u,v) = (t-a)^{\overline{\alpha-1}}(c_1, c_2)$, $c_1, c_2 \in \mathbb{R}$. Therefore, the equality $\lambda(u,v) + (1-\lambda)QN(u,v) = (0,0)$ leads us to the following:

$$c_1 \lambda (t-a)^{\overline{\alpha-1}} + \frac{(1-\lambda)}{b-a-1} \\ \times \sum_{s=a+2}^{b} f\Big(s, c_2(s-a)^{\overline{\alpha-1}}, c_2 \nabla_{a^+}^{\alpha-(n-1)}(s-a)^{\overline{\alpha-1}}, \ldots, c_2 \nabla_{a^+}^{\alpha-1}(s-a)^{\overline{\alpha-1}}\Big) = 0,$$

$$c_2\lambda(t-a)^{\overline{\alpha-1}}+\frac{(1-\lambda)}{b-a-1}$$
$$\times\sum_{s=a+2}^{b}f\left(s,c_1(s-a)^{\overline{\alpha-1}},c_1\nabla_{a^+}^{\alpha-(n-1)}(s-a)^{\overline{\alpha-1}},\ldots,c_1\nabla_{a^+}^{\alpha-1}(s-a)^{\overline{\alpha-1}}\right)=0.$$

If $\lambda=0$, then, a similar argument as represented in Lemma 3.7 gives the boundedness of Ω_3. Now, suppose $\lambda\in(0,1]$. In this case the hypothesis (C_3) and more precisely the first parts of (3.25a) and (3.25b), help us to obtain the desired result. □

REMARK 3.9. Considering the second parts of the hypothesis (C_3) (the first parts were applied in Lemma 3.8), we come to the conclusion that Ω_4 defined by (3.27d) is bounded.

Now we are ready to prove the first part of the main results that is, the existence of at least one solution for fractional-order system (1.3)–(1.4).

THEOREM 3.10. *Assume that the hypotheses (C_1)–(C_4) are satisfied. Then, the fractional-order ∇-difference system (1.3) and (1.4) has at least one solution in $\mathfrak{B}$.*

Proof. Let $\Omega\supset\cup_{i=1}^{3}\overline{\Omega_i}\cup\{0\}$ $\left(or,\ \Omega\supset\cup_{i=1}^{2}\overline{\Omega_i}\cup\Omega_4\cup\{0\}\right)$ be an open bounded subset of $\mathfrak{B}$. It follows from Lemma 3.2 that N is a L-compact operator on Ω. Also, by means of Lemma 3.6-Remark 3.9, we have:

(1) $L(u,v)=\lambda N(u,v)$ for every $((u,v),\lambda)\in[\text{dom}\,L\setminus\ker\,L\cap\partial\Omega]\times(0,1)$.

(2) $N(u,v)\notin\text{Im}\,L$ for every $(u,v)\in\ker\,L\cap\partial\Omega$.

So, we just need to prove:

(3) $\deg(JQN|_{\ker L},\ \Omega\cap\ker\,L,\ 0)\neq 0$.

Let us consider the isomorphism $J(u,v)$ as the identity operator Id(u,v) and define

$$H((u,v),\lambda)=\pm\lambda\text{Id}(u,v)+(1-\lambda)JQN(u,v).$$

So, thanks to the the degree property of the invariance under a homotopy, if $u\in\ker L\cap\partial\Omega$, then

$$\begin{aligned}&\deg(JQN|_{\ker L},\ \Omega\cap\ker\,L,\ 0)\\&=\deg(H(.,0),\ \Omega\cap\ker\,L,\ 0)\\&=\deg(H(.,1),\ \Omega\cap\ker\,L,\ 0)\\&=\deg(\pm\text{Id},\ \Omega\cap\ker\,L,\ 0)\neq 0.\end{aligned}$$

Hence, the assumption (iii) in Theorem 2.9 is also fulfilled, that completes the proof, that is the fractional-order system (1.3)–(1.4) has at least one solution. □

This position is the beginning of the second part of the main results, where, we have to present a uniqueness criterion for the fractional-order ∇-difference system (1.3)–(1.4). To this aim, we have the following theorem.

