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ON THE ANALYSIS OF BLACK–SCHOLES EQUATION FOR EUROPEAN CALL OPTION INVOLVING A FRACTIONAL ORDER WITH GENERALIZED TWO DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

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Abstract. This paper proposes alternative approach for the valuation of a European call option via Generalized Two Dimensional Differential Transform Method (GTDTM). The analysis of the Black-Scholes equation for a European call option involving fractional order with GTDTM is discussed. The fractional derivative is considered in the sense of Caputo. Also, it is assumed that the underlying asset price pays no dividend and follows the Geometric Brownian Motion (GBM). The fractional Black-Scholes equation for a European call option has been solved successfully using GTDTM. The valuation formula of a European call option with fractional order has been obtained. An illustrative example is also presented to measure the performance of GTDTM in terms of accuracy, effectiveness and suitability in the context of the Black-Scholes Model (BSM). The results show that GTDTM compares favourably and agrees with BSM. Moreover, GTDTM is found to be accurate, effective, suitable and a good approach for the valuation of a European call option with fractional order.

1. Introduction

Most of the problems in computational finance entail the computation of a particular integral. In many cases these integrals can be solved analytically and in some cases by means of numerical methods. An option is defined as a contract that grants its holder the right, without obligation to buy or sell a specific underlying asset on or before a given date in the future for an agreed price. Options come in two ways; a call option gives the holder the right to buy while a put option gives the right to sell. The history of options extends back to several centuries, it was not clear until 1973 that trading of options was formalized by the establishment of the Chicago Board of Options Exchange (CBOE) with more than one million contracts per day. In the same year, there was a turning point for research in option valuation. Black and Scholes [1] also described a mathematical framework for obtaining the fair price of a European call option. Fadugba [3] applied the Mellin transform to obtain the solution of

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fractional order equations. The solution of the Black-Scholes partial differential equation for the vanilla option via reduced differential transform method was obtained by [4]. A new generalization of two dimensional differential transform method for pricing barrier options of fractional version of the Black-Scholes model was presented by [5]. The motivation of this paper is based on the work of [5]. In this paper, the analysis of Black-Scholes equation for a European call option involving a fractional order with Generalized Two Dimensional Differential Transform Method (GTDTM) is presented. The emphasis is given to the Caputo fractional operator which is more suitable for the study of differential equations of fractional order. The organization of the rest of the paper is as follows; Section 2 presents some preliminaries used in this work. Section 3 presents GTDTM and its fundamental properties. The analysis of GTDTM for the solution of the fractional Black-Scholes equation for a European call option is discussed in Section 4. The valuation formula for a European call option with fractional order via GTDTM is also presented. Section 5 presents the application of GTDTM, discussion of results and concluding remarks.

2. Preliminaries

2.1. Fractional Black-Scholes equation for European call option

The Black-Scholes partial differential equation for the price of European call option with non-dividend yield is given by [1]

$$\frac{\partial E^c}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 E^c}{\partial S^2}+rS\frac{\partial E^c}{\partial S}-rE^c=0, \tag{1}$$

subject to the terminal and boundary conditions:

$$\lim_{S_T\to\infty} E^c=S_T,\ \lim_{S_T\to 0} E_c=0, E^c=\max(S_T-K,0) \tag{2}$$

on $[0,T]$, where $(S,t)\in R^+\times(0,T)$, E^c is the European option price at the underlying asset price S and time t, T is the maturity date, r is the risk-free interest rate and σ is the volatility. By means of change of variables, let

$$S=Ke^x\Rightarrow x=\ln\left(\frac{S}{K}\right) \tag{3}$$

$$\tau=\frac{\sigma^2}{2}(T-t) \tag{4}$$

and

$$E^c=Kv(x,\tau) \tag{5}$$

where K is the exercise price. Substituting (3), (4) and (5) into (1) and simplifying further, yields

$$\frac{\partial v}{\partial\tau}=\frac{\partial^2 v}{\partial x^2}+\left(\frac{2r}{\sigma^2}-1\right)\frac{\partial v}{\partial x}-\frac{2r}{\sigma^2}v \tag{6}$$

