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A NOTE ON GENERALIZED FRACTIONAL DIFFUSION EQUATIONS ON POINCARÉ HALF PLANE

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Abstract. In this paper we study generalized time-fractional diffusion equations on the Poincaré half plane $\mathbb{H}_2^+$. The time-fractional operators here considered are fractional derivatives of a function with respect to another function, that can be obtained essentially by means of a deterministic change of variable in the Caputo derivative. We obtain an explicit representation of the fundamental solution of the generalized-diffusion equation on $\mathbb{H}_2^+$ and provide a probabilistic interpretation in terms of a time-changed hyperbolic Brownian motion. We finally include an explicit result regarding the non-linear case admitting a separating variable solution.

1. Introduction

In this paper we study generalized time-fractional diffusion equations on the hyperbolic Poincaré half-plane

$$\mathbb{H}_2^+ = \left\{ (x,y) \in \mathbb{R}^2 \middle| y > 0 \right\}.$$

The generalization considered here is based on the application of time-fractional derivatives of a function with respect to another function (see [1] for the definition and main properties), an interesting approach that permits us to take into account both time-varying coefficients and memory effects (see e.g. [3] for a physical discussion about this). In the previous paper [7] the authors studied for the first time the time-fractional diffusion equation on the hyperbolic space involving the classical Caputo derivative. Moreover, in the more recent paper [4], an interesting probabilistic interpretation of the fundamental solution of the time-fractional telegraph-type equation on hyperbolic spaces has been provided. In particular, a relevant connection with time-changed hyperbolic Brownian motions has been proved.

The main aim of this paper is to provide a rigorous analysis of the generalized time-fractional diffusion equation on the hyperbolic space $\mathbb{H}_2^+$. We find an explicit representation of the fundamental solution by means of the method of separation of variables. Moreover, we obtain a probabilistic interpretation as the law of a stochastic

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process represented by a time-changed hyperbolic Brownian motion. In the first part of the paper we provide some necessary preliminaries about the Poincaré half-plane and the definition and basic properties of the fractional operators here considered. We decided to provide some preliminaries about the Poincaré half-plane since many non-trivial computations are involved and we think that this short guide can be of help for the potential reader.

Then, we analyze the generalized time-fractional diffusion equation on $\mathbb{H}_2^+$ providing the representation of the fundamental solution and the related probabilistic meaning. Finally, we also consider a nonlinear generalized time-fractional diffusion equation on $\mathbb{H}_2^+$ admitting a solution obtained by means of the method of separation of variables.

Few papers (for example [4] and [7]) are concerned with the probabilistic analysis of time-fractional diffusive equations on hyperbolic spaces and the present work should inspire further works to establish a connection among hyperbolic geometry, fractional calculus and stochastic processes.

2. Preliminaries

2.1. A short survey on hyperbolic geometry

We here give some necessary mathematical preliminaries about the model of the Poincairé half-plane i.e the set $\mathbb{H}_2^+ = \left\{(x,y) \in \mathbb{R}^2 | y > 0\right\}$ with the following metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{1}$$

First of all, in order to characterize the geometry of the Poincaré half-plane, we study the form of the geodesics, by using the variational principle.

We consider the family of curves on the hyperbolic plane passing through two arbitrary points (x_1,y_1) and (x_2,y_2) with parametric representation i.e.

$$\gamma = \left\{ (x(t),y(t)) \middle| t_1 \leqslant t \leqslant t_2 \right\}, \tag{2}$$

where t_1 and t_2 are such that $(x(t_1),y(t_1)) = (x_1,y_1)$ and $(x(t_2),y(t_2)) = (x_2,y_2)$.