THEOREM 3.11. *Suppose the hypothesis* (C_2) *is replaced with the following ones:*

$(C'_{2,1})$ *There exist positive constants* $(a_i,b_i)\in\mathbb{R}^2,\ i=1,\dots,n$*, such that for all* $((x_i)_1^n,(y_i)_1^n)\in\mathbb{R}^n\times\mathbb{R}^n$*, one has*

$$|f(t,x_1,x_2,\dots,x_n)-f(t,y_1,y_2,\dots,y_n)|\leqslant\sum_{i=1}^n a_i|x_i-y_i|,\quad t\in\mathbb{N}_{a+1}^b,\tag{3.40a}$$

$$|g(t,x_1,x_2,\dots,x_n)-g(t,y_1,y_2,\dots,y_n)|\leqslant\sum_{i=1}^n b_i|x_i-y_i|,\quad t\in\mathbb{N}_{a+1}^b.\tag{3.40b}$$

$(C'_{2,2})$ *There exist positive constants* $(k_i,l_i)\in\mathbb{R}^2,\ i=1,\dots,n$*, such that for all* $((x_i)_1^n,(y_i)_1^n)\in\mathbb{R}^n\times\mathbb{R}^n$*, one has*

$$|f(t,x_1,x_2,\dots,x_n)-f(t,y_1,y_2,\dots,y_n)|\geqslant k_n|x_n-y_n|-\sum_{i=1}^{n-1} k_i|x_i-y_i|,\quad t\in\mathbb{N}_{a+1}^b,\tag{3.41a}$$

$$|g(t,x_1,x_2,\dots,x_n)-g(t,y_1,y_2,\dots,y_n)|\geqslant l_n|x_n-y_n|-\sum_{i=1}^{n-1} l_i|x_i-y_i|,\quad t\in\mathbb{N}_{a+1}^b.\tag{3.41b}$$

Then, the fractional-order system (1.3)–(1.4) *has exactly one solution in* $\mathfrak{B}$ *provided that*

$$\Lambda_1\left[\sum_{i=1}^{n-2}\frac{l_{n-i}}{l_n}+\frac{l_1}{l_n}\right]+(b-a+1)(\Lambda_1+\Lambda_2)\sum_{i=1}^n|b_i|>1,\tag{3.42a}$$

$$\Lambda_1\left[\sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}+\frac{k_1}{k_n}\right]+(b-a+1)(\Lambda_1+\Lambda_2)\sum_{i=1}^n|a_i|>1,\tag{3.42b}$$

$$\Lambda_1\left[\sum_{i=1}^{n-2}\frac{l_{n-i}}{l_n}+\frac{l_1}{l_n}\right]+(b-a+1)\left[\Lambda_1\sum_{i=1}^n|b_i|+\Lambda_2\sum_{i=1}^n|a_i|\right]>1,\tag{3.42c}$$

$$\Lambda_1\left[\sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}+\frac{k_1}{k_n}\right]+(b-a+1)\left[\Lambda_1\sum_{i=1}^n|a_i|+\Lambda_2\sum_{i=1}^n|b_i|\right]>1.\tag{3.42d}$$

Proof. First, we prove the existence of at least one solution for the fractional-order system (1.3)–(1.4). Taking $y_i=0,\ i=1,2,\dots,n$ and defining

$$d_1=\max f(t,0,0,\dots,0),\quad d_2=\max g(t,0,0,\dots,0),\quad t\in\mathbb{N}_{a+1}^b,$$

fulfilment of the hypothesis (C_2) is immediate. Thus, Theorem 3.10 ensures the existence of at least one solution for the fractional-order system (1.3)–(1.4). Next, in order to complete the proof it is enough to prove that the fractional-order system (1.3)–(1.4) has exactly one solution. To this aim, assume that $(u_i,v_i)\in\mathfrak{B}$ for $i=1,2$ are two solutions of fractional-order system (1.3)–(1.4). Thus, we get the following

$$\nabla_{a^+}^{\alpha}u_i(t)=f\left(t,v_i,\nabla_{a^+}^{\alpha-(n-1)}v_i,\nabla_{a^+}^{\alpha-(n-2)}v_i,\dots,\nabla_{a^+}^{\alpha-1}v_i\right),$$
$$\nabla_{a^+}^{\alpha}v_i(t)=g\left(t,u_i,\nabla_{a^+}^{\alpha-(n-1)}u_i,\nabla_{a^+}^{\alpha-(n-2)}u_i,\dots,\nabla_{a^+}^{\alpha-1}u_i\right).$$