Setting $\omega = \frac{2r}{\sigma^2}$, (6) becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (\omega - 1)\frac{\partial v}{\partial x} - \omega v \tag{7}$$

$$v(x,0) = \max(e^x - 1, 0) \tag{8}$$

Now, we generalize the assumption that the underlying asset follows a geometric Brownian motion. Let the stock price be driven by a marked point process as in [2]. Thus (7) turns to a fractional Black-Scholes partial differential equation of the form

$$\frac{\partial^\alpha v}{\partial \tau^\alpha} = \frac{\partial^2 v}{\partial x^2} + (\omega - 1)\frac{\partial v}{\partial x} - \omega v \tag{9}$$

subject to (8), where ω is a fractional order.

2.2. Definition of some concepts

DEFINITION 1. A real valued function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $\rho > \mu$, such that $f(t) = t^\rho f_1(t)$, where $f_1(t) \in C[0,\infty]$ and is said to be in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$.

DEFINITION 2. The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $f \in C_\mu$, $\mu \geqslant 1$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \tau > 0 \tag{10}$$

where $\Gamma(\alpha)$ is the gamma function of α.

DEFINITION 3. The Riemann-Liouville fractional derivative operator of order $\alpha > 0$ of a function $f(t)$ is defined as

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau \tag{11}$$

for $\alpha \in (n-1,n)$, $t > 0$ and $n \in N$

DEFINITION 4. The Caputo fractional derivative of the function $f \in C_{-1}^n$, $n \in N$ is defined as

$${}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau \tag{12}$$

for $\alpha \in (n-1,n]$, $t > 0$.

DEFINITION 5. For n to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(x,\tau)d\tau, & \alpha \in (0,1], \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \end{cases} \tag{13}$$

where $u = u(x,t)$, $u^{(n)}(x,\tau) = \frac{\partial^n u(x,\tau)}{\partial \tau^n}$. The relation between the Riemann-Liouville operator and Caputo fractional differential operator is given by

$$J^{\alpha} D_t^{\alpha} f(t) = D_t^{-\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \tag{14}$$

DEFINITION 6. The Mittag-Leffler function is defined as the series representation, valid in the whole complex plane [8]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} \tag{15}$$

For more details on the properties and applications of fractional calculus; see [8] and [7].

DEFINITION 7. An option is a versatile security that gives its holder the right, but not obligation, to buy (call option) or sell(put option) an underlying asset at an agreed upon price during a certain period of time.

DEFINITION 8. The discounted expected payoffs for call and put options are given by $\text{Payoff}_{call} = \max(S(T) - K, 0)$ and $\text{Payoff}_{put} = \max(K - S(T), 0)$, respectively, where $S(T)$ is the asset price at time T.

DEFINITION 9. A European option is a type of put or call option that can be exercised only on its expiration date [6].

3. Generalized two-dimensional differential transform approach

Consider a function of two variables $w(x,t)$ and suppose that $w(x,t)$ is a product of two-single valued functions, say

$$w(x,t) = g(x)h(t) \tag{16}$$

Using the properties of two-dimensional differential transform, (16) becomes

$$w(x,t) = \sum_{k=0}^{\infty} G_{\alpha}(k)(x - x_0)^{\alpha k} \sum_{k=0}^{\infty} H_{\beta}(h)(t - t_0)^{\beta h} \tag{17}$$