The length of this curve in the hyperbolic plane is

$$\mathscr{L}(\gamma) = \int_{t_1}^{t_2} \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt. \tag{3}$$

We can simplify this expression by restricting ourselves to the family of parametric curves of (2) to curves with Cartesian parameterization i.e

$$\gamma = \{(x,y(x)) | x_1 \leqslant x \leqslant x_2\}. \tag{4}$$

In this case, the integral (3) becomes

$$\mathscr{L}(\gamma) = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'(x)^2}}{y(x)} dx. \tag{5}$$

We can consider an arbitrary function $w(x) = y(x) + \varepsilon h(x)$ with $\varepsilon \geqslant 0$ and $h(x)$ is such that $h(x_1) = h(x_2) = 0$.

By applying (5) the length of the curve γ reads

$$\mathscr{L}(\gamma) = l(\varepsilon) = \int_{x_1}^{x_2} \frac{\sqrt{1+(y'(x)+\varepsilon h'(x))^2}}{y(x)+\varepsilon h(x)}\,dx. \tag{6}$$

The geodesic curve is associated with a minimum point with respect to the ε variable of the function $l(\varepsilon)$ for which the condition

$$\left.\frac{dl}{d\varepsilon}\right|_{\varepsilon=0} = 0 \tag{7}$$

is satisfied.

By direct computation we have that

$$\begin{aligned}
\left.\frac{dl}{d\varepsilon}\right|_{\varepsilon=0} &= \int_{x_1}^{x_2} \frac{d}{d\varepsilon}\left(\frac{\sqrt{1+(y'+\varepsilon h')^2}}{y+\varepsilon h}\right)|_{\varepsilon=0}dx \\
&= \int_{x_1}^{x_2}\left(-\frac{h\sqrt{1+(y'+\varepsilon h')^2}}{(y+\varepsilon h)^2} + \frac{h'(y'+\varepsilon h')}{(y+\varepsilon h)\sqrt{1+(y'+\varepsilon h')^2}}\right)|_{\varepsilon=0}dx \\
&= \int_{x_1}^{x_2}\left(-\frac{h\sqrt{1+y'^2}}{y^2} + \frac{h'y'}{y\sqrt{1+y'^2}}\right)dx = \int_{x_1}^{x_2} -\frac{h\sqrt{1+y'^2}}{y^2}dx + \left[\frac{hy'}{y\sqrt{1+y'^2}}\right]_{x_1}^{x_2} \\
&\quad - \int_{x_1}^{x_2} h\frac{d}{dx}\frac{y'}{y\sqrt{1+y'^2}}dx
\end{aligned}$$

(since the function $h(x)$ is such that $h(x_1) = h(x_2) = 0$)

$$= \int_{x_1}^{x_2}\left(-\frac{\sqrt{1+y'^2}}{y^2} - \frac{d}{dx}\frac{y'}{y\sqrt{1+y'^2}}\right)hdx \tag{8}$$

By (7), the integral (8) must be equal to zero for all functions h and therefore we have that

$$-\frac{\sqrt{1+y'^2}}{y^2} - \frac{d}{dx}\frac{y'}{y\sqrt{1+y'^2}} = 0$$

and thus

$$-\frac{1}{y^2\sqrt{1+y'^2}} - \frac{y''}{y\left(\sqrt{1+y'^2}\right)^3} = 0. \tag{9}$$

We finally obtain that

$$1 + \frac{d}{dx}(yy') = 0. \tag{10}$$

Integrating twice (10) we have the following equation

$$x^2 + y^2 - 2cx - 2d = 0 \tag{11}$$

which is the equation of the semi-circles with an arbitrary center on the x-axis and with an arbitrary radius.

Other geodesic curves in the Poincairé half-plane are the half-lines parallel to the y-axis, that emerge if $x_1 = x_2$.

In $\mathbb{H}_2^+$ the position of a point can be given either by cartesian coordinates (x,y) or by hyperbolic coordinates (η,α). By η we indicate the distance of (x,y) from the origin $(0,1)$ evaluated by means of (1). By α we mean the angle that the tangent at $(0,1)$ forms with the geodesic line passing through (x,y).