Denoting $u=u_1-u_2,\ v=v_1-v_2$, we have

$$\begin{aligned}\nabla_{a^+}^{\alpha}u(t)&=f\left(t,v_1,\nabla_{a^+}^{\alpha-(n-1)}v_1,\dots,\nabla_{a^+}^{\alpha-1}v_1\right)-f\left(t,v_2,\nabla_{a^+}^{\alpha-(n-1)}v_2,\dots,\nabla_{a^+}^{\alpha-1}v_2\right),\\ \nabla_{a^+}^{\alpha}v(t)&=g\left(t,u_1,\nabla_{a^+}^{\alpha-(n-1)}u_1,\dots,\nabla_{a^+}^{\alpha-1}u_1\right)-g\left(t,u_2,\nabla_{a^+}^{\alpha-(n-1)}u_2,\dots,\nabla_{a^+}^{\alpha-1}u_2\right).\end{aligned}\tag{3.43}$$

In the light of the equality $\operatorname{Im}L=\ker Q$, we conclude that

$$\sum_{s=a+2}^{b}\left\{f\left(s,v_1,\nabla_{a^+}^{\alpha-(n-1)}v_1,\dots,\nabla_{a^+}^{\alpha-1}v_1\right)-f\left(s,v_2,\nabla_{a^+}^{\alpha-(n-1)}v_2,\dots,\nabla_{a^+}^{\alpha-1}v_2\right)\right\}=0,$$
$$\sum_{s=a+2}^{b}\left\{g\left(s,u_1,\nabla_{a^+}^{\alpha-(n-1)}u_1,\dots,\nabla_{a^+}^{\alpha-1}u_1\right)-g\left(s,u_2,\nabla_{a^+}^{\alpha-(n-1)}u_2,\dots,\nabla_{a^+}^{\alpha-1}u_2\right)\right\}=0.$$

Accordingly, using the hypothesis (C_1) one may derive that there exist at least one pair $(t_2,t_3)\in\left(\mathbb{N}_{a+2}^{b},\mathbb{N}_{a+2}^{b}\right)$ such that

$$\begin{aligned}&f\left(t_3,v_1(t_3),\nabla_{a^+}^{\alpha-(n-1)}v_1(t_3),\dots,\nabla_{a^+}^{\alpha-1}v_1(t_3)\right)\\ =&f\left(t_3,v_2(t_3),\nabla_{a^+}^{\alpha-(n-1)}v_3(t_2),\dots,\nabla_{a^+}^{\alpha-1}v_2(t_3)\right),\\ &g\left(t_2,u_1(t_2),\nabla_{a^+}^{\alpha-(n-1)}u_1(t_2),\dots,\nabla_{a^+}^{\alpha-1}u_1(t_2)\right)\\ =&g\left(t_2,u_2(t_2),\nabla_{a^+}^{\alpha-(n-1)}u_2(t_2),\dots,\nabla_{a^+}^{\alpha-1}u_2(t_2)\right).\end{aligned}$$

Therefore, using the hypothesis $(C_{2,2}^{'})$, we arrive at the following result

$$\begin{aligned}0=&\left|f\left(t_3,v_1(t_3),\nabla_{a^+}^{\alpha-(n-1)}v_1(t_3),\dots,\nabla_{a^+}^{\alpha-1}v_1(t_3)\right)\right.\\ &\qquad\left.-f\left(t_3,v_2(t_3),\nabla_{a^+}^{\alpha-(n-1)}v_2(t_3),\dots,\nabla_{a^+}^{\alpha-1}v_2(t_3)\right)\right|\\ \geqslant& k_n\left|\nabla_{a^+}^{\alpha-1}v(t_3)\right|-\sum_{i=1}^{n-2}k_{n-i}\left|\nabla_{a^+}^{\alpha-i-1}v(t_3)\right|-k_1|v(t_3)|.\end{aligned}$$