Equation (17) can be written in a compact form as

$$w(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \Omega_{\alpha,\beta}(k,h)(x - x_0)^{\alpha k}(t - t_0)^{\beta h} \tag{18}$$

where $\alpha, \beta \in (0,1]$. Suppose $w(x,t)$ is analytic and differentiable continuously w.r.t. t in the domain of interest, GTDTM is defined as [5]

$$\Omega_{\alpha,\beta} = \frac{1}{\Gamma(\alpha k)\Gamma(\beta h + 1)} [(D_{x_0}^{\alpha})^k (D_{t_0}^{\beta})^h w(x,t)]_{(x_0,t_0)} \tag{19}$$

The properties of the GTDTM are given in Table 1 [5]:

Table 1: *The properties of GTDTM*

Function	Transformed
$w(x,t)=u(x,t)\pm v(x,t)$	$\Omega_{\alpha,\beta}(k,h)=U_{\alpha,\beta}(k,h)\pm V_{\alpha,\beta}(k,h)$
$w(x,t)=au(x,t)$	$\Omega_{\alpha,\beta}(k,h)=aU_{\alpha,\beta}(k,h)$
$w(x,t)=u(x,t)v(x,t)$	$\Omega_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U_{\alpha,\beta}(r,h-s)V_{\alpha,\beta}(k-r,s)$
$w(x,t)=(x-x_0)^{n\alpha}(t-t_0)^{m\beta}$	$\Omega_{\alpha,\beta}(k,h)=\delta(k-n)\delta(h-m)$
$w(x,t)=D_{x_0}^{\alpha},\alpha\in(0,1]$	$\Omega_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{\alpha,\beta}(k+1,h)$

4. Analysis of the GTDTM

4.1. Solution of Black-Scholes equation for European call option with fractional order

Applying the GTDTM to both sides of (9) and (8) yields,

$$\begin{aligned}\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{\alpha,\beta}(k,h+1) &= (k+1)(k+2)U_{1,\alpha}(k+2,h)\\ &\quad+(w-1)(k+1)U_{1,\alpha}(k+1,h)\\ &\quad-wU_{1,\alpha}(k,h)\end{aligned} \tag{20}$$

and

$$U_{1,\alpha}(0,0)=0,\;\; U_{1,\alpha}(k,0)=\frac{1}{k!},\;\; k=1,2,3,\ldots \tag{21}$$

respectively. Using (20) and (21), the following values of $U(k,h)$ are obtained:

$$U_{\alpha,1}(0,1)=\frac{w}{\Gamma(1+\alpha)},\;\; U_{\alpha,1}(k,1)=0,\;\; k=1,2,3,\ldots \tag{22}$$

$$U_{\alpha,1}(0,2)=-\frac{w^2}{\Gamma(1+2\alpha)},\;\; U_{\alpha,1}(k,2)=0,\;\; k=1,2,3,\ldots \tag{23}$$

$$U_{\alpha,1}(0,3)=\frac{w^3}{\Gamma(1+3\alpha)},\;\; U_{\alpha,1}(k,3)=0,\;\; k=1,2,3,\ldots \tag{24}$$

$$U_{\alpha,1}(0,4)=-\frac{w^4}{\Gamma(1+4\alpha)},\;\; U_{\alpha,1}(k,4)=0,\;\; k=1,2,3,\ldots \tag{25}$$

$$U_{\alpha,1}(0,5)=\frac{w^5}{\Gamma(1+5\alpha)},\;\; U_{\alpha,1}(k,5)=0,\;\; k=1,2,3,\ldots \tag{26}$$

Continuing this way, one gets

$$U_{\alpha,1}(0,j)=(-1)^{j+1}\frac{w^j}{\Gamma(1+j\alpha)},\;\; U_{\alpha,1}(k,j)=0,\;\; k=1,2,3,\ldots \tag{27}$$