The equation of geodesic lines is given by

$$(x-\tan\alpha)^2+y^2=\frac{1}{\cos^2\alpha} \tag{12}$$

and thus the following formula

$$\tan\alpha=\frac{x^2+y^2-1}{2x}, \tag{13}$$

gives the angle α for an arbitrary point (x,y) of $\mathbb{H}_2^+$.

Finally, the relationship between the η coordinate and the Cartesian coordinates is given by

$$\cosh\eta=\frac{x^2+y^2+1}{2y}. \tag{14}$$

Starting from the relations (13) and (14) we obtain the relationship between the cartesian coordinates (x,y) and the hyperbolic coordinates (η,α) in $\mathbb{H}_2^+$

$$\begin{cases} x=\dfrac{\cos\alpha\sinh\eta}{\cosh\eta-\sinh\eta\sin\alpha}, & \eta>0,\ 0<\alpha<2\pi \\ y=\dfrac{1}{\cosh\eta-\sinh\eta\sin\alpha}. \end{cases} \tag{15}$$

We are now able to derive the expression of the Laplacian operator in hyperbolic coordinates.

First of all, we observe that the Poincaré upper half-plane is a Riemannian manifold with the following metric tensor

$$\mathbf{g}=\begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

In general, on a Riemannian manifold with a metric tensor $\mathbf{g}$, the Laplacian is given by

$$\Delta f=\frac{1}{\sqrt{|\mathbf{g}|}}\sum_{i=1}^{n}\partial_i\Big(\sqrt{|\mathbf{g}|}\sum_{j=1}^{n}g^{ij}\partial_j f\Big), \tag{16}$$

where $|\mathbf{g}|$ is the determinant of the metric tensor and the elements g^{ij} are the components of the inverse matrix of $\mathbf{g}$ and n is the dimension of the manifold.

By observing that the inverse matrix $\mathbf{g}^{-1}$ and $|\mathbf{g}|$ are respectively given by

$$\mathbf{g}^{-1} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}, \qquad |\mathbf{g}| = \frac{1}{y^4}, \tag{17}$$

we have that in this case Δf becomes

$$\begin{aligned} \Delta f &= \frac{1}{\sqrt{\frac{1}{y^4}}} \left[\frac{\partial}{\partial x} \left(\sqrt{\frac{1}{y^4}} y^2 \frac{\partial}{\partial x} f \right) + \frac{\partial}{\partial y} \left(\sqrt{\frac{1}{y^4}} y^2 \frac{\partial}{\partial y} f \right) \right] \\ &= y^2 \left[\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} f \right) \right] \\ &= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f, \quad x \in \mathbb{R},\ y > 0 \end{aligned} \tag{18}$$

By applying (15) to (18) we can derive the expression of the Laplacian in hyperbolic coordinates (see [7] Theorem 2.1 for detailed calculations) that is given by the differential operator

$$\frac{1}{\sinh\eta} \frac{\partial}{\partial \eta} \left(\sinh\eta \frac{\partial}{\partial \eta} \right) + \frac{1}{\sinh^2 \eta} \frac{\partial^2}{\partial \alpha^2}, \quad 0 < \alpha < 2\pi,\ \eta > 0. \tag{19}$$

For further reading on hyperbolic geometry you can consult [13] (Chapter 3) and [12].

2.2. Fractional derivatives of a function with respect to another function

Fractional derivatives of a function with respect to another function have been considered in the classical monograph by Kilbas et al. [6] (Section 2.5) and date back to Holmgren (1865) and Osler (1970) (see [11]). Recently Almeida in [1] studied the Caputo-type regularization of the existing definition and some interesting properties. Starting from this paper, this topic has gained interest both for mathematical reasons (see e.g. [2]) and for physical applications (e.g. in rheology, see [3] and the references therein). The utility of these generalized fractional operators in the applications is represented by the fact that they are essentially obtained by a deterministic time-change and permits us to take into account both time-variable coefficients and memory effects. Moreover, this class of operators include as special cases classical well-known time-fractional derivatives (for example, fractional derivatives in the sense of Hadamard, or Erdélyi-Kober).