Therefore

$$\left|\nabla_{a^+}^{\alpha-1}v(t_3)\right|\leqslant\sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}\left|\nabla_{a^+}^{\alpha-i-1}v(t_3)\right|+\frac{k_1}{k_n}|v(t_3)|.$$

Consequently we have

$$\begin{aligned}\left|\nabla_{a^+}^{\alpha-1}v(t_3)\right| &\leqslant \sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}\left\|\nabla_{a^+}^{\alpha-i-1}v\right\|_F + \frac{k_1}{k_n}\|v\|_E \\ &\leqslant \left[\sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}+\frac{k_1}{k_n}\right]\|v\|_X. \end{aligned} \tag{3.44}$$

With a similar argument, it can be shown that

$$\left|\nabla_{a^+}^{\alpha-1}u(t_2)\right| \leqslant \left[\sum_{i=1}^{n-2}\frac{l_{n-i}}{l_n}+\frac{l_1}{l_n}\right]\|u\|_X. \tag{3.45}$$

On the other hand by (3.43), we deduce that

$$\begin{aligned}&\nabla_{a^+}^{\alpha-1}u(t)-\nabla_{a^+}^{\alpha-1}u(a)\\ =&\nabla_{a^+}^{-1}\left\{f\left(t,v_1,\nabla_{a^+}^{\alpha-(n-1)}v_1,\ldots,\nabla_{a^+}^{\alpha-1}v_1\right)-f\left(t,v_2,\nabla_{a^+}^{\alpha-(n-1)}v_2,\ldots,\nabla_{a^+}^{\alpha-1}v_2\right)\right\},\\ &\nabla_{a^+}^{\alpha-1}v(t)-\nabla_{a^+}^{\alpha-1}v(a)\\ =&\nabla_{a^+}^{-1}\left\{g\left(t,u_1,\nabla_{a^+}^{\alpha-(n-1)}u_1,\ldots,\nabla_{a^+}^{\alpha-1}u_1\right)-g\left(t,u_2,\nabla_{a^+}^{\alpha-(n-1)}u_2,\ldots,\nabla_{a^+}^{\alpha-1}u_2\right)\right\}.\end{aligned}$$

Now substituting $t=t_2$ in the first equality and $t=t_3$ in second one and then applying the hypothesis $(C'_{2,1})$, one has

$$\left|\nabla_{a^+}^{\alpha-1}u(a)\right| \leqslant \left|\nabla_{a^+}^{\alpha-1}u(t_2)\right| + (b-a+1)\sum_{i=1}^{n}a_i\left\|\nabla_{a^+}^{\alpha-i+1}v\right\|_{F_{i-1}}, \tag{3.46a}$$

and

$$\left|\nabla_{a^+}^{\alpha-1}v(a)\right| \leqslant \left|\Delta_{a^+}^{\alpha-1}v(t_3)\right| + (b-a+1)\sum_{i=1}^{n}b_i\left\|\nabla_{a^+}^{\alpha-i+1}u\right\|_{F_{i-1}}. \tag{3.46b}$$

Now, (3.44) and (3.45), lead us to

$$\left|\nabla_{a^+}^{\alpha-1}u(a)\right| \leqslant \left[\sum_{i=1}^{n-2}\frac{l_{n-i}}{l_n}+\frac{l_1}{l_n}\right]\|u\|_X + (b-a+1)\sum_{i=1}^{n}b_i\|v\|_X, \tag{3.47a}$$

$$\left|\nabla_{a^+}^{\alpha-1}v(a)\right| \leqslant \left[\sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}+\frac{k_1}{k_n}\right]\|v\|_X + (b-a+1)\sum_{i=1}^{n}a_i\|u\|_X. \tag{3.47b}$$