Substituting (22)–(27) into (18), one obtains

$$\begin{aligned} u(x,\tau) &= \frac{w\tau^{\alpha}}{\Gamma(1+\alpha)} - \frac{w^2\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{w^3\tau^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{w^4\tau^{4\alpha}}{\Gamma(1+4\alpha)} \\ &+ \frac{w^5\tau^{5\alpha}}{\Gamma(1+5\alpha)} + \ldots + (-1)^{j+1}\frac{w^j\tau^{j\alpha}}{\Gamma(1+j\alpha)} + \ldots (1+0+\ldots)x \\ &+ \left(\frac{1}{2}+0+\ldots\right)x^2 + \left(\frac{1}{6}+0+\ldots\right)x^3 + \left(\frac{1}{24}+0+\ldots\right)x^4 \\ &+ \left(\frac{1}{120}+0+\ldots\right)x^5 + \ldots + \left(\frac{1}{n!}+0+\ldots\right)x^n + \ldots \end{aligned} \tag{28}$$

Rearranging terms, (28) becomes

$$\begin{aligned} u(x,\tau) &= \frac{w\tau^{\alpha}}{\Gamma(1+\alpha)} - \frac{w^2\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{w^3\tau^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{w^4\tau^{4\alpha}}{\Gamma(1+4\alpha)} \\ &+ \frac{w^5\tau^{5\alpha}}{\Gamma(1+5\alpha)} + \ldots + (-1)^{j+1}\frac{w^j\tau^{j\alpha}}{\Gamma(1+j\alpha)} + \ldots \\ &+ \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots + \frac{x^n}{n!} + \ldots\right) \end{aligned} \tag{29}$$

Hence,

$$\begin{aligned} u(x,\tau) &= (e^x - 1) - \left[-1 + \sum_{m=0}^{\infty}\frac{(-w\tau^{\alpha})^m}{\Gamma(1+m\alpha)}\right] \\ &= e^x - e^{-w\tau^{\alpha}} \end{aligned} \tag{30}$$

Equation (30) is the closed form solution of (9) subject to (8) for $x \in Z^+$.

4.2. Valuation formula for European call option with fractional order via GTDTM

Substituting (3)–(5) into (30), one obtains

$$E^c(S,t) = \left\{S - K\exp\left[-w\left(\frac{(T-t)\sigma^2}{2}\right)\right]^{\alpha}\right\} \tag{31}$$

Substituting $w = \frac{2r}{\sigma^2}$ into (31) and simplifying further, one gets

$$E^c(S,t) = \left\{S - K\exp\left[-r\left(\frac{\sigma^2}{2}\right)^{\alpha-1}(T-t)^{\alpha}\right]\right\} \tag{32}$$

Equation (32) is the valuation formula for the price of a European call option with fractional order α. For a special case, setting $\alpha = 1$ in (32), one obtains the valuation formula for the price of a European call option with classical Black-Scholes equation via GTDTM as

$$E^c(S,t) = S - Ke^{-r(T-t)} \tag{33}$$

The exact solution or the so called the classical Black-Scholes Model (BSM) [1] for the valuation of a European call option is given by

$$E^c(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{34}$$

with

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}}, \quad d_2 = d_1 - \sigma\sqrt{(T-t)} \tag{35}$$

5. Application, discussion of results and concluding remarks

5.1. Application

Consider the valuation of a European call option with non-dividend-paying stock with the following parameters. The option values via GTDTM with $\alpha = 1.0$, Implicit

Table 2: *The Parameters*

Variables	**Values**
Underlying asset price, S_t	120
Strike price, K	$10, 20, 30, 40, 50, 60, 70, 80, 90, 100$
Risk-free interest rate, r	5%
Volatility, σ	50%
Dividend yield, q	0%
Time to expiry, T	0.0822
Fractional order, α	$0.2, 0.4, 0.6, 0.8, 1.0$

Finite Difference Method (IFDM) with time grid points of 100, space grid points of 200 [9] and BSM [1] are shown in Figures 1-3. The comparative results analyses of GTDTM, IFDM and BSM are displayed in Figure 4. The absolute errors incurred in GTDTM and IFDM are shown in Figures 5 and 6, respectively. The comparative error analysis of GTDTM and IFDM is shown in Figure 7. The comparative study of GTDTM and BSM, for different values of α is captured in Figure 8. The effect of the fractional order on the option prices obtained via GTDTM is displayed in Figure 9.