Here we recall the basic definitions and properties for the reader's convenience.

Let $\nu > 0$, $f \in C^1([a,t])$ an increasing function such that $f'(t) \neq 0$ in $[a,t]$, the fractional integral of a function $g(t)$ with respect to another function $f(t)$ is given by

$$\left(I_{a^+}^{\nu,f} g \right)(t) := \frac{1}{\Gamma(\nu)} \int_a^t f'(\tau)(f(t) - f(\tau))^{\nu-1} g(\tau) d\tau. \tag{20}$$

Observe that for $f(t) = t^\beta$ we recover the definition of Erdélyi-Kober fractional integral recently applied, for example, in connection with the Generalized Grey Brownian

Motion [9]. For simplicity hereafter we will consider $a = 0$ (as usual) and suitable functions f such that $f(0) = 0$. All the results can be simply generalized.

The corresponding Caputo-type evolution operator (see [1]) for $0 < \nu < 1$ is given by

$$\left(\mathscr{O}^{\nu,f} g\right)(t) := \frac{1}{\Gamma(1-\nu)} \int_0^t (f(t) - f(\tau))^{-\nu} \frac{d}{d\tau} g(\tau) d\tau \tag{21}$$

$$= I_{0^+}^{1-\nu,f} \left(\frac{1}{f'(t)} \frac{d}{dt} \right) g(t). \tag{22}$$

For the general case $\nu \in \mathbb{R}$ we refer to [1]. In this paper we are interested to the case $0 < \nu < 1$ interpolating as a limit case the ordinary first order derivative, while the higher order cases can be treated in a similar way. We have used the symbol $\mathscr{O}^{\nu,f}(\cdot)$ in order to underline the generic integro-differential nature of the time-evolution operator, depending on the choice of the function $f(t)$ and the real order ν.

A relevant property of the operator (21) is that if $g(t) = (f(t))^{\beta-1}$ with $\beta > 1$, then (see Lemma 1 of [1])

$$\left(\mathscr{O}^{\nu,f} g\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\nu)} (f(t))^{\beta-\nu-1}. \tag{23}$$

Indeed, by direct calculation we have that

$$\left(\mathscr{O}^{\nu,f} f^{\beta-1}\right)(t) = \frac{(\beta-1)}{\Gamma(1-\nu)} \int_0^t (f(t) - f(\tau))^{-\nu} f'(\tau) (f(\tau))^{\beta-2} d\tau$$

and taking $y = f(\tau)/f(t)$ we have that

$$\begin{aligned}\left(\mathscr{O}^{\nu,f} f^{\beta-1}\right)(t) &= \frac{(\beta-1)\, f^{\beta-1-\nu}(t)}{\Gamma(1-\nu)} \int_0^1 (1-y)^{-\nu} y^{\beta-2} dy \\ &= \frac{\Gamma(\beta)\, f^{\beta-1-\nu}(t)}{\Gamma(\beta-1)\Gamma(1-\nu)} \frac{\Gamma(1-\nu)\Gamma(\beta-1)}{\Gamma(\beta-\nu)} \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta-\nu)} (f(t))^{\beta-\nu-1}.\end{aligned}$$

Therefore, the composite Mittag-Leffler function

$$g(t) = E_\nu(\lambda (f(t))^\nu) \tag{24}$$

is an eigenfunction of the operator $\mathscr{O}^{\nu,f}$, when $\nu \in (0,1)$ and f is a well-behaved function such that $f(0) = 0$. This means that

$$\mathscr{O}^{\nu,f} E_\nu(\lambda (f(t))^\nu) = \lambda E_\nu(\lambda (f(t))^\nu). \tag{25}$$