Considering (3.44), we have

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &= \|P(u,v)+(I-P)(u,v)\|_{\mathfrak{B}}\\ &\leqslant \|P(u,v)\|_{\mathfrak{B}}+\|(I-P)(u,v)\|_{\mathfrak{B}}\\ &\leqslant \max\left\{\left\{\Lambda_1\left|\nabla_{a^+}^{\alpha-1}u(a)\right|+(b-a+1)\Lambda_2\|N_1v\|_E\right\},\right.\\ &\qquad\left\{\Lambda_1\left|\nabla_{a^+}^{\alpha-1}v(a)\right|+(b-a+1)\Lambda_2\|N_2u\|_E\right\},\\ &\qquad\left\{\Lambda_1\left|\nabla_{a^+}^{\alpha-1}u(a)\right|+(b-a+1)\Lambda_2\|N_2u\|_E\right\},\\ &\qquad\left.\left\{\Lambda_1\left|\nabla_{a^+}^{\alpha-1}v(a)\right|+(b-a+1)\Lambda_2\|N_1v\|_E\right\}\right\},\end{aligned} \tag{3.48}$$

that in combination with (3.47a) and (3.47b) consequences the following cases:

i.

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1\left|\nabla_{a^+}^{\alpha-1}u(a)\right|+(b-a+1)\Lambda_2\|N_1v\|_E\\ &\leqslant \frac{(b-a+1)\Lambda_2|d_1|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{n-2}\frac{l_{n-i}}{l_n}+\frac{l_1}{l_n}\right]+(b-a+1)(\Lambda_1+\Lambda_2)\sum_{i=1}^{n}|b_i|\right\}}.\end{aligned} \tag{3.49}$$

ii.

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1\left|\nabla_{a^+}^{\alpha-1}v(a)\right|+(b-a+1)\Lambda_2\|N_2u\|_E\\ &\leqslant \frac{(b-a+1)\Lambda_2|d_2|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}+\frac{k_1}{k_n}\right]+(b-a+1)(\Lambda_1+\Lambda_2)\sum_{i=1}^{n}|a_i|\right\}}.\end{aligned} \tag{3.50}$$

iii.

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1\left|\nabla_{a^+}^{\alpha-1}u(a)\right|+(b-a+1)\Lambda_2\|N_2u\|_E\\ &\leqslant \frac{(b-a+1)\Lambda_2|d_2|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{n-2}\frac{l_{n-i}}{l_n}+\frac{l_1}{l_n}\right]+(b-a+1)\left[\Lambda_1\sum_{i=1}^{n}|b_i|+\Lambda_2\sum_{i=1}^{n}|a_i|\right]\right\}}.\end{aligned} \tag{3.51}$$

iv.

$$\begin{aligned}\|(u,v)\|_{\mathfrak{B}} &\leqslant \Lambda_1\left|\nabla_{a^+}^{\beta-1}v(a)\right| + (b-a+1)\Lambda_2\|N_1 v\|_E \\ &\leqslant \frac{(b-a+1)\Lambda_2|d_1|}{1-\left\{\Lambda_1\left[\sum_{i=1}^{n-2}\frac{k_{n-i}}{k_n}+\frac{k_1}{k_n}\right]+(b-a+1)\left[\Lambda_1\sum_{i=1}^{n}|a_i|+\Lambda_2\sum_{i=1}^{n}|b_i|\right]\right\}}.\end{aligned} \tag{3.52}$$

Combining (3.42a)–(3.42d) with the inequalities (3.49)–(3.52), we come to the conclusion that $u=v=0$, i.e. $(u_1,v_1)=(u_2,v_2)$. Therefore, the fractional-order system (1.3)–(1.4) has a unique solution in $\mathfrak{B}$. This completes the proof. □