5.2. Discussion of results and concluding remarks

In this paper, the Black-Scholes equation for a European call option involving a fractional order with generalized dimensional differential transform method is analysed. The Black-Scholes equation for a European call option with fractional order has been solved via GTDTM. To measure the performance of GTDTM in terms of the accuracy, effectiveness and suitability, an illustrative example is presented. The European call option prices generated via GTDTM, IFDM and BSM are displayed in Figures 1, Figure 2 and Figure 3, respectively. It is observed from the Figures 4 that GTDTM agrees with the BSM for different values of the strike price K. It is clearly seen from Figure 4 that

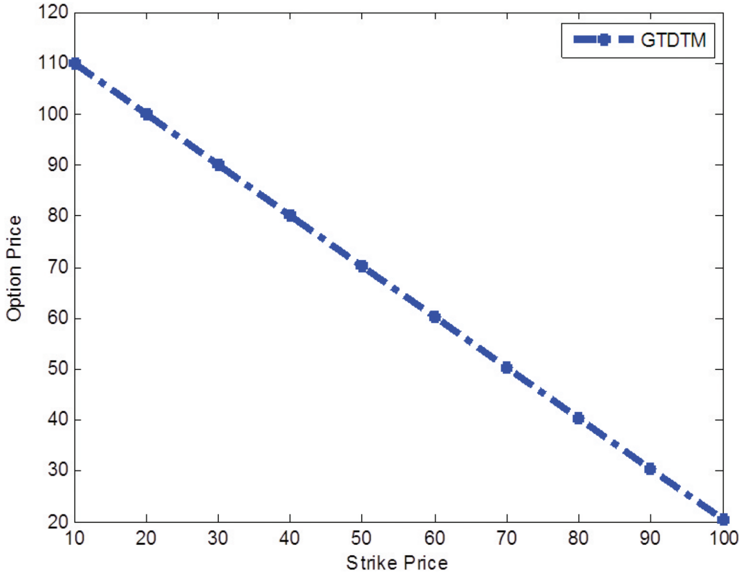

Figure 1: *Option prices generated via GTDTM*

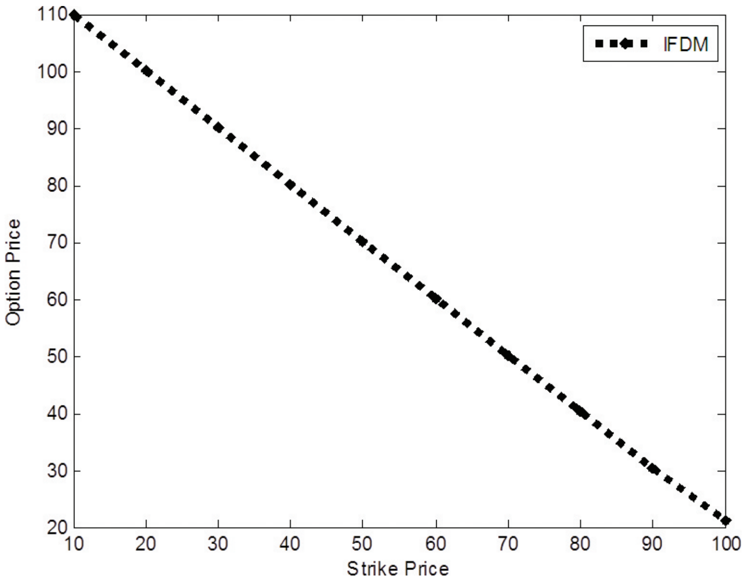

Figure 2: *Option prices generated via IFDM*

GTDTM compared favourably with BSM. It is also observed that GTDTM performs better than IFDM for $K \leqslant 90$. Figures 5 and 6 show the absolute errors generated via