3. Generalized linear and nonlinear fractional diffusion on Poincaré half-plane

3.1. The linear case

In a previous paper [7], the authors considered the following diffusion-type equation on $\mathbb{H}_2^+$

$$\frac{\partial^\beta}{\partial t^\beta}u(\eta,t)=\frac{1}{\sinh\eta}\left(\frac{\partial}{\partial\eta}\sinh\eta\frac{\partial}{\partial\eta}\right)u(\eta,t),\quad \beta\in(0,1), \tag{26}$$

where $\frac{\partial^\beta}{\partial t^\beta}$ is a fractional derivative of order β in the sense of Caputo. We here analyze the more general case involving the fractional derivative w.r.t. another function. First of all, we have the following result

THEOREM 1. *Let be $f\in L^1[0,t]$ such that $f(0)=0$, the fundamental solution for the generalized time-fractional diffusion equation*

$$\left(\mathscr{O}^{\beta,f}u\right)(\eta,t)=\frac{1}{\sinh\eta}\left(\frac{\partial}{\partial\eta}\sinh\eta\frac{\partial}{\partial\eta}\right)u(\eta,t) \tag{27}$$

is given by

$$u(\eta,t)=\frac{2}{\pi}\int_0^\infty xE_\beta\left(-\frac{f(t)^\beta}{4}-x^2f(t)^\beta\right)dx\int_\eta^\infty d\varphi\frac{\sin(x\varphi)}{\sqrt{2\cosh\varphi-2\cosh\eta}}. \tag{28}$$

Proof. We find the solution to (27) by means of the separation of variables and transform the Laplacian operator by using the change of variable $y=\cosh\eta$ which leads to

$$u(y,t)=F(y)\cdot T(t)$$

and therefore we get

$$\left(\mathscr{O}^{\beta,f}T\right)=-\omega T, \tag{29}$$

$$(y^2-1)F''+2yF'+\omega F=0. \tag{30}$$

The solution of the first equation is given by

$$T(t,\omega)=E_{\beta,1}(-\omega f(t)^\beta). \tag{31}$$

The spatial part of the solution remains the same as in the classical hyperbolic diffusion equation and we refer to [7] for the details. □

REMARK 1. We denoted by *fundamental solution*, the solution of the Cauchy problem for (27) under the initial condition $u(\eta,0)=\delta(\eta)$. It is possible to prove that the solution (28) becomes for $t\to0$ a source function in the form of a Dirac delta function (see the similar case in [7]). Therefore, the construction based on the classical method of separation of variables leads exactly to the fundamental solution for (27).

REMARK 2. Observe that for $f(t)=t$ and $\beta=1$ we recover the transition function of the hyperbolic Brownian motion, firstly studied by Gertsenshtein and Vasiliev in [5].

Moreover, for $f(t)=t$ and $\beta\in(0,1)$ we recover the results obtained in [7].

Let us introduce the process

$$T^{\beta}(f(t)) = B^{hp}(\mathscr{L}^{\beta}(f(t))),$$

where B^{hp} is the hyperbolic Brownian motion on $\mathbb{H}_2^+$ independent from $\mathscr{L}^{\beta}(t)$ which is the inverse of the stable subordinator $H^{\beta}(t)$, that is

$$\mathscr{L}^{\beta}(t) = \inf\{s>0 : H^{\beta}(s)\geqslant t\}, \quad \beta\in(0,1).$$

We have the following

THEOREM 2. *The distribution $p(x,t)$ of the process $T^{\beta}(f(t))$ coincides with the fundamental solution of the equation* (27)*.*

Proof. We observe that, by means of the deterministic time-change $f(t)\to t$, we can essentially go back to a time-fractional diffusion equation involving the Caputo derivative. Then, by means of the time-Laplace transform method, it can be proved that the fundamental solution of (26) coincides with the distribution of the process $T^{\beta}(f(t))$. □

Observe that this paper is devoted to diffusive models in the Poincaré half-space $\mathbb{H}_2^+$ but the generalizations to $\mathbb{H}_n^+$ can be obtained in a similar way from the probabilistic point of view and will be the object of a further detailed analysis.