4. An application

We are going to demonstrate that the presented existence and uniqueness criteria in Theorem 3.10 and Theorem 3.10, can be implemented in practice. To this aim, let us consider the fractional ∇-difference system

$$\begin{cases}\nabla_{1^+}^{\frac{3}{2}}y(t)=f\left(t,z,\nabla_{1^+}^{\frac{1}{2}}z\right), & t\in\mathbb{N}_1^4,\\ \nabla_{1^+}^{\frac{3}{2}}z(t)=g\left(t,y,\nabla_{1^+}^{\frac{1}{2}}y\right), & t\in\mathbb{N}_1^4,\end{cases} \tag{4.1}$$

such that $f,g\in\mathbb{N}_2^4\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+\cup\{0\}$. Besides, the corresponding boundary conditions are as follows:

$$\begin{cases}\nabla_{1^+}^{-\frac{1}{2}}y(2)=0, & \nabla_{1^+}^{\frac{1}{2}}y(2)=\nabla_{1^+}^{\frac{1}{2}}y(4),\\ \nabla_{1^+}^{-\frac{1}{2}}z(2)=0, & \nabla_{1^+}^{\frac{1}{2}}z(2)=\nabla_{1^+}^{\frac{1}{2}}z(4).\end{cases} \tag{4.2}$$

Indeed, aforementioned system is the reduced version of the primitive fractional-order system (1.3)–(1.4) under the setting $n=2$, $\alpha=\frac{3}{2}$ and $a=1$, $b=4$. Also, the functions f and g in system (4.1) read as follows

$$f(t,v,w)=\sin\left(t-\frac{1}{2}\right)+\frac{1}{30}\left\{\ln(1+|v|)+\frac{|w|}{1+|w|}\right\}, \tag{4.3}$$

$$g(t,v,w)=\sinh\left(t-\frac{1}{2}\right)+\frac{1}{20}\left\{|v|+|w|e^{-|w|}\right\}. \tag{4.4}$$

Choosing $d_1=2$, $b_i=\frac{1}{60}$, $\theta_i=1$ for $i=1,2$ and $d_2=4$, $c_i=\frac{1}{64}$, $i=1,2$, $\lambda_1=1$, $\lambda_2=\frac{1}{2}$, it is easy to check that the nonnegative nature of f and g for given positive parameter B, the hypothesis (C_3) is also satisfied. At the end, considering the parameters $\alpha=\frac{3}{2}$

and $n=2$, it follows that Λ_1 and Λ_2 defined by (3.20) and (3.22), respectively, take the following values

$$\Lambda_1 = \frac{15}{8}, \quad \Lambda_2 = \frac{3}{2}. \tag{4.5}$$

Therefore, with a direct calculation one can identify that the hypothesis (C_4) holds. So, based on Theorem 3.10, the fractional-order system (4.1)–(4.2) admits at least one solution in $\mathfrak{B}$.

Now it is time to evaluate the uniqueness of solutions for fractional-order system (4.1) and (4.2) as follows. Taking $a_i = b_i = 1,\ i = 1,2$ and $k_1 = l_1 = 1,\ k_2 = l_2 = -1$, the hypotheses $(C^{'}_{2,1})$ and $(C^{'}_{2,2})$ hold. Also, using this traditional convention that $\sum_{k=i}^{j} h_k \equiv 0,\ j > i$, we find that (3.42a)–(3.42d) are satisfied for $a=1,\ b=4$ and $\Lambda_1 = \frac{15}{8},\ \Lambda_2 = \frac{3}{2}$. Then, the fractional-order coupled system (4.1)–(4.2) admits exactly one solution in $\mathfrak{B}$.

5. Concluding Remarks

We finalize this investigation by making a summary of the presented solvability precess to find a unique solution for the fractional-order ∇-difference boundary value problems (1.3)–(1.4). So, we sort this investigation step by step as follows:

- ▶ In this paper, we have considered the higher-order coupled system (1.3)–(1.4) powered by the fractional-order ∇-differences;
- ▶ the fractional-order system (1.3)–(1.4) includes full nonliniraities, that is the independent variable, spatial variable and fractional ∇-differences having the orders $\alpha - i,\ i = 1,2,\ldots,n-1$ of the spatial variable with respect to the independent variable are included in the nonliniearities;
- ▶ the coincidence degree theory as the main solvability key has applied;
- ▶ in frame of Theorem 3.10, an existence criterion for at least one solution of the fractional-order system (1.3)–(1.4) has presented;
- ▶ Theorem 3.11 has enabled us to reach the unique solution of the fractional-order system (1.3)–(1.4);
- ▶ in section 4, a numerical application has presented to guarantee that the obtained theoretical existence and uniqueness criteria can be applied in practice.

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