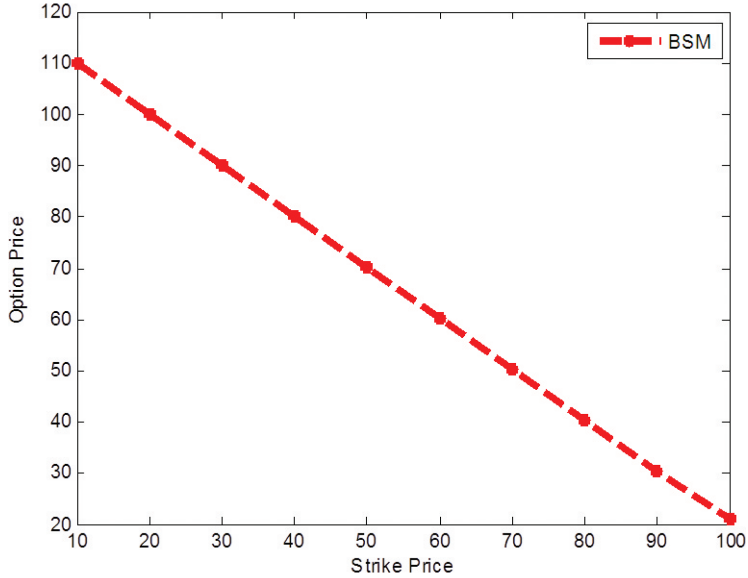

Figure 3: *Option prices generated via BSM*

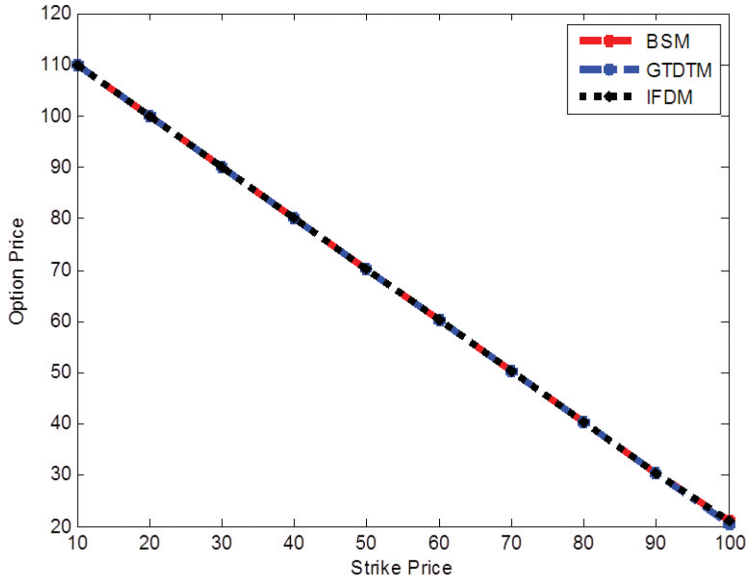

Figure 4: *The comparative results analyses of GTDTM, IFDM and BSM*

GTDTM and IFDM for different values of the strike price K. The comparative error analysis of GTDTM and IFDM is shown in Figure 7. Figure 8 shows the performance