Finally, by means of similar methods, we can generalize the recent results obtained in [4] about time-fractional telegraph-type equations in $\mathbb{H}_n$. In particular, we have that

THEOREM 3. *The distribution of the composition*

$$\mathscr{T}^{\beta}(t) = B^{hp}(L^{\beta}(f(t))), \tag{32}$$

where

$$L^{\beta}(t) = \inf\{s>0 : H_1^{2\beta}(s) + (2\lambda)^{1/\beta}H_2^{\beta}(s)\geqslant t\},$$

and $H_1^{2\beta}$, H_2^{β} are independent stable subordinators (with $\beta\in(0,1/2)$, coincides with the fundamental solution of the equation

$$\left(\mathscr{O}^{2\beta,f}u\right)(\eta,t) + 2\lambda\left(\mathscr{O}^{\beta,f}u\right)(\eta,t) = \frac{1}{\sinh\eta}\left(\frac{\partial}{\partial\eta}\sinh\eta\frac{\partial}{\partial\eta}\right)u(\eta,t), \quad \beta\in(0,1/2). \tag{33}$$

The main idea of the proof is essentially the same of that of the previous theorem. The result can be generalized to a multi-term fractional equation involving a finite number of fractional derivatives w.r.t. another function of order less than one (see [8]).

3.2. The nonlinear case

We recall that the construction of the explicit representation of the fundamental solution of the linear diffusion equation is based (also in the fractional case) on the classical method of separation of variables. We observe that particular solutions for nonlinear equations can be constructed by means of the generalized method of separation of variables (see [10]). On the other hand, we underline that in the recent literature, the analysis of the porous medium equation in hyperbolic space attracted the attention of different authors, we refer in particular to the relevant paper [14]. Indeed, the analysis of nonlinear reaction-diffusion equations in hyperbolic spaces leads to interesting and new mathematical problems and the time-fractional counterpart should be completly investigated.

Based on this motivation, a final result on non-linear fractional reaction-diffusion equation in $\mathbb{H}_2^+$ is here considered.

THEOREM 4. *The generalized time-fractional nonlinear diffusive equation in* $\mathbb{H}_2^+$

$$\left(\mathscr{O}^{\beta,f}u\right)(\eta,t)=\frac{1}{\sinh\eta}\left(\frac{\partial}{\partial\eta}\sinh\eta\frac{\partial}{\partial\eta}\right)u^n(\eta,t)-u(\eta,t),\quad n>0,\ (\eta,t)\in\mathbb{R}^+\times\mathbb{R}^+ \tag{34}$$

admits as a particular solution

$$u(\eta,t)=g(\eta)\cdot E_\beta\left(-(f(t))^\beta\right), \tag{35}$$

where $g(\eta)$ *is such that* $\frac{dg^n}{d\eta}=\frac{1}{\sinh\eta}$.

Proof. We first determine a solution by means of the generalized separation of variables in the simple form

$$u(\eta,t)=r(t)\cdot g(\eta).$$

We observe that if $g(\eta)$ is such that

$$\frac{dg^n}{d\eta}=\frac{1}{\sinh\eta},$$

then

$$\frac{1}{\sinh\eta}\left(\frac{\partial}{\partial\eta}\sinh\eta\frac{\partial}{\partial\eta}\right)u^n(\eta,t)=0$$

and therefore by substitution we have that

$$g(\eta)\left(\mathscr{O}^{\beta,f}r\right)(t)=-g(\eta)r(t)$$

and therefore $r(t)=E_\beta\left(-(f(t))^\beta\right)$. □

The study of nonlinear diffusive equations in $\mathbb{H}_2^+$ is not the main object of this paper, but we observe that by starting from this simple result, it is possible to construct exact solutions for many different classes of generalized time-fractional nonlinear equations in $\mathbb{H}_2^+$, a completly new topic of research.

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