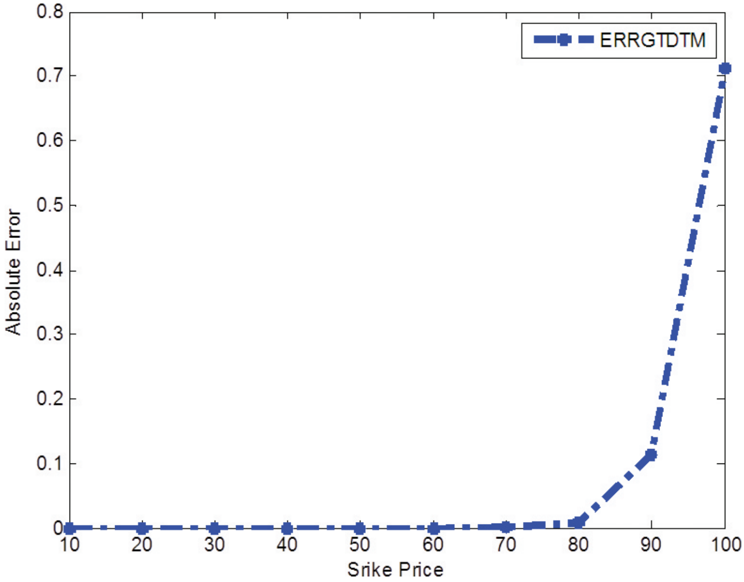

Figure 5: *Error incurred via GTDTM*

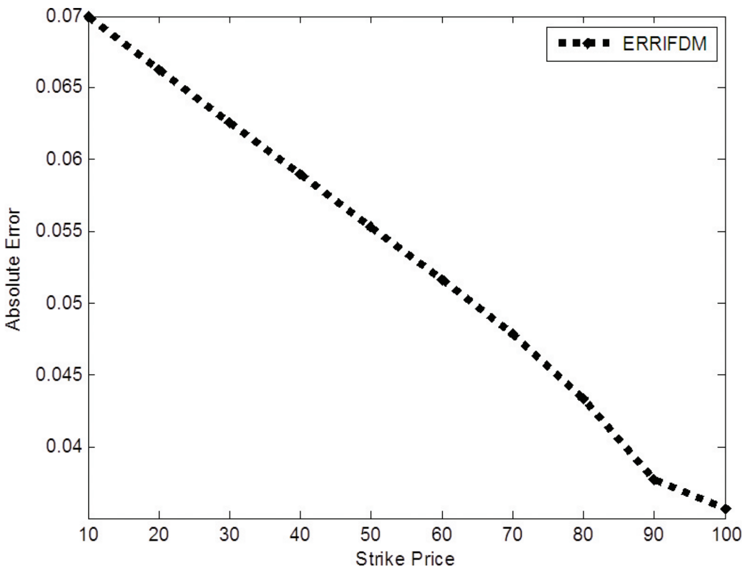

Figure 6: *Error incurred via IFDM*

of GTDTM for different values of fractional order α in the context of the BSM. It is observed from Figure 8 that GTDTM agree with BSM for $\alpha = 0.6, 0.8, 1.0$. It is also observed that when $\alpha = 0.2, 0.4$, European call option is overpriced. It is observed

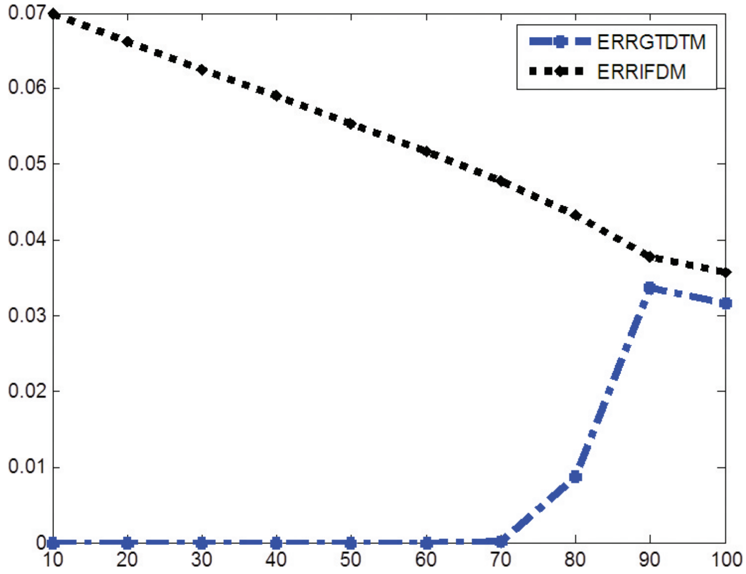

Figure 7: *The comparative error analysis of GTDTM and IFDM*

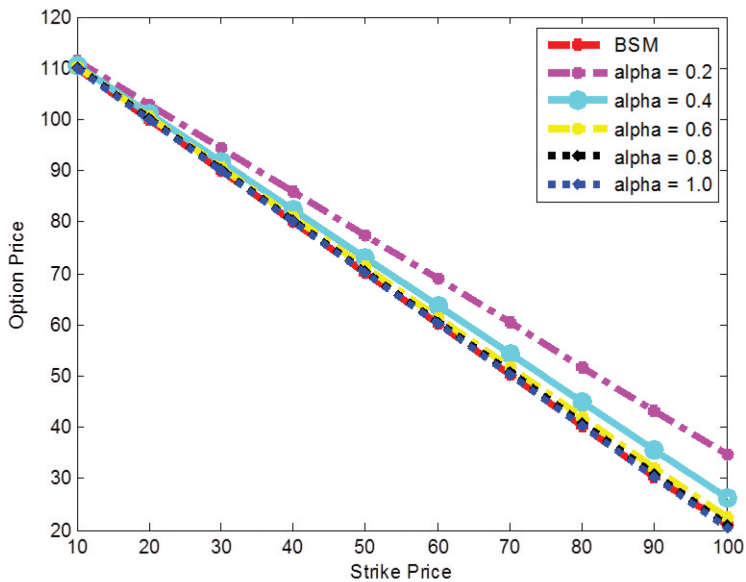

Figure 8: *GTDTM versus BSM for different values of* α

from Figure 9 that, as α decreases, the payoff of the option increases. It is also observed that European call option has the lowest price in exercise time T when $\alpha = 1.0$. Hence,

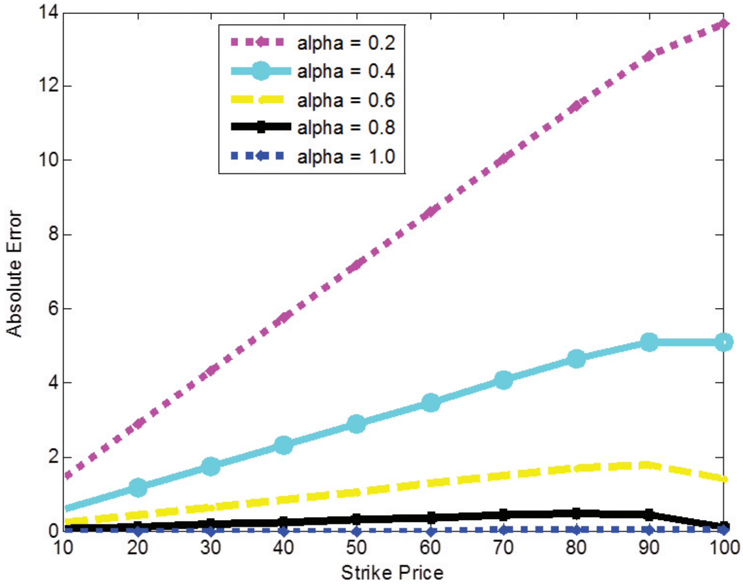

Figure 9: *Option prices via GTDTM with different values of* α

it can be concluded that GTDTM is a very efficient tool in obtaining both exact and approximate solutions for the Black-Scholes equation for a European call option with fractional order. It is also a good approach for the valuation of European call option with fractional values. A natural extension is to use GTDTM to solve fractional Black-Scholes equation for basket options with dividend yield under the geometric Brownian motion.

Conflicts of Interest. The authors declare that all the information provided in this paper has no conflict of interest.

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